STRUCTURAL ENGINEERING

STRESSES, GRAPHICAL STATICS AND MASONRY

BOOKS

BY

GEORGE FILLMORE SWAIN

STRENGTH OF MATERIALS
570 pages, 6 × 9, illustrated

FUNDAMENTAL PROPERTIES OF MATERIALS
200 pages, 6 × 9, illustrated

STRESSES, GRAPHICAL STATICS AND MASONRY
525 pages, 6 × 9, illustrated

DESIGN OF FRAMED STRUCTURES
In Press

HOW TO STUDY
65 pages, 5½ × 8, paper

STRUCTURAL ENGINEERING

STRESSES, GRAPHICAL STATICS AND MASONRY

BY

GEORGE FILLMORE SWAIN, LL.D.

Gordon McKay Professor of Civil Engineering in Harvard University; formerly Chairman of the Boston Transit Commission and Consulting Engineer of the Massachusetts Railroad Commission; Past President Am. Soc C. E., Member Inst. C. E., Am. Soc. M. E., Eng. Inst. of Canada, Am. Soc. for Testing Materials, Am. Ry. Eng. Assoc., Boston Soc. C. E., Am. Acad. Arts and Sciences, National Acad. of Sciences, Etc.

FIRST EDITION

McGRAW-HILL BOOK COMPANY, INC.
NEW YORK: 370 SEVENTH AVENUE
LONDON: 6 & 8 BOUVERIE ST.; E. C. 4
1927

PRINTED IN THE UNITED STATES OF AMERICA

THE MAPLE PRESS COMPANY, YORK, PA.

Dedicated

to the memory of my Father, who taught me never to be afraid of the truth, or to evade it, however unpleasant it may be

PREFACE

This volume, which is the third in my work on Structural Engineering, comprises:

1. The theory of statically determined framed structures, by analytical and graphical methods. The subject of graphical statics is presented in a comprehensive manner, showing how any problem of statics may be solved graphically as well as analytically.

2. The theory and design of masonry structures, giving the fundamental principles and the methods of their application, and the theory of earth pressure.

Reinforced concrete is not treated in this volume.

The next volume, which is nearly completed, will treat of the design of framed structures and of beams and girders, with the principles governing dimensioning; and it will also contain an extended chapter in which the theory and design of structures of reinforced concrete are discussed. Statically undetermined structures, such as the reinforced concrete arch, will be treated in the fifth and final volume of the series, which is about half completed.

The present volume and the succeeding one contain essentially the course which the writer has been giving for many years at the Massachusetts Institute of Technology and at Harvard University, considerably extended and brought to date. It is hoped that they will meet the needs of practicing engineers as well as of students in technical schools.

A course of four years in a technical school, in the writer's opinion, does not permit of doing more than two things, or rather, of attempting to do them, namely:

1. To train the student to think and to acquaint him with the principles governing scientific investigation.

2. To show him the elementary fundamental principles of the branches of engineering.

Unless he has learned these two things, the student is not qualified to pursue the higher or more complicated branches of the subject with understanding. Realizing this, the writer has never attempted to teach statically undetermined structures in an undergraduate course. Many schools do attempt this; but from the writer's experience it results in making the student do things which he is shown how to do, without clearly understanding why he is doing them; or else it results in a narrow specialization which prevents a student from being a broadly cultivated engineer. It is impossible to build securely on an insufficient foundation.

This seems to the writer to be what is too often attempted. The school cannot make an engineer; it can only give fundamentals, if it does its work thoroughly. Indeed, all that any teacher can do is to help the student to teach himself. The only real education is self-education, as was long ago pointed out by William Ellery Channing, in his address on Self Education. The teacher may of course give rules and formulae, and show the student how to use them; but this is merely vocationalizing education and making a man a machine which performs certain motions without understanding the reason and purpose of them. It is not true education. True education is *drawing out*, or developing the innate powers of the student, not putting facts into him. The writer is concerned only with true education.

The writer will be grateful to any reader who will point out any errors that may be found in this work. Undoubtedly there are some, for no human work is or can be perfect. I am much indebted, for assistance in connection with this volume, to my former assistants, Albert Haertlein and J. G. Peter.

George Fillmore Swain.

Harvard University,
Cambridge, Mass.
September, 1927.

CONTENTS

Choose a sensible man to a responsible place rather than a man versed in the particular art which is to be taught (or practiced), inasmuch as a *method of acquiring truth is better than the truth it has already ascertained.* Let your discipline liberalize the mind of a boy rather than teach him science (*i.e.*, facts), *that he may have means, more than results.*

The Indian will give his bow for the knife with which it was made.

From "Journals of Ralph Waldo Emerson,"
Vol. 2, p. 67. Written in 1825 at the age of 21
(slightly amended by parentheses and italics).

"Books once were men." Men put themselves into books as they write them; and their readers, to a certain degree, do the same when they read them. About all a writer puts into his book is what he gets out of himself; and about all a reader gets out of a book is what he himself can put into it. The painter paints what he sees with his eye, the writer what he sees with his mind.

Adapted from G. B. McCutcheon.

In experimental science it is always a mistake not to doubt when facts do not compel affirmation.

Pasteur.

When you meet with a fact opposed to a prevailing theory, you should adhere to the fact and abandon the theory, even when the latter is supported by great authorities and generally adopted.

Claude Bernard.

Ce que l'on conçoit bien s'énonce clairement,
Et les mots pour le dire arrivent aisément.

Boileau, L'Art Poètique
Chant I.

STRUCTURAL ENGINEERING

STRESSES, GRAPHICAL STATICS AND MASONRY

CHAPTER I

INTRODUCTION

1. Scope of Subject.—A thorough knowledge of the subject of Strength of Materials, with its dependent subjects, Mechanics, and Properties of Materials, as presented in the first and second volumes of this work, the latter supplemented by the more elaborate treatises therein recommended, should enable the reader to find the maximum stresses produced by given loads *in any solid piece* of any engineering material; and, further, to know what material is best suited to his purposes, what qualities are requisite, and how to test them.

But engineering structures are not in general made of one solid piece of material. A beam, tie, column, or shaft may be such a solid piece; or it may be composed of several single solid pieces connected together. The Theory and Design of Structures, treated in the present volume and the one following, studies the computation and design of such pieces and of the connections of their several parts.

Most structures, however, consist of a number of ties, columns, and beams, connected together to form a *framed structure*, like a bridge truss or a steel-frame building. A large part of the subject of Structures consists in the study of the computation and design of such structures, and of the connections between the main members or parts. It also concerns itself with the proper outer forces or loads which should reasonably be assumed to act on a structure, which are more uncertain than would at first sight appear, and with the effects produced by motion of those loads and the impact and vibration which they produce. Further, it studies the methods of proportioning, that is to say, the application of the results found by static or other tests (as studied in the previous volumes) to the actual design of safe structures.

2. Classification.—Structures may be classified in various ways, as, for instance, according to material, into structures of

(*a*) wood,

(*b*) metal (steel or iron),

(*c*) natural or artificial stone (masonry), and

(*d*) combinations of material.

Like most systems of classification, this one is somewhat arbitrary and unsatisfactory. Some structures, for instance, may be constructed equally well of wood or of steel, and the sole difference in the two cases will be that of material, which is rather a minor distinction. Only masonry structures constitute a class by themselves, with peculiar properties independent of mere material; and even these distinctions are being gradually eliminated, in modern forms of construction, as will be seen in the course of this work.

Structures of wood include, among others, beams, pile and trestle bridges, wooden bridges and roofs, scaffolding of all kinds, and the framing of buildings. Large wooden structures, however, seldom occur without the use of metal in combination with the wood, as for nails or fastenings, or even for some of the main parts. Some wooden structures, however, are made with wooden dowels instead of nails or bolts, and are entirely of wood.

Structures of metal comprise all structures which may be built of wood, as enumerated above, and, in addition, many others for which wood is not suitable, such as long bridges and the framing for high buildings.

Structures of masonry include walls of buildings, retaining walls to support the pressure of earth, piers and abutments of bridges, culverts, arch bridges, domes, masonry dams, etc. Under this class is included structures of concrete, which is an artificial stone.

Combined structures are those in which various materials are combined. Wood and metal are frequently combined, as noted above; and of late years a large and very important class of structures has been developed, in which steel and concrete are combined, forming what is termed *reinforced concrete.*

Another classification is according to the character of the *external forces acting,* into pieces or structures subject to

(*a*) direct tension (ties),

(*b*) compression (struts, or columns, or piers),

(*c*) flexure (beams),

(*d*) torsion (shafts), and

(*e*) a combination of the above forces.

Again, classification may be made according to the *form of construction,* into structures composed

(*a*) of one solid or homogeneous piece, like a wooden beam or column, or one of steel rolled in one piece,

(*b*) of several pieces riveted or otherwise connected together continuously, so as to act as one solid piece, like a plate girder or a built-up column, and

(*c*) of several pieces connected together at their ends, forming an open framework, a so-called *truss.*

This last classification indicates that a structure may be composed of parts, each of which is in itself a structure of a simpler kind, just as a building may be composed of parts each of which is itself a building.

3. Forces Acting.—The forces which have to be studied with reference to a structure are of two kinds which should be carefully distinguished, namely, the *outer forces* and the *inner forces*.

The *outer forces* are those which are applied to the structure from without. They are the active forces, and it is the function of the structure to sustain them. But these outer forces are themselves of two kinds: the *loads* and the *reactions*.

The *loads* are known or must be determined as a preliminary step to the study of the structure; for instance, a railroad bridge must be designed so that it may carry safely a load consisting of the heaviest train that may have to pass over it; and a roof must be able to withstand the heaviest weight of snow or the greatest wind pressure which may ever act upon it. The determination of the loads frequently requires considerable study, investigation, and inquiry, besides the exercise of much judgment; but this is essentially preliminary to the study of the structure itself. Every structure is designed with a so-called *factor of safety*, which represents the number of times the assumed load which would be required to destroy it. For instance, if the factor of safety is five, this means that it would require five times the assumed load to destroy the structure.

Now if this margin is present, it is clearly not necessary to provide for a very remote and certainly infrequent contingency. In proportioning a bridge to resist wind pressure, for instance, it is not necessary to assume that a gale is blowing such as would only occur once in a century, and to allow a factor of safety of five with this load. If the heaviest usual or probable load is used with a factor of five, an occasional heavier load does not require so large a factor; but no load which can be foreseen should be so large that it would actually wreck the structure. This explanation is to illustrate the fact that the determination of the proper loads to use is not only uncertain, but requires insight and good judgment. Further discussion of the factor of safety will be given later.

The loads, however, are always of two kinds, namely, the weight of the structure itself (known as the *dead load* because it is unvarying and immovable), and the outwardly applied loads. These last may be unvarying like the pressure of earth against a wall, but generally are applied in a varying and intermittent manner, like the load upon a railway bridge or the wind pressure upon a roof. For this reason they are generally called *live loads*. Now, after a structure has been designed, its dead load may be computed with any desired degree of precision, because all the dimensions and materials are known, whereas until it is designed its dead load is unknown. The engineer in designing is therefore always con-

fronted with this predicament that he cannot design the structure until he knows its dead load or weight, and he cannot know this weight until he has designed the structure. He must therefore assume the dead load beforehand, by experience, rules, or tables, and then design the structure to support both dead and live loads. Afterward he must not forget to compute the dead load and see whether it agrees with his assumption. If the divergence is too great, it may be necessary to redesign the structure. The neglect of this precaution has in some cases led to very serious consequences, as in the case of the first Quebec bridge, which collapsed. The dead load, being merely the weight of the structure, always acts vertically, and is sometimes called the *dead weight.*

The *reactions* or *supporting forces* are the forces exerted at the places where the structure is supported, and are outer forces called into action by the applied loads and forming with those loads a system of forces in equilibrium. So far as the mechanics of the problem is concerned, the loads and reactions are indistinguishable in character; but the reactions are always determined after the loads have been assumed, and are such that they must balance those loads, subject to any conditions imposed by the construction of the supports, or by other conditions. For instance, a reaction acting through a frictionless roller must act perpendicular to the surface upon which the roller rests.

If the reactions may be found by the principles of statics alone, the structure is said to be *statically determined* (*or determinate*) *with respect to the outer forces.* If they are not, the structure is *statically undetermined with respect to the outer forces,* and the reactions must be found by the aid of elastic conditions, that is to say, conditions based on the conditions of deformation. Such cases will be studied in a succeeding volume.

The determination of the inner forces, or stresses, follows, in general, the determination of the reactions. The reactions being given, or found, the inner forces may be found by statics alone, in which case the structure is termed *statically determined with respect to the inner forces;* or they cannot be so found, in which case the structure is *statically undetermined with respect to the inner forces,* in which case the latter must be found by elastic conditions. Such cases will, in general, be left to a following volume.

4. In tabular form, therefore, we have:

<table>
<tr><td rowspan="4">Forces to be considered in the design and computation of structures</td><td rowspan="3">Outer</td><td rowspan="2">Loads</td><td>Dead, or weight of structure (fixed)</td></tr>
<tr><td>Live (to be assumed)</td></tr>
<tr><td colspan="2">Reactions to balance loads</td></tr>
<tr><td colspan="3">Inner, or stresses</td></tr>
</table>

The first step in the design of a given structure, in general, having determined the type by economic and practical considerations, is to decide upon the live loads to be taken, then to fix on the dead loads, next to find the reactions, then to find the stresses in the various parts, and finally to design those parts. Afterward the actual dead load should be computed and compared with the assumption made, and, if necessary, the design revised. If the designer has experience, his first assumption of the dead load will be close; but, if the structure is a large or novel one, his assumption may be much in error. The above procedure may be modified in certain cases, as will be seen later.

CHAPTER II

DETERMINATION OF THE REACTIONS

1. The live loads which a structure must be designed to support depend upon the character of the structure, and their determination is often a matter of difficulty and requires the exercise of much judgment. The difficulties, however, are not mathematical, but arise from essential uncertainties in the problem. A discussion of these loads will be given in Chap. III; for the present they may be classified simply as follows:

a. Live loads on building floors; weight of a crowd of people, of furniture or other contents of rooms.

b. Live loads on roofs: weight of snow or pressure of wind.

c. Live loads on highway bridges: weight of a crowd of people, of snow, of heavy teams, cars, or road rollers, and lateral pressure of the wind.

d. Live loads on railroad bridges: weight of trains, centrifugal force on curves, lateral pressure of the wind.

e. Live loads on retaining walls: pressure of the earth.

f. Live loads on dams: pressure of the water.

In any of these cases there may be special loads, and there are many other structures not mentioned above: chimneys, for instance, are exposed to wind pressure; coal pockets and bins to the pressure of coal and the load of towers for handling, and so on.

When the loads applied to a structure are given or have been ascertained, the next step is to determine the reactions which balance those loads.

These reactions are not determined by chance, but are fixed by the conditions existing, and it is important to inquire first when they are determined by the laws of statics alone. *If, for any given system of loads, the reactions are fixed by the statical conditions of equilibrium between the loads and reactions, the latter are said to be* **statically determined,** *and the structure is said to be statically determined as regards the outer forces.* But in some cases there may be a number of sets of reactions, any one of which would balance the loads (*i.e.*, fulfil the statical conditions). One of these sets is the true set of reactions. In this case the reactions are said to be *statically undetermined.* Reactions which are statically undetermined may, however, be found by imposing the statical conditions and also a proper and necessary number of *assumptions*, and this is the usual procedure; they may also be found by imposing suitable elastic conditions,

or conditions relating to the deformation of the structure; but the accuracy of these is often questionable, as will be seen, and they are not at the command of most students.

2. When Is a Structure Statically Determined as Regards the Outer Forces?—A structure may be supported at any number of places, and the loads may be applied in any manner. Let us consider here only one simple and most usual case, that in which there are two supports, and in which all the outer forces, both loads and reactions, lie in a plane, which we shall take as the plane of the paper. The two reactions are R_1 and R_2, and the loads are supposed to be given. In order to determine the reactions completely, six quantities must be found, namely, two components and a point of application for each reaction. The surface on which the structure is supported is of course given by the conditions of the problem; in Fig. 1, for example, the planes *ab* and *cd* are known, so that one coordinate of the point of support s_1 completely fixes that point, and so for s_2.

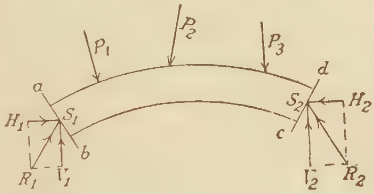


FIG. 1.

The problem, then, is one in algebra; there are six unknown quantities to be determined, *viz.*, two points of support, and the four forces, H_1, V_1, H_2, and V_2; and the question is, whether the laws of statics furnish equations enough to determine these quantities. If they do, the structure is *statically determined regarding the outer forces;* if they do not, the structure is *statically undetermined regarding the outer forces.*

Now the laws of statics furnish just three independent equations between the loads and reactions, namely, the three equations of equilibrium of forces in a plane, $\Sigma H = 0$, $\Sigma V = 0$, and $\Sigma M = 0$. Since we must, therefore, have as many independent conditions as there are unknown quantities to be determined, it follows that unless three additional conditions are given by the manner of support or of construction, the structure is statically undetermined as regards the outer forces.

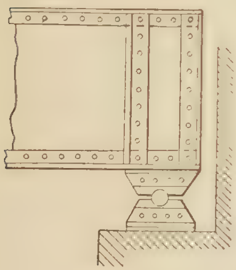


FIG. 2.

3. Usual Methods of Giving Conditions Relating to Outer Forces.—The conditions necessary to make a structure statically determined as regards the outer forces may be given by various forms of construction.

a. If either end is supported upon a knife-edge, the point of application of the reaction is thereby fixed. Knife-edges are used in scales for weighing, but are inadmissible in engineering structures. If, however, one end rests upon a pin, about which the structure is free to turn, the

point of application of the reaction is practically fixed at the center of the pin (see Fig. 2).

b. If either end rests upon rollers which are free to roll along the supporting surface, then, if we neglect friction, the supporting force must act at right angles to that known surface, *i.e.*, the relation of H to V for that reaction is fixed. If the surface of the support is horizontal, $H = 0$; if it is inclined to the horizontal at an angle α, then $H = V \tan \alpha$ (see Fig. 3). If both pin and rollers are used (see Fig. 4), the reaction is fixed as regards both point of application and line of action, whereas, if the pin only is used, the line of action is not fixed.

It must be noted that if we say that simply the direction is fixed by rollers, the structure must be supposed held to the supports in such a way that the reaction may, if necessary, fall outside the rollers; thus in Fig. 5, if the reaction should be R' acting at c normal to the surface, the condition imposed simply by the rollers would be met, but in this case the left-hand part of the bearing surface near a would have to be held down to the supporting surface. If there were but two rollers, for example, one at

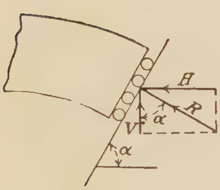


Fig. 3.

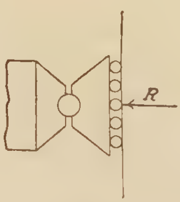


Fig. 4.

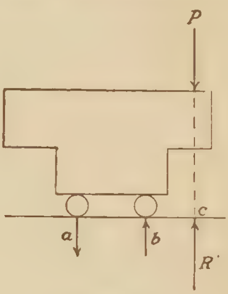


Fig. 5.

b and one at a, the former would be pressed against the surface, and the latter pulled away, the resultant of the two forces being the reaction R'. If the structure is not held against such tipping as would occur in the previous case if a is not held down, and a nest of rollers is simply inserted between the structure and the support, then this construction really gives two conditions: first, that the reaction is perpendicular to the supporting surface and, second, that the reaction must not fall outside of ab. If, under the first condition alone, the structure is statically determined and the reaction thus determined falls *within* ab, then the second condition, being otherwise fulfilled, does not affect the problem; but if, under the first condition alone, the structure is statically determined and the reaction thus determined falls *outside* ab, then, since this last condition is not fulfilled, as it must be for stability, and since without it the structure is statically determined, it follows that with it the structure has too many conditions, one of them unfulfilled, and that it will collapse.

In other words, care must be taken to observe how many conditions are really given by any specific form of construction.

When we speak of the *direction* of a force, we mean that it acts parallel to a given line, but we do not mean that the *sense*, *i.e.*, the direction in which it acts along that line, is fixed. Thus when we say that one reac-

tion is vertical, we do not mean that it acts upward or downward, but simply in a vertical direction.

c. If a structure is separated at any point into two distinct parts which are connected by a hinge about which either part is free to rotate, the condition is thereby introduced that the moment about that hinge of all the outer forces acting on either side of it must be zero (Fig. 6). Thus, in the figure, the resultant moment about the pin of H_1, V_1, and P_1, must be zero, P_1 being the resultant load to the left of the pin. In other words, the resultant outer force on either side of the hinge must act through the hinge.

Usually both points of support are fixed by pins, or in some similar manner, and the direction (but not the *sense*) of one reaction is fixed by rollers, or by a sliding support in which friction is neglected; but other methods are sometimes used. The examples, some of which are purely imaginary and impractical, and inserted only for illustration, will show

Fig. 6.

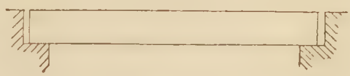


Fig. 7.

various possibilities. The student must accustom himself to see at once whether a structure is statically determined, for if it is not, assumptions must generally be made to solve it. He who makes an assumption deliberately and carefully, knowing that it is necessary, may be said to work intelligently, and may make a close approximation to the truth; but he who makes assumptions without knowing that he is doing so is unsafe and unreliable. If he is right, it is by chance.

4. It is not only necessary to have a *sufficient* number of equations, but it is equally necessary not to have too many. Evidently *a structure may be statically undetermined in two ways: either by having too few conditions given, or by having too many conditions given.* If *too few* conditions are given, then the necessary number must be obtained by making assumptions, unless elastic conditions are used; for instance, if but one point of application is fixed by a pin and the direction of but *one* reaction is fixed by rollers, then one condition is lacking, and this may be supplied by assuming the point of application of the other reaction. An ordinary beam supported at the two ends (Fig. 7) is statically undetermined, having all three conditions lacking, none given but the statical equations. Here the usual practice is to assume each reaction to act in the center of its bedplate, and to assume one reaction to be either vertical or parallel

to the resultant load. A structure statically undetermined, with some conditions lacking, is stable, but may have various reactions consistent with equilibrium.

If, however, a structure is statically undetermined with *too many* conditions given, it is in general unstable and will collapse *unless the system of loads happens to be such as will satisfy the superfluous conditions.* For instance, suppose that each of the two reactions has its point of application fixed by a pin and that each end is placed on rollers on a horizontal surface. In this case, besides the three statical equations, there are four conditions given by the manner of support; that is, there is one superfluous condition. The structure is therefore unstable, and in general will be rolled from its supports; it will be stable only when the resultant load is vertical, and hence has no tendency to move it horizontally.

5. Advantages and Disadvantages of Statically Undetermined Structures.—The principal advantage of having the reactions statically determined is that they may be found by simple and immutable laws, so that there is no doubt regarding their values. If they are statically undetermined, they must be found by means of assumptions which may or may not be true, or else they must be found by laborious calculations which are themselves based upon suppositions which may be rendered entirely erroneous by slight and possible changes in physical conditions. For instance, we may assume that the reactions at the ends of the beam in the last figure act at the center of the bedplates, but a slight settling of the abutments, or a slight tipping due to a yielding foundation, may throw the actual bearing upon one edge. In general we may say that *the reactions, if statically undetermined, may be materially altered by circumstances beyond our control.*

6. Process of Finding the Reactions When Statically Determined.—It has been explained that the usual case is that in which both points of application and the direction of one reaction are fixed. In all cases, however, the unknown reactions are found by determining their components by means of the three equations:

Sum of all horizontal components of outer forces = 0, ($\Sigma H = 0$);
Sum of all vertical components of outer forces = 0, ($\Sigma V = 0$);
Sum of all moments of outer forces about any point = 0, ($\Sigma M = 0$).

Resolve each reaction into components, assume their directions, place arrows upon them, and apply the equations. The true directions in which they act may not be obvious, but the arrows may be assumed in any direction (consistent with each other), and if the summations are taken properly, the result will be correct, a negative value for any component meaning that it acts in a direction contrary to that assumed. If all unknown forces, positive in value, are assumed to act to the right or upward, then a negative vertical force will always mean a downward force, and a negative horizontal force will always mean a force acting to

the left. The proper signs must be carefully observed in making the summations, but the entire process is extremely simple, as a few examples will indicate. In the above, by direction is meant sense. The components are horizontal and vertical.

In finding reactions for loads which are distributed, a diagram showing the loads should always be drawn in such a way that the load per running foot at each point is laid off as an ordinate. The resultant load will then be the area of this figure, and will act through the center of gravity of the figure. Thus a uniformly distributed load will be represented by a parallelogram.

Example I.—A beam supported at both ends (Fig. 8).

Since both points of support are given and one reaction is vertical, the outer forces are statically determined.

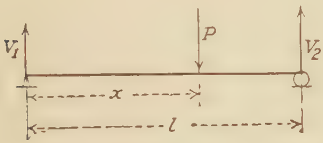


FIG. 8.

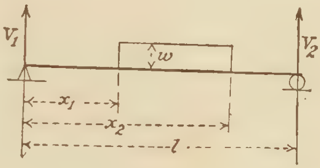


FIG. 9.

Case 1.—If the load consists of a single concentrated load P as shown,

$$\Sigma M = 0 \text{ gives at once } V_1 = P\frac{l - x}{l}$$

$$\Sigma V = 0 \text{ gives } V_2 = P\frac{x}{l}$$

There are no horizontal components of the reactions.

Case 2.—If the load P is inclined, resolve it into horizontal and vertical components H and V. Then V causes reactions as above, while H causes an equal horizontal reaction at the fixed end of the beam.

Case 3.—If the load is a uniformly distributed load of w pounds per foot, covering a distance as shown in Fig. 9, we find by similar methods

$$V_1 = \omega(x_2 - x_1)\frac{2l - x_1 - x_2}{2l}$$

$$V_2 = \omega(x_2 - x_1)\frac{x_1 + x_2}{2l}$$

Case 4.—If both ends are fixed, *i.e.*, able to exert a horizontal reaction, the outer forces are statically undetermined. In this case the assumption is usually made that the reactions are parallel to each other and to the resultant (Fig. 10).

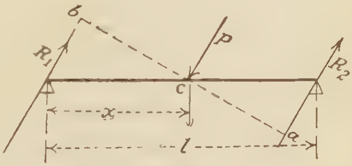


FIG. 10.

In this case we find

$$R_1 = P\frac{ac}{ab} = P\frac{l - x}{l}$$

$$R_2 = P\frac{bc}{ab} = P\frac{x}{l}$$

so that not only is each vertical component of the reactions proportional to the distance from the load to the other point of support, but the horizontal components are in the same proportion.

The laws of reactions on simple beams should be perfectly familiar to every student before he proceeds further.

Example II.—A projecting roof or awning is loaded as shown (Fig. 11) and supported at A and B. At B let us suppose that there is a roller between the roof and the wall; this roller may be supposed attached to the roof, and yet roll in a cage so that the roof could not be pulled away from the wall.

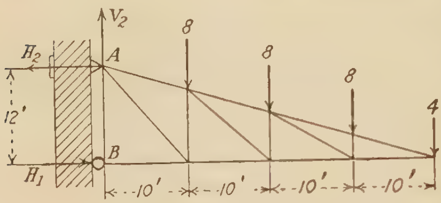


FIG. 11.

Now with the loads as shown, anyone can see that the reaction at B must act to the right, and it must be horizontal on account of the roller. Both points of application and the direction of one reaction are given; hence the structure is statically determined. We indicate the reaction components as shown, and we then write immediately

$$8 + 8 + 8 + 4 = V_2 \quad (\Sigma V = 0)$$
$$H_1 - H_2 = 0 \quad (\Sigma H = 0)$$
$$80 + 160 + 240 + 160 - 12H_1 = 0 \quad (\Sigma M = 0 \text{ about } A)$$

From these we have at once

$V_2 = 28$ (acting upwards, as shown)
$H_1 = 53.33$ (acting to the right as shown)
$H_2 = 53.33$ (acting to the left as shown)

Example III.—A series of roof trusses 20 feet apart (Fig. 12).

Load consists of wind pressure on the left, of 50 pounds per square foot on the vertical surface, and 20 pounds per square foot on the inclined surface.

Both points of application of reactions are given, and we suppose the right-hand end, as an illustration, to roll upon a surface inclined at 30° as shown.

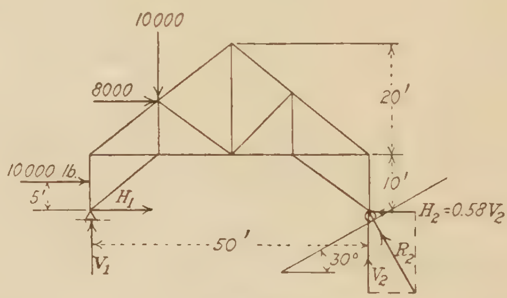


FIG. 12.

Knowing the relations between the lengths of the sides of a 30° triangle to be 1, 2, and 1.73, we proceed as follows:

Length of inclined roof	$= \sqrt{400 + 625} = 32$ feet
Load on left vertical side	$= 20 \times 10 \times 50 = 10{,}000$ pounds
Load on left inclined area	$= 20 \times 32 \times 20 = 12{,}800$ pounds
Vertical component of this	$= 12{,}800\frac{25}{32} = 10{,}000$ pounds
Horizontal component of this	$= 12{,}800\frac{20}{32} = 8{,}000$ pounds

These components may and should be found directly:

Vertical component $= 20 \times 25 \times 20 = 10{,}000$
Horizontal component $= 20 \times 20 \times 20 = 8{,}000$

Since we are simply finding reactions, we deal only with the resultant forces acting at the centers of the areas, as shown, since the force is uniformly distributed. We assume the left reactions to act as shown; those at the right we must assume with the resultant acting normal to the inclined surface, but we may assume it acting in either

direction; that is, if we assume V_2 acting up, we must assume H_2 toward the left, or if we assume V_2 acting down, we must assume H_2 toward the right; we cannot assume V_2 acting up and H_2 toward the right, or V_2 down and H_2 toward the left. We label V_2, and see at once that $H_2 = \frac{V_2}{1.73} = 0.58V_2$. Hence we write

$$H_1 - 0.58V_2 + 18{,}000 = 0 \quad (\Sigma H = 0)$$
$$V_1 + V_2 - 10{,}000 = 0 \quad (\Sigma V = 0)$$
$$50{,}000 + 160{,}000 - 375{,}000 + 50V_1 = 0 \quad (\Sigma M = 0 \text{ about right support})$$

From these we find at once

$$V_1 = 3{,}300 \text{ (acting } up)$$
$$V_2 = 6{,}700 \text{ (acting } up)$$
$$H_2 = 0.58V = 3{,}886 \text{ (acting toward the } left)$$
$$H_1 = -14{,}114 \text{ (acting toward the } left)$$

It is clear that the solution of all such problems requires simply arithmetic, a little geometry, and a knowledge of the equations of mechanics. Trigonometry is seldom

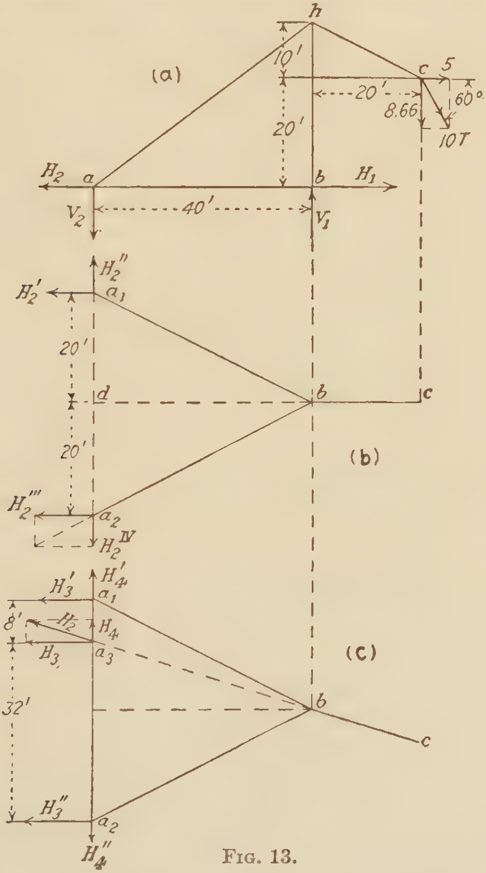


FIG. 13.

required, and sines, cosines, or other functions should not be introduced unless the object is to confuse that which is simple.

Example IV.—A crane is shown in elevation and plan in Fig. 13 and is held by two guys. In this case, although the reactions to be found are those at b, a_1, and a_2,

the structure is really to be considered as exemplifying a case of forces in a plane; for the load must lie in the vertical plane bhc; otherwise, if it had a component perpendicular to this plane, it would swing the crane about the post bh; and the structure may be considered as supported at b and h, with the condition that the reaction at h, which acts in the vertical plane bhc, is to be resolved along the two guys. This reaction at h, therefore, not only acts in the vertical plane bhc, but also in the Inclined plane a_1a_2h. If, then, we choose to consider the reactions as acting at the level of the ground, they act at b and at that point in the line a_1a_2 where it is intersected by the plane bhc, whatever may be the position of the crane; *i.e.*, however it may be swung around the post bh.

Case 1.—Let the crane bisect the angle between the guys, in plan. Then each guy has equal reactions (Figs. 13*a* and 13*b*). All numerical results are in tons.

We first perceive that

$$V \text{ component of load } = \frac{10\sqrt{3}}{2} = 8.66$$

$$H \text{ component of load } = \frac{10}{2} = 5$$

$$H_2 = \frac{4}{3}V_2$$

$$H_2' = H_2''' = \tfrac{1}{2}H_2$$

$$H_2'' = H_2'''' = \tfrac{1}{2}H_2$$

$$\Sigma M = 0 \text{ about } a \text{ gives}$$

$$8.66 \times 60 + 5 \times 20 - 40V_1 = 0$$

$$V_1 = 15.49 \text{ (acting up)}$$

$$\Sigma V = 0 \text{ gives}$$

$$-V_2 - 8.66 + 15.49 = 0$$

$$V_2 = 6.83 \text{ (or 3.42 at } a_1 \text{ and the same at } a_2)$$

$$H_2 = \frac{4}{3}V_2 = 9.11$$

$$H_2' = H_2''' = 4.56$$

$$H_2''' = H_2'''' = 2.28$$

$$\Sigma H = 0 \text{ gives}$$

$$-9.11 + H_1 + 5 = 0$$

$$H_1 = 4.11 \text{ (toward the right)}$$

Case 2.—If the crane is swung so as to be in the plane of one guy a_1b, we have no reaction at a_2.

Distance $a_1b = 44.72$ feet

$$\Sigma M = 0;\quad V_1 = \frac{8.66 \times 64.72 + 5 \times 20}{44.72} = 14.8 \text{ (by slide rule)}$$

$$\Sigma V = 0;\quad V_2 = 6.1 \text{ (at } a_1)$$

$$H_2 = \frac{44.72}{30}V_2 = 9.1 \text{ (by slide rule)}$$

$$\Sigma H = 0;\quad H_1 = 4.1$$

After some practice the student will be able to write the values directly as above, without first writing them in the general form $\Sigma(-) = 0$; but he should not do this until he has trained himself to careful habits in observing the directions of the forces. All of the numerical work is given above, the slide rule being used to obtain results. This useful instrument gives results which are close enough for most structural work, the above values being correct to the nearest tenth.

Case 3.—If the crane be swung into the plane a_3bc (Fig. 13*c*), we may find the reactions at a_3 and b, and then resolve that at a_3 into two components at a_1 and a_2 proceeding thus:

$$\text{Distance } a_3b = \sqrt{40^2 + 12^2} = 41.76 \text{ feet}$$

$$\Sigma M = 0 \text{ gives } V_1 \text{ (at } b) = \frac{8.66 \times 61.76 + 5 \times 20}{41.76} = 15.20 \text{ (acting up)}$$

$$\Sigma V = 0 \text{ gives } V_2 \text{ (at } a_3) = 6.54 \text{ (acting down)}$$

$$H_2 \text{ (at } a_3) = \frac{41.76}{30} V_2 = 9.10 \text{ (acting to the left along } ba_3)$$

$$H_1 \text{ (at } b) = 9.10 - 5 = 4.10 \text{ (acting to the right along } ba_3)$$

Now resolving V_2 and H_2 into components at a_1 and a_2, we have

$$V_2' \text{ (at } a_1) = \frac{32}{40} V_2 = 5.23 \text{ (acting down)}$$

$$V_2'' \text{ (at } a_2) = \frac{8}{40} V_2 = 1.31 \text{ (acting down)}$$

We must first resolve H_2 into H_3 and H_4.

$$H_3 = 9.10 \frac{40}{41.76} = 8.71; H_4 = 9.1 \frac{12}{41.76} = 2.61$$

$$H_3' = 8.71 \frac{32}{40} = 6.97; H_3'' = 8.71 \frac{8}{40} = 1.74$$

$$H_4' = 6.97 \frac{20}{40} = 3.48; H_4'' = 1.74 \frac{20}{40} = 0.87$$

We see that $H_4' - H_4'' = 3.48 - 0.87 = 2.61 = H_4$ as it must for equilibrium.

7. Examples of Reactions.—The student should solve as many of the following examples (many of which are purely imaginary) as may be necessary to give him confidence in his complete mastery of the method. They may all be solved by simple processes of geometry and arithmetic.

In each of the following cases, the student should decide whether the reactions are statically determined:

Case 1.—At points of support shown thus,  the point of application of the reaction is fixed and the reaction may be considered to have both a horizontal and a vertical component.

Case 2.—At points of support shown thus,  the point of application of the reaction is fixed and the reaction is to be considered as normal to the supporting surface and may act either toward it or from it.

Case 3.—At points of support shown thus,  the point of application is not fixed and may be *within* or *without* the limits of the rollers shown, but acts perpendicular to the surface upon which the rollers rest.

The case shown in Fig. 14*n* is worthy of careful study. Here it might appear that three conditions are given, and that the structure is statically determined, the three given conditions being the direction of each reaction, and the condition that the resultant of all outer forces on either side of the center (including one reaction) is horizontal. But it will be obvious that if a single load acts on either side of the center, the reactions, whose (inclined) directions are given, may be found by the triangle of forces, and that consequently the force on the vertical plane at the center, which must be the same as the reaction on the unloaded side, will not be horizontal, and the structure will collapse. In other words, there must be *more*

conditions given than necessary, or more than three. In any solid body acted upon by given outer forces, the resultant force acting upon any plane A cutting the body in two may be found. If, now, one reaction R_1 be fixed in direction, and also the direction of the resultant on A, by the manner of construction, we have not only given a relation between R_1 and the loads between R_1 and A, but we have also imposed the condition

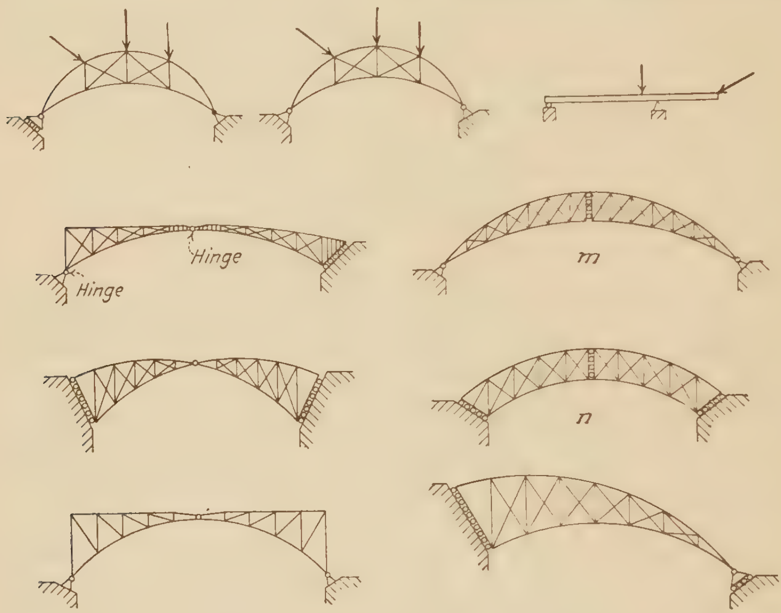
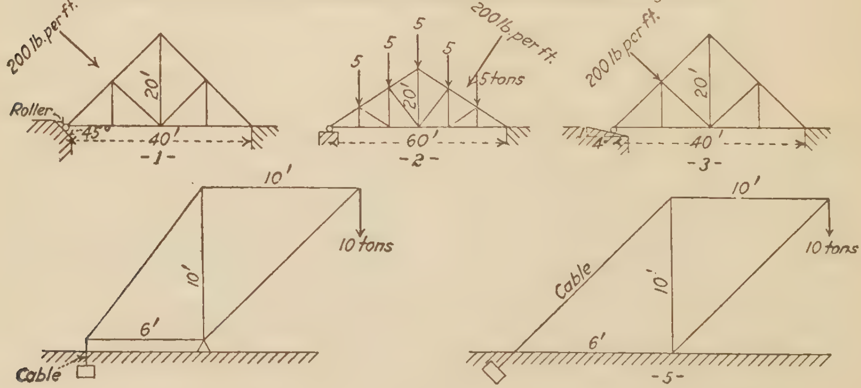
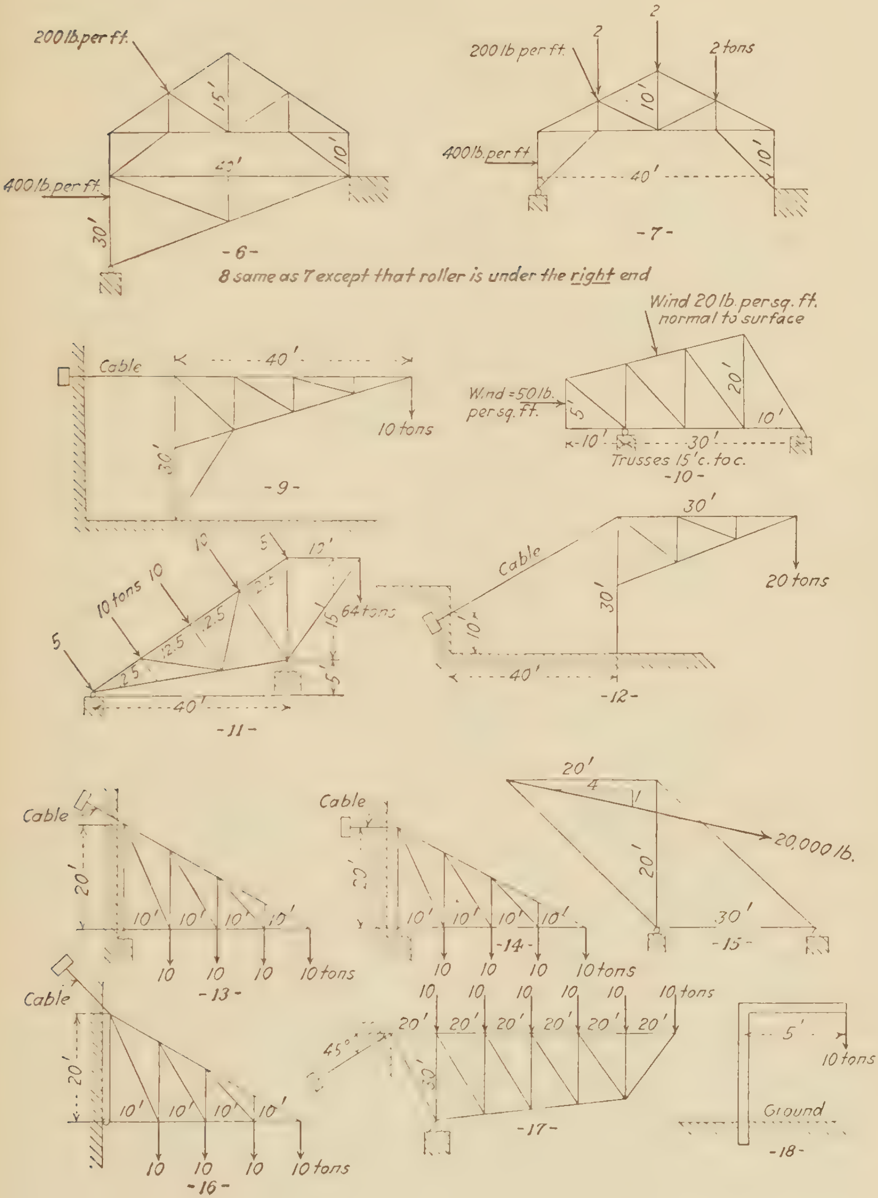


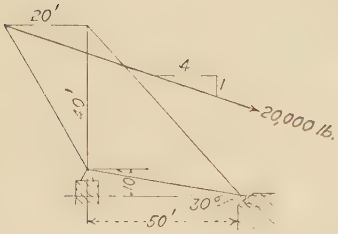
FIG. 14.

that *there must be loads between* R_1 *and* A (if A is not perpendicular to R_1) or that there must be no loads between R_1 and A (if A is perpendicular to R_1), which is really another condition. Do similar remarks hold with reference to Fig. 14*m*?

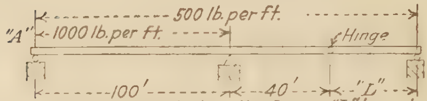
Find the horizontal and vertical components of the reactions in the following cases.




200 lb. per ft.
15'
20'
10'
400 lb. per ft.
30'
-6-
2
2
2 tons
200 lb per ft.
10'
400 lb. per ft.
40'
10'
-7-
8 same as 7 except that roller is under the right end
Wind 20 lb. per sq. ft. normal to surface
Cable
40'
10 tons
30'
-9-
Wind = 50 lb. per sq. ft.
5'
20'
10'
10'
30'
Trusses 15' c. to c.
-10-
10
5
10 tons 10
12'
2.5
2.5
2.5
2.5
15'
64 tons
5'
5
40'
-11-
30'
Cable
20 tons
30'
10'
40'
-12-
Cable
20'
10' 10' 10' 10'
10 10 10 10 tons
-13-
Cable
10' 10' 10' 10'
-14-
10 10 10 10 tons
20'
4
1
20,000 lb.
20'
30'
-15-
Cable
20'
10' 10' 10' 10'
10 10 10 10 tons
-16-
10 10 10 10 10 10 tons
20' 20' 20' 20' 20' 20'
45°
-17-
5'
10 tons
Ground
-18-

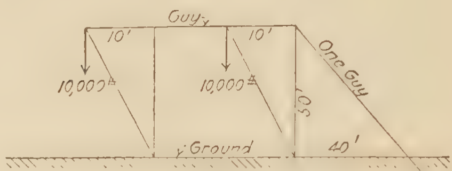


-19-

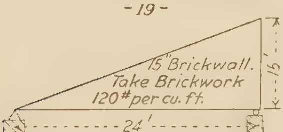


What must be the length of span "L" to make the reaction of "A" downward?

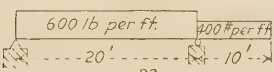
-20-



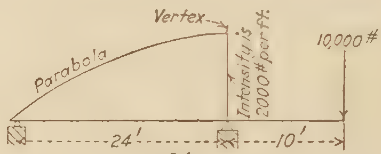
-21-



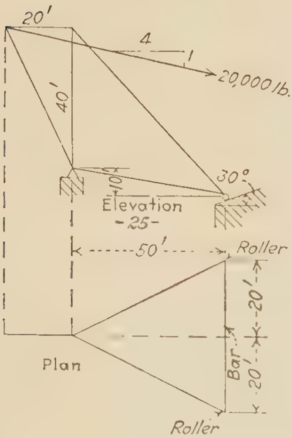
-22-

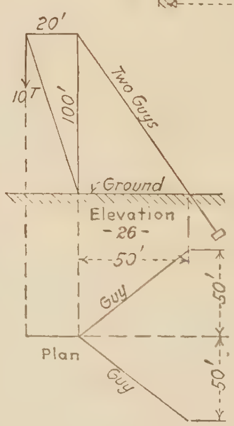


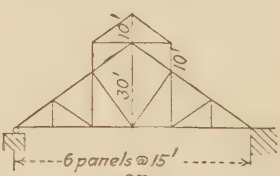
-23-



-24-

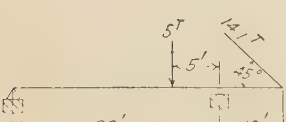




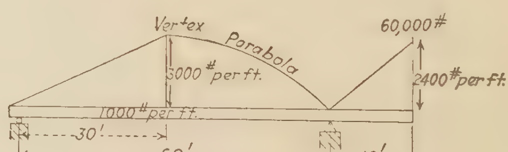


-27-

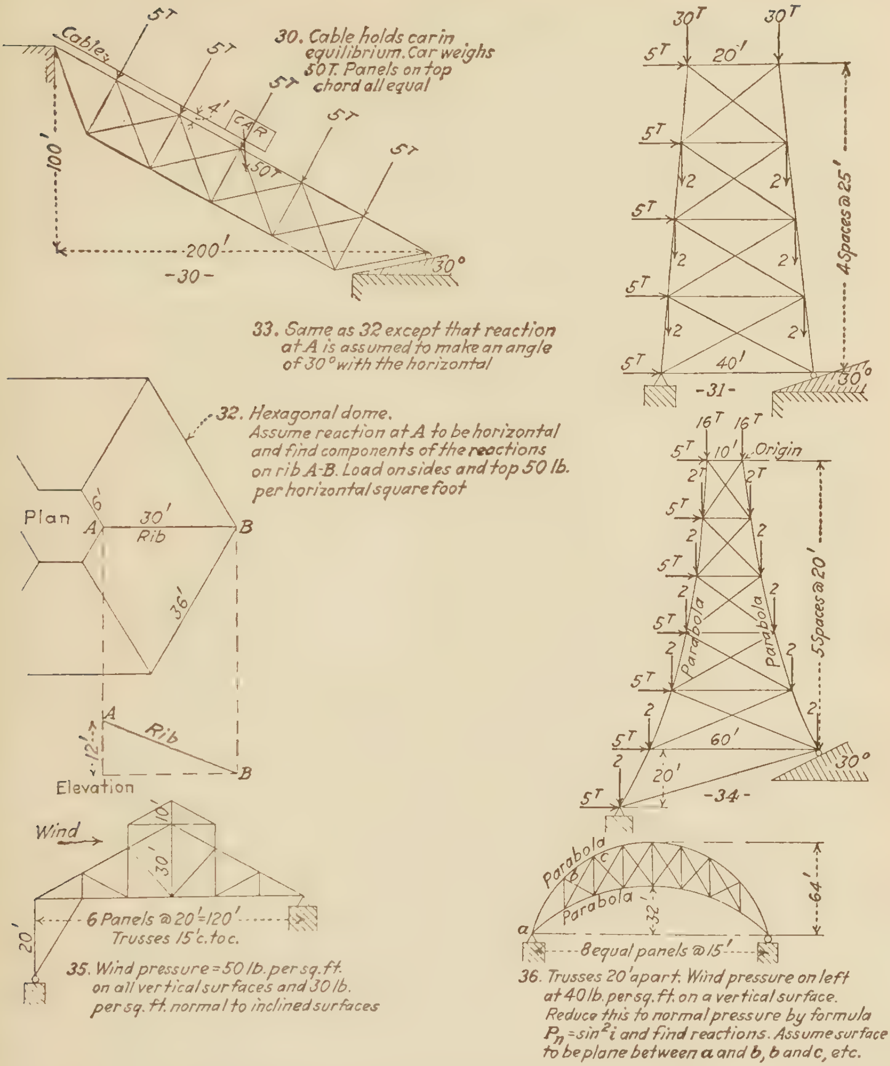
Wind on left at 40# per sq. ft. on vertical surface, and 20# per sq. ft. on inclined surfaces. Trusses are 20′ apart. Take wind on inclined surfaces normal to those surfaces



-28-



-29-



The above examples are taken, by permission, from a set of examples prepared, for the use of students, by Prof. F. P. McKibben.

8. Forces Not in a Plane. Three Supports.—This case may be studied in its general aspect as follows: At each point of support four unknown quantities must be found in order completely to define the reactions, namely, one point of support, and three force components along rectangular axes in space, making, therefore, 12 quantities to be determined. The equations of statics give six equations of equilibrium, namely:

$$\Sigma X = 0;\ \Sigma Y = 0;\ \Sigma Z = 0$$
$$\Sigma M_x = 0;\ \Sigma M_y = 0;\ \Sigma M_z = 0$$

or, expressed in words, the algebraical sum of the components along each of the rectangular axes X, Y, and Z must equal zero, and the algebraic sum of the moments about each of those axes must equal zero.

In order to have the reactions statically determined, therefore, six conditions must be given by the methods of support or construction. If the three points of support are given, three more conditions are needed, which may be given by fixing the direction of reactions. The point of support being given, to fix a reaction so that it acts in a given plane through that point is to impose one more condition; to fix it so that it acts parallel to a given line in that plane is to impose two more conditions. If, therefore, one support is so constructed that the given point of support is freely permitted to move along a given line, the reaction must then act in a plane perpendicular to that line, and one condition is imposed. If a reaction is absolutely fixed in direction, and point of application, three conditions will be present; if two reactions are so fixed, and the point of application of the third is also fixed, the structure will be unstable except under a particular combination of loads. For instance, if a three-legged table, supported on points, has two legs on rollers which are free to roll in any direction on a horizontal surface, it is evident that the table may be rotated about the fixed leg, and will be stable only under loads whose resultant is either vertical, or if not vertical, intersects the vertical line through the foot of the fixed leg. In other words, in such a case, three points of application of reactions are fixed, and two conditions are given regarding each of two reactions, or seven conditions in all, so that there is one superfluous condition which must be satisfied if there is to be equilibrium, and this condition is that the loads must have no moment about a vertical axis through the foot of the fixed leg. A structure supported at three points will be statically determined as regards the reactions if the three points of support are fixed, and if in addition one support be arranged so as to allow motion along a line (thus confining the reaction to a plane perpendicular thereto) and another support be free to move in any direction upon a plane surface (thus confining the reaction to a line perpendicular thereto).

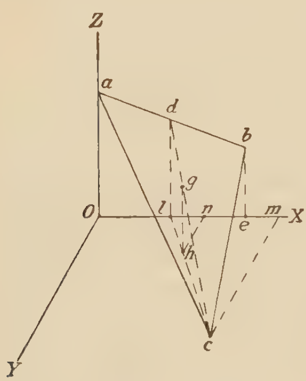


FIG. 15.

9. Illustrations of Structures Supported at More than Two Points.— Let a triangular surface be supported at its three vertices, and loaded with a vertical load uniformly distributed over the surface. Assume three rectangular axes, one vertical OZ and passing through the highest vertex, the other two horizontal and in the plane through the lowest

vertex (Fig. 15). Take the horizontal axis OX in the vertical plane passing through one side ab of the triangle. The resultant load P will act through the center of gravity g of the triangle, $cg = \frac{2}{3}cd$; $ad = db$. The projection of g on the XY plane is h, and $hn = \frac{1}{3}cm$.

Suppose all the reactions to be vertical.

The equation $M_x = 0$ about OX shows at once that the reaction at c equals one-third the load, for $P(hn) - R_c(cm) = 0$. Similarly, using the equation $\Sigma M_y = 0$, we find

$$P(On) - \frac{P}{3}(Om) - R_b(Oe) = 0$$

$$R_b = P\frac{On - \frac{1}{3}\,Om}{Oe}$$

But

$$On = \frac{1}{2}Oe + \frac{1}{3}lm$$

$$Om = \frac{1}{2}Oe + \ lm$$

Hence

$$On - \frac{1}{3}Om = \frac{1}{3}Oe$$

or

$$R_b = \frac{1}{3}P$$

This result could be more directly obtained by remembering that if the axes are changed so that the plane ZY contains the line ac, by taking moments about OY we shall find R_b to be $\frac{1}{3}P$ just as we found R_c to have this value.

From the above it follows that *if a triangular surface is loaded vertically and supported by vertical reactions at the vertices, all the reactions are equal, each being one-third the load.* The same will be true for a load in any direction, with reactions parallel to that direction.

For a good illustration of the determination of reactions on inclined surfaces, the student is referred to the design of rafters and purlins for a hip roof.

The present chapter has dealt only with the determination of reactions when statically determined. Other cases, such, for instance, as a beam supported by more than two reactions in the same plane, will be treated in subsequent chapters.

CHAPTER III

THE LOADS ON STRUCTURES

1. The first step in the design or study of a structure from the structural point of view is the determination of the loads which it is or may be required to support. This might be supposed to be a very simple matter of fact. On the contrary, it is often one of the most uncertain matters in the design and one of the principal reasons for requiring a factor of safety. It requires foresight and judgment to assume the loads properly.

The loads differ, of course, according to the character of the structure. They will be classified in this chapter according to type of structure. Loads are also either vertical, horizontal, or inclined. Loads are also fundamentally divided into two kinds: (1) the weight of the structure itself, or the *dead load,* and (2) the applied loads from the outside. These last are in general varying from time to time, being sometimes applied and then removed, and varying also in magnitude, and hence they are called *live* or moving loads, while the dead load is constant in magnitude and always acting. The study of the loads involves, therefore, a knowledge of the effects of time and of repetition of load, which are discussed in Chap. XVIII. If a heavy load is to be applied only once, or at rare intervals, it may not be necessary to design the structure for it, though it should be ascertained.

2. The loads may therefore be classified as follows:

1. Dead load, on all structures.
2. Snow, on roofs and in some cases on bridges.
3. Wind, on roofs and bridges.
4. A crowd of people, on floors of buildings and highway bridges.
5. Concentrated loads on floors of buildings.
6. Merchandise, on floors of buildings.
7. Vehicles, on all bridges.
8. Centrifugal force, on railroad bridges which are on a curve.
9. Tractive and braking force, on railroad bridges.
10. Impact, on all bridges. This is discussed in Chap. XVIII.
11. The pressure of a liquid, as of water on the face of a dam, or on a containing tank, or of other liquids or their containers. These loads are discussed in Chap. XXVIII.

12. The pressure of earth, as on the back of a retaining wall. This is discussed at length in Chap. XXI.

13. Miscellaneous forces, such as the pull from a cable attached to a block connected to a structure.

3. Dead Loads.—The dead loads are always vertical, and they depend only upon the dimensions and character of material of the completed structure. It might, therefore, be thought that they are easy of exact determination. But when a structure is to be designed, its dimensions are not known. The dimensions cannot be known till the structure is designed, and it cannot be designed till the dimensions are known. Sometimes, but very rarely, the dimensions may be expressed as functions of the live and dead loads, and found by analytical processes. This, however, is never done. The dead weights are first *assumed*, from experience or rules, and the structure designed; afterwards the dead weights are computed, and if the original assumption was sufficiently accurate, no change need be made; otherwise the computations and design must be revised, perhaps several times. In the original Quebec bridge, which collapsed, one error made by the designers was that, although the structure was a novel one, for which no precedent existed, the dead loads originally assumed were not afterward computed and compared with the assumptions. The actual dead loads were later found to be 20 to 30 per cent in excess of those assumed.

One conclusion may be drawn from the above regarding the proper order of procedure in designing a structure; namely, to design the parts, if practicable, in the order in which they act in carrying the load from its point of application to its ultimate destination. That is, if the load is carried directly by part A, while A is supported by B, and B by C, and so on, design first A, then B, then C. In designing B, all the loads it has to carry, except its own weight, will then be known. Thus, the stringers of a railroad bridge would usually be designed first, then the floor beams, then the trusses or girders, then the piers and abutments, and lastly the foundations. Similarly, in designing a building, begin at the top and work down. The student will find, however, that no part of a structure can be considered as finally designed until all parts are designed. At any moment the design of some part may require a change of dimensions or details of the parts which have gone before.

4. Formulae for the dead weight of the various kinds of structures will be given in the chapters dealing with those structures. The weight will vary with the type and the details used, the assumed loads and unit stresses, and the dimensions. Since the economical depth of a truss or beam is a fairly constant proportion of the span, the principal dimension which affects the dead weight is the span. The weight *per foot of span* will be a fairly constant coefficient times the span, the coefficient varying with the kind of structure, details, and specification used. The weight

of the floor or roof covering, per foot of span, will be nearly constant. Thus the weight of a beam or truss span, per foot, including floor, will be approximately $c_1 + c_2l$, and the total weight will be

$$W = c_1l + c_2l^2 \tag{1}$$

where l is the span, and c_1 and c_2 are constants. Values of the constants will be given in the appropriate chapters. In some cases, a different form of equation is used.

5. Rule for Weight of Iron and Steel.—The following rule should be memorized: Taking the weight of iron as 480 pounds per cubic foot, or, 1,728 cubic inches, 36 cubic inches would weigh 48 pounds; hence, since 36 cubic inches is the volume of a rod 1 inch square and 1 yard long, the rule is obtained that for a prismatic piece of iron of any area of cross-section, *ten times the area in square inches equals the weight in pounds per yard; to this add* 2 *per cent for steel;* otherwise expressed, if the area of a prismatic piece of steel is a, its weight per foot of length will be $3.4a$.

6. Loads to Be Assumed in Designing.—The loads to be assumed in designing, with the usual unit stresses, are not always the greatest loads that the structure may possibly be exposed to, because such *possible* loads may be so improbable or may occur so rarely, if they occur at all, that it would be absurd to make the structure able to carry them as if they were usual loads. The maximum possible loads should, if practicable, be estimated, and the structure so designed that such loads would not cause collapse or injury, but the loads used in designing should be those which may be repeated indefinitely without injury. A structure so designed will, in general, have the stresses not much over about one-half the elastic limit; so that the loads could be nearly doubled without exceeding the elastic limit, could be increased in general still more without causing any injury except a small permanent deformation, and could be nearly quadrupled without causing collapse. It is in the assumption of the proper loads to be assumed that judgment and experience are necessary. See Chap. XIX for discussion of facts bearing on this subject.

7. Impact.—In computing the stresses in a member of a structure, the live loads are supposed to be placed upon the structure in the most unfavorable position, but are assumed to be stationary. In reality they are in motion, and the stresses they produce are greater than as if they were stationary. A railroad train passing over a bridge at high velocity causes vibration and shock, due to the suddenness of application, and to unevenness of track, unbalanced moving parts, etc. The same is true of a truck passing over a highway bridge, especially if the pavement is uneven. The additional stress due to the fact that the load is moving, above what it would be if it were stationary, is called *impact*, and should be allowed for by adding some percentage to the live-load stresses, and then taking the total stress as the sum of that due to dead load, live load,

impact, and any other loads which may act, such as wind and centrifugal force.

The allowance for impact is arbitrary. Sometimes a formula is used to find it, and sometimes definite percentages of the live-load stress are given. The fraction of the live-load stress will depend upon the length of the structure which is covered by the live load when in the most unfavorable position for the piece in question, and it may also depend to some extent upon the speed of the vehicle. The speed cannot be included in the formula, because it not known. A formula which has been generally used[1] is the following:

$$I = S_l \frac{300}{300 + L} \tag{2}$$

in which S_l = computed maximum live-load static stress.

L = length in feet of the portion of the span that must be loaded to produce S_l.

I = amount to be added for impact.

Thus, for L = 300 feet, I = 0.5. For small values of L, I is practically S_l; for L = 50 feet, $I = \frac{6}{7} S_l$.

The formula recently adopted by the A. R. E. A., *for railroad bridges*, is

$$I = S_l \frac{300}{300 + \frac{L^2}{100}} \tag{2a}$$

For buildings, no impact allowance is made. For highway and electric railway bridges, fixed percentages are generally given for the different parts of the structure, though by the specifications of the A. R. E. A. the allowance is one-half of that given by Eq. (2*a*).

The subject of impact is further discussed in Chap. XVIII.

8. Snow Load.—Snow may cover highway bridges and roofs to a considerable depth. For railway bridges it may generally be neglected, for, though a bridge with a solid floor may have quite a load of snow, the requirements of traffic will cause most of it to be quickly removed. In the case of a highway bridge, too, the snow must be removed if the traffic is to move freely. The presence of snow, then, will either prevent the application of the full maximum loads assumed, or will reduce the impact with which it is usually accompanied.

For bridges, then, the snow load may generally be entirely neglected. The principal case where it should be considered is for roofs.

The snow load on roofs will depend upon the climate and the slope of the roof. On a flat surface, in northern latitudes, snow may fall to a depth of 6 to 8 feet, and in drifts it may be even deeper. On a flat roof, the wind will generally prevent its accumulation of any such depth. The weight of snow varies with its condition, the deepest snow being generally

[1] Originally proposed in 1895 by the late Fred Thompson, then Bridge Engineer of the Southern Railroad, and adopted by the late C. C. Schneider, and known as the Pencoyd formula.

the driest and lightest. Ordinarily 10 to 12 inches of dry snow make 1 inch of water; that is, a cubic foot of snow weighs 5 to 6 pounds; wet snow will weigh more, up to 8 pounds per cubic foot, or over. On a flat roof the snow load may possibly be as large as 20 pounds per square foot. It is sufficient, however, to allow 15 to 20 pounds, as anything above this would be rare. More may be necessary in some latitudes or under certain conditions. The Boston Building Laws require, on roofs with a pitch (ratio of rise to horizontal distance, or tangent of angle of slope, on the line of maximum slope) of less than 4 inches per foot, a live load of 40 pounds per square foot of horizontal projection.

On a sloping roof, the snow load will be less, for snow will not ordinarily remain to any considerable depth on a roof with an inclination of over 45°. The live loads on a roof are only two, snow and wind, and the greater the wind pressure the less snow will remain, though a layer of it may freeze and remain even in a high wind. If the maximum wind load is assumed, it is often proper to assume no snow load at all.

Snow may lie on one side of a gently sloping roof to a considerable depth while it is entirely melted on the other, but generally the snow load assumed is taken to cover either half or the entire roof. If the maximum wind pressure is assumed, it is often assumed that there is no snow on the side acted on by the wind, though there may be the maximum snow load on the other side.

Sometimes the snow load prescribed is varied according to the slope. Thus, the Boston Building Laws prescribe minimum live loads as follows:

a. On roofs with a pitch of 4 inches or less to the foot, only the vertical load, above mentioned, of 40 pounds per square foot, on half the roof or on the whole.

b. On roofs with a pitch of more than 4 and not over 8 inches per foot, a vertical load of 15 pounds per square foot of horizontal projection, and a wind load of 10 pounds per square foot normally, these two loads acting together or separately.

c. On roofs with a pitch of 8 to 12 inches per foot, a vertical load of 10 pounds per square foot of horizontal projection and a normal wind pressure of 15 pounds per square foot acting together or separately.

d. On roofs with a pitch of over 12 inches per foot, a vertical load of 5 pounds and a normal wind load of 20 pounds per square foot, acting together or separately.

The maximum wind pressure, on a vertical surface, is taken (in the Boston Building Laws) at 20 pounds, so that these requirements are not quite logical, for the wind pressure on a roof sloping at 45° will certainly be less than on a vertical surface, although, as will be later shown, the wind does not always blow horizontally.

These facts will indicate that, in assuming snow and wind loads, the engineer should use judgment based on the local conditions; and he is

in any case bound by building laws, if they exist. They also show the uncertainties of the subject, and the futility of expecting mathematical exactness in the computation of structures, owing to the inherent variations and uncertainties of the data. Engineering is not a mathematical science, though it requires a knowledge of mathematics, as a tool, in solving its problems. It is far more a science requiring judgment, experience, and common sense.

LIVE LOAD ON BUILDINGS

9. Live Loads on Floors and Columns of Buildings.—The only live loads on roofs, as already stated, are snow and wind. On floors, the live loads are those arising from a crowd of people, or from the merchandise, machinery, fixtures, or other articles that the floor supports.

10. Weight of a Crowd of People.—In the design of buildings and highway bridges, the proper weight to assume for a dense crowd becomes of importance. This is purely a matter of experiment, and numerous tests have been made by packing as many people as possible, whose weights were known, into known areas. B. B. Stoney, an English engineer, long ago found in this way a load of 147 pounds per square foot. Recently, Prof. L. J. Johnson has obtained a load of 181.3 pounds per square foot.[1] It must be borne in mind, however, that such a load would mean a crowd so dense that people could not move, and that such a load would not cover any large area, and would be unaccompanied by impact. Even a load of 100 pounds per square foot means a very dense crowd. It should not be assumed that even this load would be apt to cover a large area. The larger the floor area, the smaller the average load that is reasonable to assume. It has long been customary to assume 100 pounds per square foot as the maximum load of a crowd, which may cover an entire bridge or a floor of a building, and this is probably sufficient, though it should be remembered that heavier loads over small areas may occur quite frequently.

For small areas where dense crowds may occur, such as balconies, some staircases, and some assembly rooms, a load of 160 pounds would not be too great to provide for. It must be remembered, however, that if a structure is proportioned properly for a load of 100 pounds per square foot, it will have a considerable factor of safety, and that it could carry 200 pounds per foot with safety, and considerably more before it would collapse. If extreme loads will not occur except at long intervals, this factor of safety may be encroached upon by such loads, and a load of 100 or 125 pounds assumed for designing.

11. Reductions Allowed.—It is assumed that the assumed uniformly distributed load may cover any one entire floor, or any part of it. It is

[1] *Trans. Am. Soc. C. E.*, vol. LIV, p. 441 June, 1905. This discussion, illustrated by photographs, will be found interesting.

not considered reasonable, however, that all the floors of a building should be assumed as carrying, simultaneously, so large a load (see Art. 5). Consequently, for office buildings, or other buildings except those used for storage or machinery, but in which the principal load may be from a crowd, it is customary to proportion columns, walls, or other parts receiving a load from more than one floor, on the supposition that one floor, the top one, for instance, in proportioning columns, may be loaded with this load, but that the load on each succeeding floor may be progressively diminished.

A common requirement for office buildings is now 125 pounds per square foot for the first floor and 75 pounds for floors above, but these are made subject to arbitrary reductions, depending upon the number of floors carried by the structure in question, and sometimes depending upon the floor area. Thus, the Boston Building Law of 1918, after specifying the floor loads just stated, allows the following reductions "in all buildings except storage buildings, wholesale stores, and public garages, for all columns, girders, trusses, walls, piers, and foundations":

	Per Cent Reduction
Carrying one floor	0
Carrying two floors	25
Carrying three floors	40
Carrying four floors	50
Carrying five floors	55
Carrying six floors or more	60

This means that if a column carries three floors, for each of which separately the live load is 75 pounds per square foot, the column may be proportioned for only 60 per cent of this, or 45 pounds per square foot on each of the three floors.

The New York Building Code of 1917 allows for columns in buildings more than five stories in height the full live load on the top floor, 95 per cent on the floor just below, 90 per cent on the next lower floor, and in each succeeding floor 5 per cent less, down to 50 per cent of the full live load, and the same on all floors below. This means that a column, carrying three floors with equal areas on each floor tributary to the column, would be proportioned for 2.85 times the full live load on one floor, as compared with 1.8 times by the Boston rules. Clearly, this is an arbitrary matter, depending upon judgment as to what is safe. If the required load to be assumed upon any one floor is larger, the reduction for a column carrying several floors would reasonably be greater than for small floor loads. The Boston loads required are somewhat larger than the New York loads, but the allowed reduction is much greater in Boston.

Professor C. R. Young[1] discusses this matter and gives a table showing the reduction in live load allowed by the building laws of various cities. The reader should refer to this. He suggests that even on main girders it is not necessary to assume that the entire area tributary to it is loaded with the maximum load per square foot, and that a reduction of from 10 to 20 per cent is reasonable. This is clearly logical, for except in the case of a crowd of people, the entire area would not be covered; and even in a warehouse there must be aisles which are unloaded. For columns, he suggests the following loads, which seem eminently reasonable: for the column below the roof, 100 per cent of the specified roof loading; for the next column 90 per cent of the aggregate total specified live load on roof and the floor below, on the entire tributary area; for the next columns, in order, 85, 80, 75, 70, 65, 60, 55, 50, and, for all below this tenth column counting down, 50 per cent of the total live load on all floors above and roof, on total tributary area. These loads are central, not excentric, if column connections and girders are symmetrical. There may be partial loads accompanied by excentricity (see Art. 48 of the chapter on Column Design, in the volume on Design).

Buildings to be used as warehouses, factories, etc. should have the loads assumed to conform to actual requirements. These, however, cannot always be precisely foreseen. In a warehouse, for instance, it cannot be foreseen what loads will be placed upon the floors. It is, therefore, usual for certain minimum loads to be prescribed by the Building Law, for instance, 250 pounds per square foot for heavy storage or manufacturing and 125 pounds for light storage or manufacturing. The Building Commissioner is also given power to fix the loads allowed on any existing floor, and to prescribe loads to be used in new buildings, if in excess of the minimum loads specified. The reader should refer to some good building code, such as that for Chicago, New York, Boston, or that of the National Board of Fire Underwriters, for details as to loads to be assumed. No reductions are generally allowed on columns for warehouses, wholesale stores, and the like, for every floor in such a building may be carrying its full load.

The actual load on floors in office buildings, hotels, schoolhouses, and dwellings is generally much smaller than that usually prescribed. Observations on several office buildings have shown actual average loads of less than 20 pounds per square foot. But it must always be assumed that any floor is likely to be loaded by a crowd of people, so that any such assumed load as 20 pounds would be much too small.

12. Concentrated Loads on Foors.—A single heavy concentrated load, such as that of a steel safe, is often specified for office buildings. In some cases a concentrated load of 4,000 pounds must be assumed.

Impact.—No impact addition is generally made for loads on buildings.

[1] "Possible Economies in Steel Construction," *Can. Eng.*, p. 615, Nov. 9, 1926.

13. Weight of Materials and Merchandise.—The "Pocket Companion" gives[1] the specific gravities and weights of various materials, and the weights per square foot on floors. *Report* 5 of the Insurance Engineering Experiment Station, of the Boston Manufacturers' Mutual Fire Insurance Company, gives a similar table. It does not seem necessary to reprint these, as they are voluminous, and may be easily referred to. They are subject to considerable variation. For example, the weight of sand will vary according to moisture, character of grain, percentage of voids, whether loose, shaken, or rammed. The same is true of gravel, earth, broken stone, and other materials. The weight of wood will vary greatly according to species and moisture. For wood exposed to the weather it is common to assume the weight per foot board measure as 4.5 pounds or 54 pounds per cubic foot. For weight of various woods, see the volume on "Fundamental Properties of Materials," and the references therein.

It may be well for the reader to keep in mind the following as approximate weights to be used:

Wood: 4.5 pounds per foot board measure = 54 pounds per cubic foot (wet).

Concrete: 150 pounds per cubic foot, if of gravel or broken stone.

Concrete: 112 pounds per cubic foot, if of cinders.

Ballast: 120 pounds per cubic foot.

Waterproofing: 150 pounds per cubic foot.

Rail and fastenings, steam railroads: 150 pounds per linear foot of track (A.R.E.A.).

Rail and fastenings, electric railways: 100 pounds per foot per track.

Portland cement, from 94 to 115 pounds per cubic foot depending upon the degree of compactness. One bag of cement is a quarter of a barrel, and weighs 94 pounds. It is often assumed as a cubic foot.

14. The following table, taken by permission from the "Pocket Companion," edition of 1923, gives weights on floors as specified by the building laws of several cities. Such laws, however, are frequently changed.

[1] See pp. 300–303, edition of 1923.

MINIMUM LIVE LOADS OF FLOORS AND ROOFS, POUNDS PER SQUARE FOOT
By Building Laws of Various Cities
(From "Pocket Companion," Carnegie Steel Company, revised 1923)

Description of building	New York, 1922	Chicago, 1920	St. Louis, 1917	Boston, 1921	Cleveland, 1920	Baltimore, 1908	Pittsburgh, 1914	Cincinnati, 1917	Philadelphia, 1919
Floors for rooms									
Apartments and dwellings	40	40	50	50	70[a]	60	50	40	40
Asylums, hospitals, etc	100	50	50	50[c]	....	....	70	40	40
Detention buildings, etc	100	50	....	50[c]	80	....	....	60	100
Factories:									
Light manufacture	120[d]	100[d]	100[d]	125[d]	....	125[d]	125[d]	100[d]	120[d]
Heavier manufacture	....	....	150[d]	250[d]	....	175[d]	....	150[d]	
Hotels, lodging houses	40	50	50	50[c]	70	60	70	40[b]	40
Office buildings, etc	60	50	60[b]	75[b]	70[b]	75[b]	70	50[b]	60
Public buildings:									
Municipal buildings	100	....	....	75[c]	100	....	....	100	100
Churches	100	100	75	100	80	75	125	100	100
Libraries, museums	100	....	....	100	125	....	200	....	100
Theaters	100	100	100	100	80	75	125	100	100
Schools, colleges, etc	75	75	75	50	70	75	70	60	75
Stores, light goods	120	100	100	125	100[b]	125	125	100	120
Stores, heavier goods	....	....	150	250	....	175	....	150	150
Warehouses	....	....	150	250	....	250	200	150	150
Floors for assembly halls, etc.									
Auditoriums, fixed seats	100	100	100	100	80	75	125	100	100
Auditoriums, movable seats	100	100	100	100	125	125	125	100	100
Armories, dance halls, etc	100	100	...	100	150	....	150	150	100
Miscellaneous									
Garages, stables	120	100[e]	100	150[e]	150[e]	100	....	75	120
Corridors, hallways	100	100	100	75[f]	70[g]	....	....	80[g]	
Stairways, fire escapes	100	100	100	75[f]	100[h]	....	....	80[g]	
Sidewalks	300	....	....	250	200	200	....	300	
Roofs:									
Flat, slope up to 20° (1/3)	40	25	30	40	35[i]	40	50[k]	25	30[i]
Steep, slope over 20° (1/3)	30	25	....	25[j]	30[i]	20	50[k]	25	30[i]
Wind pressure	30[l]	20	30	10-20[n]	20[o]	30	25	20[p]	30[m]

[a] Dwellings, Cleveland, 60.

[b] First floors: St. Louis, 100; Boston, 125; Cleveland, 125; Baltimore, 150; Cincinnati, 100.

[c] Public floors of hospitals, hotels, public buildings, etc.: Boston, 100.

[d] Floor loads do not include the weight or the impact load of machinery. Impact: Cincinnati, 25 per cent.

[e] Garages, private: Chicago, 40; Boston, 75. Garages, public, upper floors: Cleveland, 100. Stables: Cleveland, 80.

[f] Corridors, stairways, etc., for assembly halls, armories, etc.: Boston, 100.

[g] Except in dwellings where floor loads are less.

[h] Stairways, etc., for apartment houses, 80; dwellings, 60.

[i] Loads per square foot of superficial roof area; other roof loads are for the projected area.

[j] Loads include wind pressure: 10 pounds up to 2/3 slope, 15 up to 1/1 slope, 20 over 1/1 slope.

[k] Dead and live load; snow load 25 pounds, reduced 1 pound each degree between 20 and 45°.

[l] For buildings over 150 feet high, or where height is over four times least horizontal dimension.

[m] Wind pressure for high buildings in built-up districts: 25 pounds at tenth story, 2½ pounds less for each story below, and 2½ pounds more for each story above, up to 35 pounds.

[n] For buildings 40 feet high, 10 pounds; up to 80 feet, 15 pounds; over 80 feet, 20 pounds.

[o] Wind pressure on curtain walls, 30 pounds.

[p] For buildings over 100 feet high, or where height is over three times the average width of base.

15. Recommendations of Building Code Committee of U. S. Department of Commerce.—The U. S. Department of Commerce has recently (1925) published and approved the report of its Building Code Committee, dated Nov. 1, 1924, giving recommended minimum live loads for use in the design of buildings. This should be in the hands of every designer of buildings, and should help to unify building codes, which are extremely divergent. Variations of as much as 300 per cent in allowable floor loads for the same occupancy are found in different cities, and variations of 100 per cent are common. There is no reason why the loads required on buildings of the same character in Boston and in New York should be different. Materials, and probably men, weigh the same in both places, and a given beam will carry the same in both. The present divergence indicates, as the committee states, "either that safety is disregarded in many cases or that an unnecessary amount of building materials or labor is used." Of 106 codes in different cities, for office buildings, above the first floor, 4 require a minimum live load of 40 pounds per square foot, 2 require 150 pounds per square foot, 4 require 100 pounds per square foot, and the others are intermediate. The Committee recommend 50, with provision for a load of 2,000 pounds on a space $2\frac{1}{2}$ feet square, on an otherwise unloaded floor.

The following table gives the minimum loads recommended by the committee:

MINIMUM LIVE LOADS RECOMMENDED BY BUILDING CODE COMMITTEE OF THE U. S. DEPARTMENT OF COMMERCE, 1925

Character of occupancy	Minimum live load per square foot	Remarks
Human occupancy:		
Private dwellings, hospital rooms and wards, guest rooms in hotels, lodging and tenement houses, and similar occupancies.	40	Where floors of one- and two-family dwellings are of monolithic type, or of solid or ribbed slabs, live load to be 30 pounds per square foot.
Floors for office purposes and rooms with fixed seats, as churches, school classrooms, reading rooms, museums, art galleries, and theaters.	50	For office floors, provide for 2,000 pounds on any space 2½ feet square if this load on an otherwise unloaded floor will produce greater stresses.
Aisles, corridors, lobbies, public spaces in hotels and public buildings, assembly rooms without fixed seats, grandstands, theater stages, gymnasiums, stairways, fire escapes or exit passageways, and other spaces where crowds of people are likely to assemble.	100	This does not apply to such spaces in private dwellings, for which the minimum is 40.
Industrial or commercial occupancy:		
Storage purposes, general	250	
Storage purposes, special	100	
Manufacturing, light	75	
Printing plants	100	
Wholesale stores, light merchandise	100	
Retail salesrooms, light merchandise	75	
Stables	75	
Garages:		
All types of vehicles	100	
Passenger cars only	80	
Sidewalks	250	Or 800 pounds concentrated.
Roofs:		
On horizontal projection	30	If rise is 4 inches or less per foot horizontal.
On horizontal projection	20	If rise is 4 to 12 inches per foot horizontal.
On horizontal projection	0	If rise is over 12 inches per foot horizontal. No vertical load, but normal wind pressure 20 pounds per square foot.
		Provision must be made for possible shifting of partition walls.

Except in buildings for storage purposes the following reductions in assumed total floor live loads are permissible in designing all columns, piers or walls, foundations, trusses, and girders.

REDUCTION OF TOTAL LIVE LOADS CARRIED

	Per Cent
Carrying one floor	0
Carrying two floors	10
Carrying three floors	20
Carrying four floors	30
Carrying five floors	40
Carrying six floors	45
Carrying seven or more floors	50

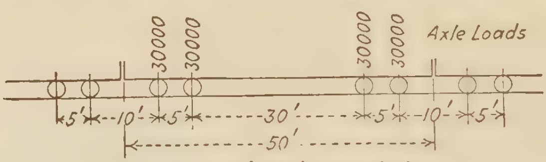
For footings, only one-half the live load as computed above shall be taken, *i.e.*, for 7 or more floors one-fourth of the full live load, and for one floor, one-half of the full live load.

These requirements cannot be followed if there is a building code, for the latter is the law. It is hoped, however, that building codes may be unified and simplified.

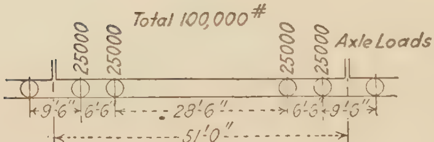
16. Where there is a building code prescribing these or other minimum loads, there should also be the requirement that changes in character of occupancy should be reported promptly to the building commissioner, and the latter should also inspect all buildings to ascertain whether the minimum or allowed loads are exceeded, and, if they are exceeded, he should specify to the owner and post a notice, stating the maximum approved safe loading. The occupant of the building should then be responsible for keeping the load below the certified limit, and the commissioner should also, so far as practicable, see that it is not exceeded.

17. Live Load on Electric Railroad Bridges or on Elevated Railroads.—This load is that of a train of the heaviest cars, or on electric roads, that of one or two cars.

The Massachusetts Board of Public utilities prescribes on each track two 50-ton cars (loaded) 40 feet long, with two trucks, and eight wheels, a total wheel base of 25 feet, and a wheel base of each truck 5 feet.



New York Public Service Commission



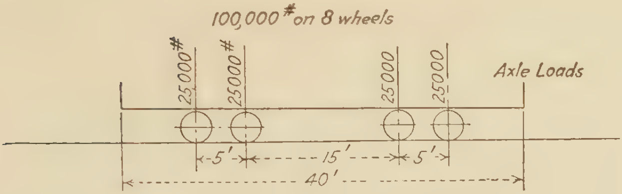
Philadelphia Department of City Transit

Fig. 16.

Loads on various elevated railways are shown in Fig. 16. These run in trains.

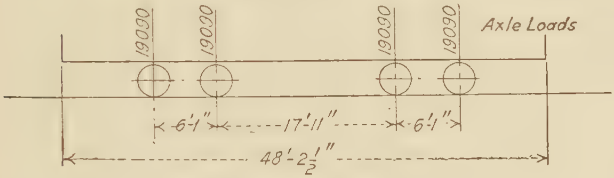
In any actual case, the loads in use or those prescribed should be ascertained.

In some cases service cars may be in use, carrying heavy materials which may weigh more than the passenger cars shown. These generally

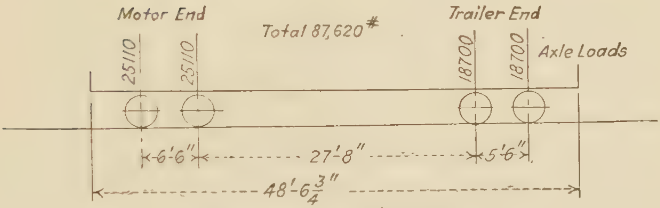


Massachusetts Public Service Commission
Two cars as above required

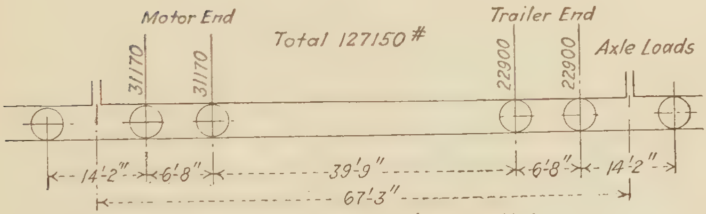
Heaviest Passenger Cars on Elevated Lines with Assumed Loads
Total 76,360#



Boston Elevated Railway Co



Chicago Metropolitan West Side Railway Co



Interborough Rapid Transit Co., New York

FIG. 16.—(*Continued.*)

run singly and perhaps infrequently. It may or may not be necessary to consider them in designing.

Interurban electric lines generally operate single cars at considerable intervals. A bridge on such a line will not generally have more than one car on it at a time, though in case of blockade or accident there may be more. Some specifications use, instead of the wheel loads, an arbitrary

uniform load. Thus, Cooper's specification of 1909 required the following:

UNIFORM LOAD IN POUNDS PER FOOT OF TRACK TO REPLACE ACTUAL CAR

Span in feet	City, suburban, or interurban bridges	Country highway bridges
100 or less	1,800	1,200
200 or more	1,200	1,000
Variation in 5 feet	30	10

It is more general, however, to use the actual car loads. For centrifugal force, see Art. 36.

LIVE LOADS ON HIGHWAY BRIDGES

18. These loads consist of the weight of a crowd of people or of vehicles. These vary with circumstances so much that it is customary to divide highway bridges into classes in which the assumed loads differ, such as the following:

1. City bridges, carrying heaviest loads.
2. Suburban or interurban bridges.
3. Town or country bridges, but likely to carry heavy loads from industrial plants.
4. Light town or country bridges.

19. Trucks.—The heaviest highway vehicle is the loaded truck. The heaviest auto truck weighs about 7 tons and its rated load is 10 tons, making a total of 17 tons. Trucks are often overloaded, sometimes 50 or even over 100 per cent. A truck weighing 20 tons, with load, is therefore sometimes met with; and the loads seem to be increasing. The rear axle of an auto truck carries from 70 to 87 per cent of the total load, averaging 78 as found by measurements in Massachusetts.

In some states there are legal restrictions, but they are often violated. Thus, in Illinois the limit of total weight is 24,000 pounds, 16,000 pounds on one axle, and 800 pounds per inch width of tire; in Iowa the total weight may be 28,000 pounds; in Massachusetts, 28,000 pounds and 800 pounds per inch of tire; in Michigan, 38,000 pounds, 18,000 pounds per axle, and 700 pounds per inch of tire.[1] It then appears that weights of 15 or even 19 tons are authorized; and in Massachusetts actual weighing showed loads of 22 tons on a truck and of even more on a truck and tractor or trailer,[2] with 1,000 to 1,500 pounds per inch of tire. In Massachu-

[1] "Bus Operating Practice," published by the International Motor Truck Company, 1925. The maximum width of tire is about 14 inches.

[2] DEAN, A. W.: "Highway Construction in Massachusetts, with Discussion," *Jour. Boston Soc. C. E.*, pp. 26–27, January, 1922.

setts a 2-ton truck was found to be carrying $6\frac{1}{4}$ tons. In New Jersey, loads much in excess, if found, must be removed to another truck.

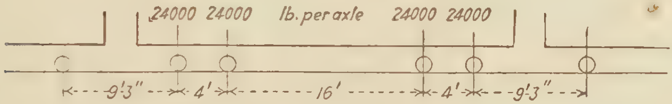
The Board of Public Utilities of Massachusetts therefore specifies, for highway bridges carrying electric railway tracks, a 20-ton truck with 6 tons on the front and 14 on the rear axle, wheels 12 feet apart and gage between wheels 6 feet; the truck covering a space 32 by 10 feet. This is used with an impact allowance of 50 per cent on steel floor members, but with no impact on wood floors.

In New York City, surface cars are considered, for stresses in trusses, as a continuous load per foot of each track equal to

$$1{,}600 \text{ pounds} - \frac{\text{span in feet}}{2} \text{ (with impact allowance)}$$

while for floor systems and short spans an eight-wheeled car is used, shown in Fig. 17. The impact allowance is

$$I = S\frac{150}{L + 150}$$



New York City for Surface Cars

FIG. 17.

S being the stress due to moving load, and L the length in feet *of load* producing S; that is, for floor beams, L is sum of adjacent panel lengths.

Roadway traffic is, for trusses, considered as a continuous load *per square foot* of

$$100 \text{ pounds} - \frac{\text{span in feet}}{20} \text{ (with impact)}$$

and for floor systems and short spans one four-wheeled truck weighing 25 tons, with 15 tons on the front axle, wheelbase 6 by 12 feet, space covered 10 by 32 feet (or 10 feet in front of and behind axles), surrounded by the above specified uniform load per square foot. The impact allowance for uniform load is as above, but no impact is added for concentrated roadway loads.

Footwalk traffic is considered, for trusses, as a continuous load per square foot of

$$100 \text{ pounds} - \frac{\text{span in feet}}{10}$$

but not less than 50, without impact; and on brackets and details, 125 pounds per square foot.

The future development of the water truck is uncertain. It is the opinion of some truck makers that the tendency is toward lighter weights.

Besides the usual trucks, there are trailers of various kinds, steam shovels, cranes, and military loads. Special loads like cannon, steam boilers, etc. may have to be transported, and may require exceptional provisions.[1]

The flooring, stringers, and floor beams of a highway bridge should be designed for the electric car or truck specified. In the case of floor beams, it may be necessary to cover the part of the adjacent panels which are not covered by the car or truck with a uniform load of a crowd, up to, say, 2 feet of the wheels of the vehicle. The electric car and the truck are assumed simultaneously in the most unfavorable positions. If there is no electric line, one of the heaviest trucks may be sufficient, or, if the traffic is heavy, one in each direction.

Also, for the main girders, if the span is less than a certain limit—50 feet by the recently adopted specification of the Bureau of Public Roads of the Department of Agriculture, October, 1924—and for members of main trusses for which the "loaded length" (length of span covered by live load to produce maximum stress) is less than 50 feet, the same procedure may be followed.

For greater lengths than this limit of 50 feet, it is not practical to try to arrange vehicles so as to produce the maximum stress, and the trusses or girders may be proportioned for one or two electric cars, as specified, and on the rest of the highway only a uniform load from a crowd of people, or an arbitrary uniform load as described below, or as in the New York City specifications.

20. Uniform Load.—This should not be less than the weight of a reasonable crowd, or less than that load which would produce the same stress as a reasonable number of vehicles. It has been pointed out in Art. 10 that the larger the area covered, the less dense a crowd. The same is true of an area covered with automobiles. If a bridge is entirely covered with automobiles, as closely as possible, the average load per foot will rarely exceed 50 pounds. Most of the cars will be passenger cars, and light. The heaviest passenger car loaded gives an average load on the wheelbase of only about 100 pounds, less on the actual space covered, and still less allowing clearance.

Even in a garage, the actual average load will rarely exceed 50 pounds per square foot, though the equivalent uniform load to produce moment and shear may be greater.

It is therefore clear that the uniform load to be assumed may diminish with the span or the loaded length. The specifications of the Department of Public Utilities of Massachusetts, originally prepared in 1901 by the writer, and unchanged since, require in city bridges a load of 100 pounds per square foot for loaded lengths of 100 feet or less, 80 pounds for 200 feet or over, and proportionally for intermediate lengths. The

[1] Spofford, C. M., "Highway Bridge Floors," *Proc. Eng. Soc. Western Penna.*, vol. XXXI, pp. 727–826, 1915.

load may also be varied for the different classes of bridges in Art. 1, for instance:

UNIFORM LOAD PER SQUARE FOOT IN PROPORTIONING MAIN TRUSSES AND GIRDERS

Span in feet	Class A	Class B	Class C	Class D
100 or less	100	80	80	80[a]
200 or more	80	60	60	55
Variation of 1 pound in	5 feet	5 feet	5 feet	5 feet

[a] For 75 feet or less.

It should also be observed that the effect of a crowd may involve some impact, the less the greater the density. A column of soldiers in step will sometimes shake a bridge and cause higher stresses than a much larger quiescent load. In going over a bridge, thought to be light, a column of soldiers should break step.

The specifications of the Department of Agriculture use four classes, namely, 1–15, based on one 15-ton truck, 1–20, based on one 20-ton truck, 2–15, and 2–20; and for trusses and girders, when the loaded length is more than 50 feet, prescribe the following uniform loads:

Loaded length, feet	Uniform load, and proportionally for intermediate spans		
	Class 1–15	Classes 1–20 and 2–15	Class 2–20
50	100	130	180
100	80	90	120
200 and over	60	70	90

21. Road Rollers.—Before the days of the auto truck, the heaviest load on a highway bridge was often a road roller. The heaviest load of this kind is said to be the 20-ton roller of the Buffalo-Springfield Roller Company, shown in Fig. 18. The front roller, 4 feet 4½ inches wide, carries nearly 7 tons; 11 feet 1 inch behind this is an axle with two rollers each 22 inches wide and 6 feet apart center to center, each carrying nearly 7 tons. Thus the front and rear rollers overlap slightly.

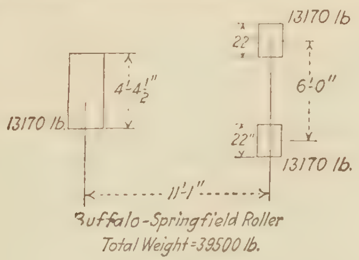


FIG. 18.

22. Impact on Highway and Electric Railway Bridges.—This depends largely on the condition of the pavement and tracks. The specifications of the Massachusetts Board of Public Utilities require the following percentage of the live stress:

	Per Cent
For auto truck:	
For stringers, floor beams, hangers, and truss members receiving their whole load from one panel point only	50
For wood flooring and wood stringers, no impact.	
For all other live loads:	
For floor beams and stringers	25
For floor beam hangers	40
For all counters	40
For other members in trusses, and for main girders, the percentage shall be $26\frac{2}{3}$ minus one-twelfth the loaded length in feet, with a maximum of 25 and a minimum of 10 per cent.	

See also Art. 7, and the New York specifications in Art. 19.

23. Distribution of Load.—A load applied to the floor of a railway or highway bridge is generally distributed on the supporting members in an uncertain manner.

A load applied on top of a fill of earth or ballast is distributed over a larger area below. It is often assumed to be distributed from the load at an angle of 45° with the vertical; that is to say, a concentrated load is assumed distributed uniformly over a square whose side is the depth of fill; and a load distributed on top over an area 2 by 3 feet would be distributed in a depth of 4 feet upon a rectangle 10 by 11 feet. If the areas for adjoining loads overlap, the overlapping portion is counted but once; that is, if two concentrated loads are 3 feet apart on a fill 4 feet deep, they would be distributed over a rectangle 8 by 11 feet. This practice, though arbitrary and inaccurate, is probably near enough (see Art. 38, Chap. XXI).

The load on the rail of a steam railroad is distributed by the rail, and may be assumed uniformly distributed on the stringer over a length covered by three ties, if the ties are not over 8 inches apart; *i.e.*, if the ties are 8 inches wide, over a length not over 48 inches. No doubt the maximum pressure on the stringer is from the tie directly under the load, but on the other hand it is probably distributed over more than three ties, if there is no overlapping. The same is true of the distribution through ties on a bracket angle supporting them on the web of a girder. Often railroad ties are but 4 inches apart, and the distribution would be over 36 inches.

A concentrated wheel load applied to a wood or concrete floor, though acting directly above a stringer, will not all be carried by that stringer, but will be distributed in part to adjoining stringers, which are usually

only a few feet distant. The distribution will depend upon the stiffness of the floor.

The Massachusetts Department of Public Utilities specifies that for the purpose of designing the stringers the following assumptions may be made:

> Where plank floors or floors resting on planks are used, each wheel load of the auto truck may be considered to be distributed over a width of floor equal in feet to the thickness in inches of the supporting layer of planking. The width over which the load is distributed shall never, however, be taken as more than 6 feet.
>
> Where solid floors are used each wheel load may be assumed to be distributed over a width of 6 feet.

This matter is more fully discussed in the chapter dealing with Floor Systems, in the volume on Design.

LIVE LOADS ON RAILROAD BRIDGES

24. The actual loads are a series of concentrated loads consisting of one or two engines and tenders followed by cars. Years ago an approximate uniform load per foot was commonly assumed, except for the floor. But while a train of the heaviest cars, which are short coal or freight cars,

MOMENT DIAGRAM — COOPER'S E-60 LOADING	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18		
Sum of Loads in thousands of pounds	30	90	150	210	270	309	348	387	426	456	516	576	636	696	735	774	813	852		
Load in thousands of pounds per axle	30	60	60	60	60	39	39	39	39	30	60	60	60	60	39	39	39	39		Train Load 6000 lb.-ft.
Distances in feet		8	5	5	5	9	5	6	5	8	8	5	5	5	9	5	6	5	5	
Sum of Distances in feet	0	8	13	18	23	32	37	43	48	56	64	69	74	79	88	93	99	104	109	
MOMENTS in thousands of foot-pounds	0	240	690	1440	2490	4920	6465	8553	10488	13896	17544	20124	23004	26184	32448	36123	40767	44832	49092	

Fig. 19.

gives nearly a constant and uniform load per foot, the engine loads, and especially the load on the drivers, is much greater than that of the train. It was obvious that the actual wheel loads of the engines, at least, should be used, though it was justifiable to use a uniform load for the cars. The difficulty was that there were many different types and weights of engines, with different arrangement and spacing of wheels. This led to a great multiplicity of loadings. The heaviest engine at that time, however, was of the Consolidation type, with four pairs of drivers, the heaviest having about 40,000 pounds on a pair of drivers. This led Theodore Cooper,[1] in 1894, to suggest the use of this type as a standard, the minimum, or E-25, having 25,000 pounds on a driving axle, less on

[1] "Train Loads for Railroad Bridges," *Trans. Am. Soc. C. E.*, p. 174, 1894.

truck and tender axles, and a uniform train load of 2,500 pounds per foot of track following two engines. For other weights, all loads were increased in proportion, the *E*-40 having 40,000 pounds on two drivers, 4,000 pounds per foot of train load, and other wheel loads in proportion. This set of standards has remained in use to the present day. The loads, however, have increased so that now *E*-60 or *E*-70 is common. The *E*-60 loading is shown in Fig. 19.

25. Locomotive Excess. Equivalent Uniform Load.—All loadings for railroad bridges are to some extent arbitrary. While the engine loads may be taken as the weights of some actual engine, the actual train loads will seldom, if ever, exactly correspond with any assumed load. Moreover, in order to provide for the future, even the engine loads as well as the train loads must be taken larger than the heaviest actual loads; or, if this is not done, the structure must be designed so that a considerable increase of live loads above the heaviest in actual use shall not cause stresses higher than safe limits. The assumed loads are therefore to some extent arbitrary.

The computation of stresses for an assumed load of two actual or assumed engines followed by a uniform train load is somewhat tedious, but not particularly so if a moment diagram is prepared. Efforts have been made, however, to use a simpler method by assuming either (*a*) an equivalent uniform load, or (*b*) a uniform train load, either the one specified for the assumed loading, *i.e.*, 6,000 pounds per foot for the *E*-60 loading, or some other, combined with either one, two, or three single concentrated loads to represent the additional weight of the engine loads over the uniform train. By method (*a*) no concentrated loads whatever would be used. By method (*b*) the only concentrated loads would be the so-called "floating loads" or "locomotive excesses." If two steam locomotives are supposed to be used, there should be, logically, two locomotive excesses, always spaced a distance apart equal to about the length of an engine and tender; it is assumed that the locomotive excess or excesses may be placed anywhere (but always at the above distance apart), and not always with the leading one at the head of the uniform load, for an engine may be in the middle of a train. Steinman suggests a floating group of three excesses.

If the shears and moments at different points of any assumed span, or the stresses in all the bars, are computed for an assumed actual loading consisting of two engines and train, such as the *E*-60 loading, it is obvious that, for each shear, moment, or stress, an equivalent uniform load in the worst position, or an equivalent excess or excesses with an assumed uniform load, may be computed, which will give the same shear, moment, or stress as the actual engine and train loading assumed. But it is also clear that the equivalent uniform load, or the excess or excesses, so determined, will be different for each point or for each bar. The equivalent

uniform load, or the excesses, which will produce the same moment at the center of a 100-foot span, will not be the same as those which will produce the same shear at the ends, or the same shear or moment at another point of the same span, or at any point of another span. To use either method with any saving of time, somebody must compute the equivalent uniform load, or the equivalent excesses, for the shear, moment, and stress, at various points of all usual spans; and even this will give only the uniform load or excesses which are equivalent to some one typical assumed loading such as *E*-60. It may be done for equivalent uniform loads for various typical loadings, and it has been done, with enormous labor, by Dr. D. B. Steinman,[1] and charts and tables given therefor.

As an illustration, taking the *E*-60 loading, the equivalent loads would be as follows:

EQUIVALENT LOADS PER TRACK

For	100-foot span			200-foot span		
	Uniform load, pounds per foot	Single excess, pounds	Each of two excesses 56 feet apart, pounds	Uniform load, pounds per foot	Single excess, pounds	Each of two excesses 56 feet apart, pounds
Moment at center....	7,607	80,340	91,300	7,114	111,390	77,300
Shear at center.......	9,446	86,160	....	8,448	122,400	85,000
Shear at ends........	9,000	150,000	104,200	8,142	214,215	124,540

It was formerly the practice, on the continent of Europe, to use a uniform load, more or less roughly determined, instead of actual wheel loads; but the mistake was common of using the same load for all the parts of a span, varying it only with the span. In 1884, Winkler attempted to give more accurate methods, with formulae, for determining the equivalent uniform load.[2]

26. Types of Locomotives.—There are many different types, varying in arrangement and spacing of wheels, the loads varying in each type. The most common method of designation is that proposed by F. M. Whyte, in which the number of wheels in the pony or forward truck is stated, then the number of driving wheels, and then the number in the rear truck; for instance, 2-8-0 is the Consolidation type with four pairs of drivers and no rear truck. The following table (Fig. 20) shows the types needed by the structural engineer:

[1] "Locomotive Loadings for Railroad Bridges," *Trans. Am. Soc. C. E.*, vol. LXXXVI, p. 606, 1923.

[1] WINKLER, "Ueber die Belastungs-Gleichwerthe der Brückenträger," from *Festschrift der Königl techn. Hochschule zu Berlin*, 1884.

Representation	Common Name	Whyte Symbol
△▫OOOO	8-Wheeled Switcher	0-8-0
△o▫OOO	Mogul	2-6-0
△o▫OOOo	Prairie	2-6-2
△o▫OOOO	Consolidation	2-8-0
△o▫OOOOo	Mikado	2-8-2
△o▫OOOOO	Decapod	2-10-0
△o▫OOOOOo	Santa Fe	2-10-2
△o▫oOO	American	4-4-0
△o▫oOOo	Atlantic	4-4-2
△o▫oOOO	10-Wheeled	4-6-0
△o▫oOOOo	Pacific	4-6-2
△o▫oOOOOo	Mountain	4-8-2
△o▫oOOOOO	Mastodon	4-10-0
△o▫OOOO▫OOOO	Articulated Mallet	2-8-8-0
△o▫OOOOO▫OOOOOo	" "	2-10-10-2
△o▫OOOO▫OOOO▫OOOOo	" "	2-8-8-8-2

FIG. 20.—Types of locomotives.

27. Locomotive Loads.—When Mr. Cooper suggested the *E*-loading in 1894, with the heaviest as *E*-40, this engine weighed somewhat more than the usual heaviest engines in use at the time. It had 40,000 pounds on a driving axle, 160,000 on all drivers, tender 104,000, engine 180,000, total weight 284,000 pounds. But soon there began a great increase in weights, as the necessity was felt of hauling greater train loads. Up to that time the principal types of engines, aside from switchers, had been the American, the Mogul, the 10-wheeler, and the Consolidation. Other types were soon developed. The Atlantic type was introduced in 1894, for fast passenger service, and later the other types in the table. To increase the tractive power, the weight on the drivers had to be increased, and in order to avoid too great an increase in the axle load, the number of drivers was increased, and the Decapod, Santa Fe, and Mastodon came into use. To increase still further the weight on drivers, without increasing unduly the axle load or the rigid driving wheelbase, which was objectionable on curves, the French Mallet engines were introduced, in which there were two sets of drivers with a swivel joint between. These were termed *articulated* engines, though generally known as Mallet engines, after the inventor. The first locomotive of this type, built by the

American Locomotive Company for the Baltimore and Ohio Railroad, was exhibited at the Louisiana Purchase Exhibition in 1904. The two groups of drivers were driven from separate sets of cylinders, the rear group by the high-pressure cylinders and the forward group by the low-pressure cylinders. The type was 0–6–6–0, and it weighed 150 tons. The first extensive use of the Mallet type was by the Great Northern Railway, about 1906, the type being 2–6–6–2. Today the largest Mallet engine is of the type 2–10–10–2, built by the American Locomotive Company for the Virginian Railway, and weighing 665,000 pounds without the tender.

In 1914 the Baldwin Locomotive Works extended the Mallet principle, and built a "triple articulated" locomotive, of type 2–8–8–8–2, weighing 864,400 pounds, with tender, for the Erie Railroad. The diagram is shown in Fig. 21.

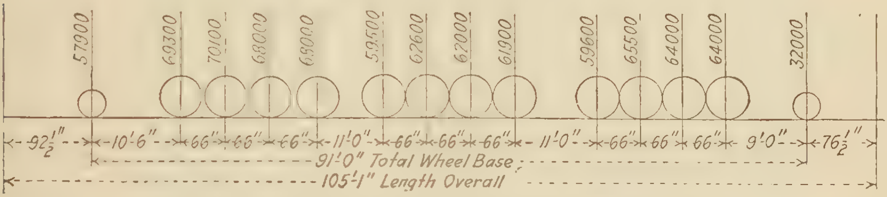


Fig. 21.—Erie Railroad Triple Articulated Engine Class P-I.

It is then obvious that since the *E*-loading was proposed, not only has there been a great increase in weight, but the types have radically changed. In 1903, Mr. Cooper increased his maximum from *E*-40 to *E*-50, which he recommended as "sufficient for future load requirements,"[1] the train load being of course 5,000 pounds per foot.

But if the stresses are calculated for some of the present-day locomotives, it will be found that to equal them the *E*-loading will require *E*-70, or even higher.

The last 30 years have seen changes in locomotive construction which may be summarized as follows:[2]

1. The development of new types, namely, the Atlantic, Prairie, Pacific, Mikado, Santa Fe, Decapod, Mastodon, and Mountain.
2. The development of the compound locomotive.
3. A system for burning oil.
4. Outside valve gears and mechanical reversing devices.
5. The trailing truck.
6. Improvements in steam generating.
7. The articulated locomotive.
8. Superheated steam.
9. The perfection of counterbalancing.

[1] "Discussion on Loadings for Railway Bridges," *Trans. Am. Soc. C. E.*, vol. LI, p. 105, 1903.

[2] See Austin, W. A., "A Quarter Century of the Steam Locomotive," a paper before the Philadelphia Section of the A.S.M.E., Nov. 27, 1923, reprinted in *Baldwin Locomotives*, 1924; also Warner, Paul T., "Motive Power Development on the Pennsylvania Railroad System, 1831–1924," in *Baldwin Locomotives*, 1924.

Kind of engine	Engine and tender: Type	Engine and tender: Weight, pounds	Engine and tender: Wheel-base	Engine and tender: Weight per foot, pounds	Engine: Weight, pounds	Engine: Wheel-base	Engine: Weight per foot, pounds	One set of driving wheels: Weight, pounds	One set of driving wheels: Wheel-base	One set of driving wheels: Weight per foot, pounds	Tender: Weight, pounds	Tender: Wheel-base	Tender: Weight per foot, pounds
E-40	2-8-0	284,000	48′	5,917	180,000	23′	7,826	160,000	15′	10,667	104,000	16′	6,500
Erie R. R. Mallet	0-8-8-0	560,000	70′ 10″	7,906	410,000	39′ 2″	10,467	206,000	14′ 3″	14,456	150,000	20′ 10″	7,201
A. T. & S. F. Frt. Mallet	2-8-8-2	700,400	98′ 5⅛″	7,116	462,800	59′ 10″	7,735	206,400	16′ 6″	12,509	237,600	31′	7,665
A. T. & S. F. Frt. Mallet	2-10-10-2	850,000	108′ 1⅝″	7,860	616,000	66′ 5″	9,274	275,000	19′ 9″	13,924	234,000	32′ 6″	7,200
Erie articulated	2-8-8-8-2	845,060	90′ / 97′ 8″	9,390 / 8,653	Combined with tender			251,200	16′ 6″	15,224	No separate tender		
Virginian Ry.[1]	2-10-10-2	875,000	95′ 8″	9,146	665,000	65′ 3″	10,192	300,000	19′ 10″	15,078	210,000	22′ 10½″	9,182
Pa. R. R. N1S[1]	2-10-2	650,840	82′ 9¼″	7,863	442,840	41′ 11½″	10,564	361,940	22′ 2″	16,326	208,000	26′ 8¼″	7,793
Erie R. R.[1]	2-10-2	603,100	78′ 4½″	7,695	415,100	41′ 5″	10,019	337,100	22′ 0″	15,323	188,000	23′ 0″	8,174
Virginian Ry.[1]	2-8-8-2	746,500	91′ 9½″	8,133	542,500	57′ 4″	9,463	242,800	15′ 6″	15,665	204,000	27′ 11½″	7,296
U. S. Standard[1]	2-8-8-2	754,000	94′ 6″	7,979	540,000	58′	9,310	240,000	15′ 9″	15,238	214,000	27′ 10½″	7,678
L. S. & M. S. Mallet[1]	0-8-8-0	623,200	74′ 4½″	8,379	466,400	40′ 3½″	11,576	236,200	14′ 9″	16,014	156,800	20′ 10″	7,528

[1] Steinman's paper.

10. Feed-water heating.
11. Reduction in steam consumption.
12. Many auxiliary devices.
13. Finally, increase in allowable weight.

As compared with the *E*-40, some of the modern locomotives show the following increases:

	Per Cent
Total weight of engine and tender	208
Weight of engine	269
Weight on drivers	111
Weight on one axle	72
Weight of tender	128

It is a striking fact that the weight on an axle has increased much less than the other weights. The increase in tractive power has been accomplished mainly by increasing the number of drivers. There are limitations to increase in axle loads, in track, ties, ballast, etc. Also, the weight of tenders has increased less than engine weights; this is due to the more economical use of fuel.

The accompanying table gives data regarding engine loads. There is some discrepancy between different statements of weights, but nothing substantial. Some weights of electric locomotives are included in the list.

The Erie articulated engine 2-8-8-8-2 has no separate tender, the tender

being carried on the wheels shown. This enables almost the entire weight to be utilized for tractive force, as the weight of tender is carried in great part on driving wheels. This tractive monster is intended to operate at or near the center of a train, with 160,000-pound cars on each side, so that it may pull and push at the same time.[1] The draw-bar pull is too great for operation at the head of a train. Dividing the total weight, not by the wheelbase of 90 feet, but by 97 feet 8 inches, which includes half the distances to adjoining car wheels, gives 8,653 pounds per foot. All loads per foot of wheelbase are of course greater than loads per foot of length of car or engine. The weight of this Erie engine is given in Steinman's paper as 864,500 pounds, with 275,400 on the heaviest group of drivers; but this may be a different engine, as the wheel spacing is not exactly the same as for the engine in the table.

28. Train Loads.—Train loads have also increased, but not so greatly as engine loads. The heaviest train is a freight train. The heaviest Pullman car weighs about 164,000 pounds, and its length overall is 82 feet, 11¼ inches, giving an average load of about 2,000 pounds per foot, or 14,000 pounds per wheel.

When Mr. Cooper proposed his loading, the maximum train load per foot was given by him as 3,779 pounds, excluding some "gun cars," and the maximum axle load was 25,350 pounds. He thought it "very improbable that our train loads would ever exceed as a rule 4,000 pounds per foot," or E-40. In 1903, he raised the limit to 5,000 pounds. Mr. Hodge, in the same discussion, stated the heaviest train load that he knew of as 6,000 pounds per foot, and he thought the limit would be 15 per cent greater or, say, 7,000 pounds per foot; but at that time there were cars in use *in the plant* of the Jones and Laughlin Steel Company weighing 7,300 pounds per foot, on the Monongahela Connecting Railway of Pittsburgh.

The heaviest cars today usually weigh, loaded, from 5,000 to 6,000 pounds per foot of total length. The heaviest known to the writer is the 90-ton coal car used in special service of the Norfolk and Western Railroad, weighing, with 10 per cent overload, 265,200 pounds, on six axles, or 44,300 pounds per axle, the total length being 46 feet, giving an average of 5,770 pounds per foot.[2] On the Duluth and Iron Range Railroad, however, there are ore cars 24 feet long between coupling pins which weigh, when loaded, from 120,000 to 160,000 pounds or from 5,000 to 6,667 pounds per foot. The scheduled capacity of cars is often exceeded by 10 per cent, or even more.

As ore cars are run on only a few railroads, it is generally safe to take the train load as 6,000 pounds per foot.

[1] *Eng. News*, May 7, 1914.
[2] *Eng. News*, p. 117, Jan. 16, 1913.

29. In assuming loads for designing, it must be remembered that while almost all freight cars are likely to be run over any railroad, some, like ore cars, are distinctly limited to certain roads. Still, different roads carry different loads. Some roads are distinctly coal carriers and are regularly loaded with the heaviest loads, which other roads may carry only occasionally.

Also, if a structure is properly proportioned for certain loads, it can safely carry 50 per cent excess.

The E-loading is faulty in making engine loads and train loads vary in the same ratio. Train loads have not increased in as great proportion as engine loads. Heavier engines may be required because grades are steep and in order to haul larger trains, not heavier cars; and greater total loads, not greater loads per foot.

Whether engine and train loads will increase still further is doubtful. Predictions are dangerous and it is amusing to read those that have been

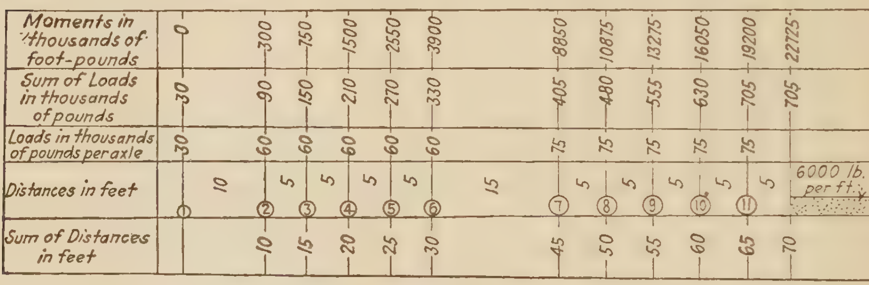


FIG. 22.—Suggested M-loading.

made in the past, some of which have been quoted. It seems now, as it seemed 20 years ago, that the limit has nearly been reached.

It should be remembered, however, that a given percentage of increase in the loads assumed means a very much smaller percentage increase in cost of bridges, since labor and some other costs will increase but little. It was stated in 1894 that an increase of 1 per cent of load meant an increase of but one-fourth of 1 per cent of cost for plate-girder bridges, and of one-third of 1 per cent for pin-connected bridges.[1] With the relatively large increase of labor costs since that time, it is probable that today the increase of cost will be still less. In other words, a 5 per cent increase of cost would give a 20 per cent increase of load capacity for plate girders, and a 6⅔ per cent of cost would give the same increase of load capacity for pin-connected bridges. It therefore does not seem wise to try unduly to limit the loads to be assumed, considering future uncertainties.

30. It seems obvious that the Cooper E-loading has been outgrown, and is not a practical standard any longer, considering the increase of weight, the disproportionate increase of train and engine loads, and partic-

[1] Blakeslee, in discussion of Mr. Cooper's paper.

ularly the introduction of new types of engines. Steinman suggests the substitution of the M-loading, shown in Fig. 22, in which the loads may be increased proportionally from a minimum, as in the E-loading. The writer approves this suggestion. The necessity for it seems conclusively proved by the fact that if E-loadings are to be taken which will give the same stresses as those caused by the modern heavy engines, not only will the equivalent E-loading vary with the span, but several different equivalents will be found for the different members of the same span. Steinman states that for a 140-span Pratt truss, eight different E-loadings will be necessary. If the modern types of heavy engines are not used on some particular railroad, and are not likely to be, the E-loadings may still be sufficient for that road. The M-loading is shown to give results varying less from those produced by modern types of engines than the E-loading. The precise M-load, whether M-50 or M-60 or M-40, is to be decided on as suits the railroad company, as in the case of the E-type.

31. The use of the equivalent uniform load, which, when so placed as to cause maximum effect, will cause the same effect as the desired concentrated load system, has been strongly advocated by many engineers.[1] If this is to be done, somebody must first compute all the shears, moments, or stresses caused by the actual load system. This is not such a difficult matter as might be supposed, owing to certain simple forms of the usual influence lines, as will be explained, and certain formulae which may be deduced for the equivalent uniform load in many cases. Dr. Steinman, in his admirable paper, has indeed made these computations, for the engines in his list and up to the longest usual spans (see table on p. 46). This computation has nevertheless involved enormous labor. The tables and charts which he presents might be incorporated in specifications; but, if so used, they must be accepted as correct, and many engineers will be slow to rely upon any other man's computations. On some roads, also, it will not be necessary to use such heavy loads, or even the types shown. Tables may be prepared, however, which will give equivalent uniform loads for any system of concentrated loads selected; and such tables, if many bridges are to be calculated, would be decidedly useful, convenient, and time saving.

Also, tables might be constructed giving the locomotive excess or excesses which might be used with an assumed uniform load, to be equivalent to a concentrated load system. This method has the approval of some distinguished engineers, though it does not seem to meet with general approval. It is logical in view of the fact that the most striking characteristic of an engine and train load is the large concentration at the drivers. Most engineers seem to think that if a concentrated load

[1] See WADDELL, "Some Disputed Points in Railway Bridge Designing." *Trans. Am. Soc. C. E.*, vol. XXVI, p. 77, 1892; also WADDELL, "The Compromise Standard System of Live Loads for Railway Bridges, and the Equivalents for Same," Steinman, *loc. cit.*

system for engine loads is to be given up, it would be best to take merely an equivalent uniform load with no concentration at all.

Many engineers, however, prefer to adhere to the use of a concentrated load system, because the system chosen can be varied according to the circumstances and because such a system is more concrete, objective, and easily understood by the layman or the railroad manager than any equivalent loading. But if the latter correctly gives the same results, it may be used in the stress computation all the same, while the concentrated load system may be shown, if necessary, as the loads used. This is not deception; it is merely using, in computation, a uniform load which has previously been found to give the same results as an actual engine and train.

32. Relations between Actual Shear, Moment, or Stress and Equivalent Uniform Load.—The influence line for the moment or shear at any point or for the stress of a given kind in any bar, forms with the axis in many cases a triangle (Fig. 23), and the maximum stress of the given

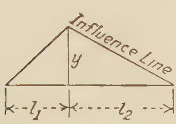


FIG. 23.

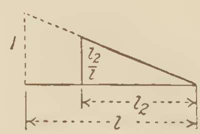


FIG. 24.

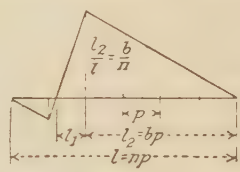


FIG. 25.

kind, or the maximum shear or moment, produced by a uniform load w per running foot is w multiplied by the area of that triangle, or

$$S = \frac{y}{2}(l_1 + l_2)w \tag{3}$$

This is true no matter what the value of y, the maximum produced by a unit load. Hence if the actual maximum stress, shear, or moment produced by the assumed typical loading, as M-60, is S, the equivalent uniform load which will produce the same stress is

$$w = \frac{2S}{y(l_1 + l_2)} \tag{4}$$

If S is the shear on a simple beam of span l without floor beams, the influence line is that in Fig. 24; $l_1 = 0$, $y = l_2/l$;

$$w = \frac{2Sl}{l_2^2};\ S = \frac{1}{2}w\frac{l_2^2}{l} \tag{5}$$

If S is the shear in a panel with floor beams, or the vertical component of a diagonal or vertical in a bridge with parallel chords, the influence line is that in Fig. 25; $y = l_2/l = b/n$, and l_1 is determined by the condition

$$\frac{l_1}{p} = \frac{l_1 + l_2}{np};\ l_1 = \frac{l_2}{n - 1}$$

and

$$w = \frac{2S(n-1)}{b^2p} \tag{6}$$

If S is the vertical component of the stress in the diagonal of a curved-chord bridge (Fig. 26),

$$\frac{x + ap}{h} = \frac{x + ap + p}{h'};\ \text{or}\ x + ap = \frac{x + ap + p}{r}$$

$$x = \frac{ap + p - apr}{r - 1}$$

$$y = \frac{l_2 x}{l(x + ap + p)} = \frac{b}{n} \cdot \frac{a + 1 - ar}{r}$$

The value of l_1 is found from the condition

$$\frac{(l_1 + l_2)x}{l} = \frac{l_1(x + ap)}{p} = \frac{l_1 h}{h' - h} = \frac{l_1}{r - 1}$$

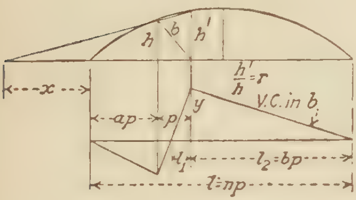


FIG. 26.

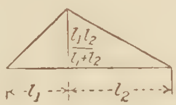


FIG. 27.

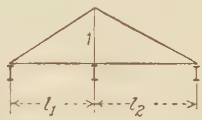


FIG. 28.

from which

$$l_1 = l_2\left(\frac{n}{ar + b} - 1\right)$$

$$w = \frac{2Sr}{b^2p} \cdot \frac{1}{\dfrac{n}{ar + b} - 1} \tag{7}$$

For $r = 1$, this reduces to Eq. (6).

If S is the moment at any point of a beam, or the stress in a chord piece, the influence line is Fig. 27 and

$$w = \frac{2S}{l_1 l_2} \tag{8}$$

If S is the maximum floor-beam reaction (Fig. 28),

$$w = \frac{2S}{l_1 + l_2} \tag{9}$$

If S is the maximum end shear,

$$w = \frac{2S}{l} \tag{10}$$

33. Computation for Wheel Loads Followed by Uniform Load.—The computation of shears and moments for this loading has been fully

explained in Chap. XI of "Strength of Materials," and also the use of the so-called "moment diagram." Figure 22 is the moment diagram for the M-60 loading, one of those suggested by D. B. Steinman in his paper, which is approximately equivalent to Cooper's E-75 for short spans and to Cooper's E-60 for long spans.

Only one remark need be here made, supplementary to the explanations in Chap. XI.

For the maximum moment at any point C of a simply supported span AB (Fig. 29), the influence line has the form of a triangle ADB. The shorter segment of the length is l_1 and the longer l_2. To find the maximum moment at C, or the maximum value of any function for which the influence line has this shape, the distance AB must be loaded so that one wheel is at C, and when that wheel is just to the right of C, the average load per foot on the right must be greater than the average load on the left, while if the load is just to the left of C, the reverse must be true. This criterion fixes the load which must lie at C.

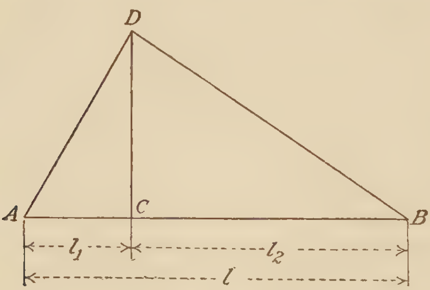


FIG. 29.

Now suppose that the shorter segment l_1 exceeds the total distance from the first wheel to the head of the uniform train load (70 feet in the diagram). Let the total weight in front of the uniform load be ΣP, and the uniform train load w pounds per foot. If the head of the uniform load be placed at C, the average load per foot on the left will be $\Sigma P/l_1$, and that on the right will be w. If $\Sigma P/l_1 > w$, the function will be increased by moving to the right, and the maximum value will occur when one of the engine loads lies at C, according to the criteriod. If $\Sigma P/l_1 < w$, the loads must be moved to the left, and the maximum value will be when C is at some point of the train load. This point will be fixed if we find how far to the right of A the head of the uniform loan must be placed. Call this distance x. Then the equation

$$\frac{\Sigma P + (l_1 - x)w}{l_1} = w$$

will give the value of x, or

$$x = \frac{\Sigma P}{w} \tag{11}$$

Hence in this case, for maximum moment, the head of the uniform load is at a distance $\Sigma P/w$ from A, no matter what the value of l_1 or l_2. If, when the train is placed in this position, some of the forward wheels of the engine are off the span (the moment being the function to be found) ΣP is the sum of the concentrated loads still on the span.

For the M-60 loading $\Sigma P = 705$, $w = 6$; hence $x = 117.5$ feet, and when $l_1 > 117.5$ the head of the uniform load is always to be placed 117.5 feet from A, for maximum moment if $\frac{\Sigma P}{l_1} < w$.

34. Every reader should study the excellent paper by Dr. D. B. Steinman, already referred to. After showing that the Cooper E-loading fails to represent adequately the modern types of locomotives, the author presents three alternatives:

1. The typical M-diagram (Fig. 22). He shows that the M-60 gives a close agreement with the maximum given by any of seven modern heaviest locomotives, for any type of influence line.

2. A uniform load of 6,000 pounds per foot and three floating loads of 90,000 pounds each, spaced 12 and 48 feet apart, which is shown to agree closely with the M-60 diagram.

3. An equivalent uniformly distributed load, for which formulae, tables, and charts are given.

In order to find the equivalent uniform load which will produce the same maximum moment, shear or stress as a given concentrated loading, such as the M-loading, it is necessary to find first the value of that maximum moment, shear, or stress, by using the moment diagram, and then from the influence line to find the uniform load which will give the same value. The maximum stress produced by the uniform load is the area of the moment diagram (taking only the part on one side of the axis) multiplied by the uniform load per foot. If the maximum stress produced by the system of concentrated loads and also that produced by the uniform load can both be expressed by a formula, then a formula may be found for the equivalent uniform load. Thus, for finding the maximum moment at a point, for the case in the last article, where the uniform load covers l_2 and part of l_1, if an expression for the moment is found for the diagram (Fig. 22), and then the equivalent uniform load, w found by Eq. (8), it will be found to be

$$\omega = 6 + \frac{37{,}387.5}{l_1(l_1 + l_2)} \tag{12}$$

and for midspan, where $l_1 = l_2$,

$$\omega = 6 + \frac{18{,}694}{l_2^2} \tag{13}$$

and for the end shear, similarly, using Eq. (10),

$$\omega = 6 + \frac{630}{l_2} - \frac{18{,}150}{l_2^2} \tag{14}$$

The writer gives a formula for equivalent uniform load, obtained by plotting the accurate equivalents of the typical M-60, but the formula gives results differing in some cases by 10 per cent from those equivalents, and is not satisfactory to me. I prefer to use the typical diagram (Fig.

22), or the equivalent with floating concentrations, which, as shown above, should and will agree with the results of the diagram better than any single uniform load can.

35. Electric Locomotive.—Such locomotives, used on few lines at present, generally consist of units which may be coupled together, like multiple-unit cars in subway trains. The heaviest types are indicated in Fig. 30. These are probably not exact, as there are discrepancies in

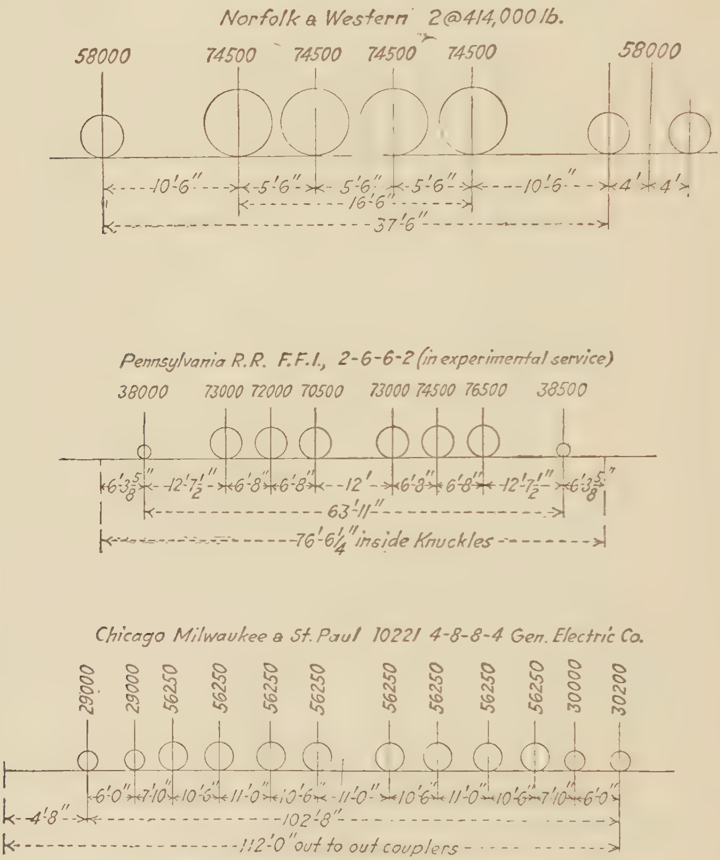


FIG. 30.—Heavy electric locomotives.

the weights as given in different places. The effect of these engines is closely the same as that of the M-loading properly rated, generally at less than M-60.

36. Centrifugal Force.—A bridge on a curve must be proportioned to resist the centrifugal force. The only case of importance is a railway bridge, where loads move at high speed, though it may be necessary to consider it in some cases on a highway bridge.

The centrifugal force on a body of weight W moving with a velocity v around a curve of radius r is

$$C = \frac{Wv^2}{gr} \tag{15}$$

If d is the "degree of the curve," or the angle subtended by a chord 100 feet in length,

$$\sin \frac{1}{2}d^\circ = \frac{50}{r}$$

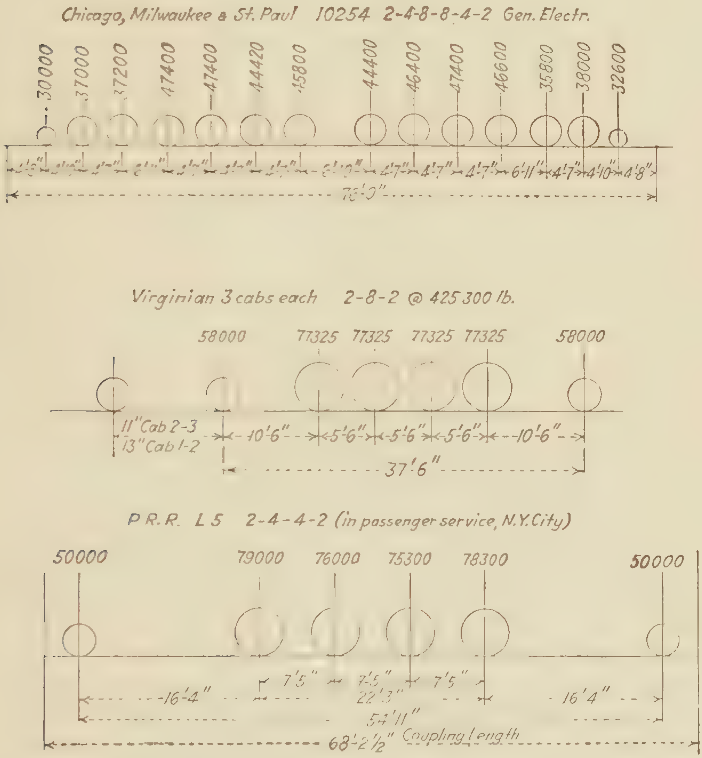


FIG. 30.—Heavy electric locomotives.—(*Continued.*)

The radius of a 1° curve is 5,730 feet, and that of a d° curve is $5{,}730/d$ with sufficient accuracy. In the above equation, v is in feet per second, and

$$v \text{ (in feet per second)} = \frac{1}{1.47} v \text{ (in miles per hour)}$$

Hence if v is in miles per hour, C is the following percentage of W.

$$C \text{ (per cent of } W) = \frac{100v^2d \times 1.47 \times 1.47}{32.2 \times 5{,}730} = 0.00117v^2d \tag{16}$$

For a 1° curve at 80 miles per hour,

$$C \text{ (per cent of } W) = 7.49$$

For a 2° curve at 65 miles per hour the percentage is 9.89.

The A.R.E.A. specifications give 7.5 and 10 for these two cases. The speed must be taken as the greatest speed at which the train assumed would pass around the curve. The A.R.E.A. specifications should be followed unless there is specific reason for not doing so.

The centrifugal force acts at the center of gravity of the train, or about 6 feet above the rail, according to the above specifications. It results in an increased load on the outer girder or truss.

For electric railroad bridges, the Massachusetts Department of Public Utilities requires 10 pounds per running foot for each degree of curve, applied 5 feet above base of rail.

37. Tractive or Braking Force.—When a vehicle is being accelerated, there is a longitudinal tractive force applied to the driving or motor wheels (acting backward on the structure), and when it is being brought to a stop, there is a retarding force applied to the wheels to which brakes are applied, acting forward on the structure. These forces are a percentage of the loads on the wheels affected, depending on the coefficient of friction. In a railroad train, generally all the wheels are braked, and the A.R.E.A. specifications prescribe a force of 20 per cent of the live load on one track only, applied 6 feet above the top of the rail. For electric roads the Massachusetts Department of Public Utilities specifies 20,000 pounds applied at the top of the rail (20 per cent of the weight of one car); and the Standard Specifications for Highway bridges, of the U. S. Department of Agriculture, specify 20 per cent. This force should really be applied at the center of gravity of the load. Everybody knows that a sudden stop will throw a passenger forward. This results in an increased load on the forward truck or wheels. In the Massachusetts car, 20 feet center to center of trucks, the increase, if the center of gravity is 4 feet above the rail or, say, 2 feet above the truck bearing, and if the weight above the truck is 40 tons and the total weight 50 tons, would be $20{,}000 \times \frac{4}{5} \times \frac{2}{20} = 1{,}600$ pounds. This is neglected, or included in the impact allowance, and the force of 20,000 pounds is taken at the top of the rail.

WIND PRESSURE

38. Theory.—The pressure exerted by any perfect liquid, whether at rest or in motion, against a surface with which it is in contact, is always normal to that surface. This results from the definition of a perfect fluid.

Air is not a perfect fluid, but is very nearly so, and should be so assumed here.

If a *jet* of a perfect fluid is directed against a plane surface of indefinite extent at right angles to it (Fig. 31), the total pressure against that surface may be found from the principle that *force multiplied by time equals change in momentum* (mv); in other words, the force acting upon a given mass m for a time t changes the momentum mv of the mass by an amount equal to the force multiplied by the time it acts. If A is the area of the jet, v its velocity, and w the weight of a cubic unit of the fluid, the mass passing any given area A in one second is wAv/g. This mass is deflected by the plane surface and ultimately, if the surface is large, moves parallel to the surface. Hence in each second a momentum wAv^2/g, in a direction perpendicular to the surface, is destroyed. The force perpendicular to the surface, acting on this mass, is P; hence, P being in pounds, and A in square feet,

$$P = \frac{\omega}{g} A v^2 \tag{17}$$

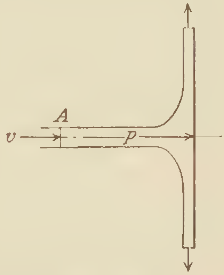


Fig. 31.

The weight of a cubic foot of air, at the normal barometric pressure of 760 millimeters of mercury and at a temperature of 0°C., is 0.08071 pound, and g being 32.2, this formula reduces to

$$P = 0.0025Av^2 \text{ (} v \text{ in feet per second)} \tag{18}$$

or

$$P = 0.0054Av^2 \text{ (} v \text{ in miles per hour)} \tag{18a}$$

This pressure P is, of course, not distributed uniformly over the plate.

The case of wind blowing against a plane surface, however, is not the same as the jet. The body of air is larger than the plate. If the area of the jet A is taken as the area of the plate, the intensity of pressure, assumed distributed uniformly, if Eq. (18*a*) is taken to be correct, would be, if v is in miles per hour,

$$p = \frac{P}{A} = 0.0054v^2$$

But this is not correct. There are two causes of error. In the first place, the mass wAv/g does not have its velocity perpendicular to the plate destroyed in a second. A good part of this mass is only partly deflected and passes around the edges of the plate instead of moving parallel to it. This reduces the value of P and of p. In the second place, a partial vacuum is formed back of the plate, by the eddying air. This increases the value of P and p.

To what extent these effects will counterbalance, is unknown. It has sometimes been assumed that they would just counterbalance, and the formulae given have been:

$$\left.\begin{aligned} p &= \frac{v^2}{400} \text{ (} v \text{ in feet per second)} \\ &= \frac{v^2}{185} \text{ (} v \text{ in miles per hour)} \end{aligned}\right\} \tag{18b}$$

The value of the coefficient is uncertain, depending upon such variables as the size and shape of the exposed surface. The preceding facts have been given only to show the basis for the principle, now generally admitted, that the intensity of the wind pressure, on a surface normal to its direction, varies as the square of the velocity.

The formula

$$p = \frac{v^2}{200} = 0.005v^2 \tag{19}$$

which agrees closely with Eq. (18*a*) if v is in miles per hour, is known as Rouse's formula, and was quoted by Smeaton as based on experiments made in the eighteenth century.

This equation was generally regarded favorably until the time of the construction of the Forth bridge, about 1884. It was soon found that the standard anemometers registered too great a velocity, and this, coupled with some change in the relation between velocity and pressure, led the U. S. Weather Bureau, in 1894, to give the formula

$$p = 0.004v^2\frac{B}{30}$$

where B is the height of the barometer in inches. Even this coefficient is now considered to be too great.

39. Later Formulae.—Extended discussion of the subject may be found in the *Proceedings* of the Institution of Civil Engineers of Great Britain.[1]

In the first paper, measurements in a wind tunnel are described, on plates of various shapes, by means of a Pitot tube apparatus, on both windward and leeward sides. The intensity on the leeward side was practically uniform except near the edges, where it was greater. The maximum pressure on the windward side of a circular plate was 2.1 times the pressure on the leeward side, and on a rectangular plate 1.5 times the pressure on the leeward side. The plates were all small, the circular plates being from ½ to 5 inches in diameter, and the rectangular plates from 1.77 × 1.77 inches to 9 × 0.15 inches. Larger plates would probably give different results.

If p is the resultant pressure per square foot, and v the velocity in miles per hour, the formula found was

$$p = 0.0027v^2 \text{ (for circular and square plates)}$$

Other experimenters have found the following results:

Dines: $p = 0.0029v^2$ (square plate)
Froude: $p = 0.00366v^2$
Langley: $p = 0.00326v^2$ to $0.0039v^2$ (square plate)

[1] STANTON, "On the Resistance of Plane Surfaces in a Uniform Current of Air," vol. CLVI, p. 78, 1903; STANTON, "Experiments on Wind Pressure," vol. CLXXI, p. 175, 1907; REMFRY, "Wind Pressures and Stresses Caused by the Wind on Bridges," vol. CCXVI, p. 3, 1922; STANTON, "Report on the Measurement of the Pressure of the Wind on Structures," vol. CCXIX, p. 125, 1925.

The total pressure was found to be proportional to the area, but this would not be true for large areas.

Ferrel,[1] in 1885, proposed, as the true theoretical formula, the following, in which there are corrections for temperature and for barometric pressure, since both these factors affect the weight per cubic foot.

$$p \text{ (on windward side only)} = \frac{0.002698v^2}{1 + 0.004T} \cdot \frac{P}{P_0} \qquad (20)$$

in which I = the temperature in degrees Centigrade.

P = actual barometric pressure.

P_0 = standard barometric pressure (760 millimeters).

He thought that $\frac{1}{4}$ or $\frac{1}{5}$ should be added to allow for the vacuum on the leeward side. Adding say, $\frac{2}{9}$, the formula becomes, for total p in pounds per square foot, and v in miles per hour,

$$p = \frac{0.0033v^2}{1 + 0.004T} \cdot \frac{P}{P_0} \qquad (21)$$

Professor F. E. Nipher gives, for p in pounds per square foot and v in miles per hour, for the pressure on the windward side alone,

$$p = 0.00258v^2$$ [*]

40. Measurement of Wind Velocity and Pressure.—Such instruments are called anemometers. The velocity may be measured and the pressure found from it by one of the formulae already given, or the pressure may be measured.

Robinson Anemometer.—This is the usual instrument for measuring wind velocity. It consists of four hemispherical cups set on the ends of two horizontal arms at right angles to each other, the open diametral sections of the caps all facing in the same direction around the circle, so that the wind will strike the concave side of a cup and the convex side of the other cup on the same arm. The standard Robinson or Kew anemometer has nine-inch cups on 24-inch arms. Dr. Robinson assumed that the velocity of the wind was three times the velocity of the cups, and found this substantiated by theoretical calculations.[2] By experiments with better whirling machines, in which the anemometer was placed on the end of a long arm which could be rotated with a known speed, it was found that the coefficient 3 was too great, and that it should be about 2.2; each instrument, however, has a factor of its own, depending on dimensions, weight, lubrication, etc. Owing to the above-mentioned difference in the factor, many reported wind velocities are too great. In this country the factor 3 has been generally retained, to facilitate comparison of records.

The shaft of the Robinson anemometer is geared to a counter, from which the number of revolutions in a given time can be read, and the *average* velocity in that time can be found. Some instruments are record-

[1] See Ferrel, "A Popular Treatise on the Winds," John Wiley & Sons, Inc., 1889.

[*] *Trans. Am. Soc. C. E.*, vol. XXXVII, p. 301, 1897.

[2] *Trans. Royal Irish Academy*, vol. XXII, 1850.

ing, and trace a curve of velocity, but still they give the average velocity because the inertia of the instrument prevents it from responding accurately to very short gusts. Some instruments have electric connections, and are either recording or not.

Dr. J. Patterson, assistant Director of the Meteorological Service, Toronto, Ontario, has found that an anemometer with three cups is better than one with four cups, and both the United States and Canada have adopted the three-cup instrument as the new standard.

Pressure Instruments.—Instead of obtaining the pressure from the velocity by using an uncertain formula, there are instruments for measuring the pressure directly. The *pressure plate anemometer*, originally invented by Sir Hiram S. Maxim, in 1909, has a pressure plate 13 inches in diameter, kept directed against the wind by a vane, the motion of the plate indicating the pressure or the velocity by moving an arm around a graduated scale. Another instrument is a pendulum which is blown from the vertical position by the wind, the divergence indicating velocity or pressure. C. F. Casella and Company make a pressure instrument in which the plate is kept perpendicular to the wind even if it does not blow horizontally. *Dines'* anemometer utilizes the principle of the Pitot tube. A vertical tube is bent horizontally at the top and the open end is kept facing the wind in azimuth, this tube being connected to the inside of a hollow float, which is made to rise by an increase of pressure. The vertical tube passes inside of a larger vertical tube having perforations, and as the wind blows across these perforations, it sucks air out of the larger tube which at the bottom is connected with a chamber *above* the float, so that suction causes the float to rise just as pressure in the other tube causes it to rise. The *anemo-biograph* made by Negretti and Zambra, is also made on the Dines principle and registers the velocity on a revolving cylinder.[1]

Direct measurements of pressure are seldom made. When wind pressures are reported, they are generally obtained from measurements of velocity, and it is important to know how they were made, what formula was used, and whether the recorded velocity or the real velocity was used. Most anemometers indicate too high a velocity.

41. Experiments of B. Baker at the Forth Bridge.—In the construction of the 1,700-foot spans of the Forth bridge, the question of wind pressure was of great importance. On account of the large size of the members the wind forces were of greater magnitude than the train loads. Following the collapse of the Tay bridge, in 1879, a Board of Trade Committee was appointed to study and report on wind pressures on railway structures. This committe advised the adoption of the following rules:

[1] For further information regarding anemometry, consult a long paper by Brazier, in the *Annales du Bureau Central Metrologique de France*, for 1914; also the article on anemometers in the "Dictionary of Applied Physics," and the catalogues of the following makers: C. F. Casella and Company, 49 and 50 Parliment St., London; Negretti and Zambra, 38 Holborn Viaduct, London.

1. The wind pressure to be taken as 56 pounds per square foot.
2. The surface assumed to be from one to two times the vertical projection of the structure.
3. The factor of safety to be 4 for the stresses in ironwork alone, and 2 for the entire bridge, against overturning.

The engineers of the Forth bridge, Messrs. Baker and Fowler, considered it necessary to test the wind pressures by direct experiment. They therefore installed three pressure boards, so arranged that the pressures on them could be registered. One was a large board 20 by 15 feet, fixed in direction facing the direction of the heaviest gales; one was a small fixed board with an area of 1.5 square feet; the third was a small swiveling board with 1.5 square feet area. On this bridge the wind pressure assumed was 56 pounds per square foot over an area equal to twice the vertical projection of one truss on account of the considerable distance between trusses, deducting 50 per cent for cylindrical surfaces. The results of the measurements convinced the engineers that this loading was in excess of anything likely to be realized.

In order to study the relative pressure on different surfaces, an ingenious method was adopted. A horizontal arm was suspended at its center by a cord. On one end of the arm was placed a model of the surface to be studied, and on the other was an adjustable vertical plane surface. If the arm was pulled to one side, released, and allowed to swing, it would twist the cord unless the pressure on the model and on the plane surface was equal; and by changing the area of the plane surface, until it would swing without twisting, the relative pressure on the model could be found.[1]

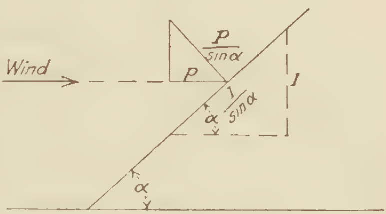


FIG. 32.

42. Pressure on Inclined Surfaces. The previous formulae have referred to the pressure on a surface normal to the direction of the wind. If the surface is inclined at an angle α to the direction of the wind, the real pressure on it will be normal, since air is nearly a perfect fluid, but the intensity will be less than on a normal surface. Calling the pressure intensity on the inclined surface p_n and that on the normal surface p, the relation between the two must be found.

Formerly, this relation was found as follows (Fig. 32): If an area of the surface be taken whose vertical projection is unity, the wind blowing horizontally, its area will be $1/\sin\alpha$. If the horizontal pressure on this surface be taken as the same as on its vertical projection p, and if this be

[1] BAKER, B., "The Forth Bridge: Paper read before the British Association in Montreal," *Engineering*, vol. XXXVIII, pp. 213, 223, and 478, 1884; also *Engineering*, vol. XXXIV, p. 219, 1882; and for a full description of the Forth bridge, an article by W. Westhofen, *Engineering*, p. 213, Feb. 28, 1890.

resolved into components normal and tangential to the surface, the former will be $p \sin \alpha$, and the intensity will be $p \sin^2 \alpha$, or

$$p_n = p \sin^2 \alpha \tag{22}$$

This is not very satisfactory, because it is uncertain whether the horizontal force will be p, and it is not clear that it can be resolved in this way, because, if it is, the tangential component would have a vertical component and a horizontal component. But if p is resolved vertically and normally, the normal component intensity would be p, or the same as on a surface perpendicular to the wind, which is impossible.

This formula does not agree with some experiments, though it does with others. It must be remembered that accurate experiments are difficult to make, and that they may be entirely wrong if the wind should not blow in the direction assumed, but at an angle above or below that direction. The usual vane, which is supposed to keep the surface at right angles to the wind, rotates with the surface about a vertical axis, and so only responds to changes in the point of the compass from which the wind blows, not to changes above or below the horizontal. We all know that the wind does not always blow horizontally, in fact, that it rarely does. For this reason, experiments on wind pressure may be very unreliable unless the instrument revolves about a universal joint.

Hutton's formula, which has often been used, but which has no theoretical justification, is

$$p_n = p \sin \alpha^{1.84\cos\alpha - 1} \tag{22a}$$

By the method of deducing Eq. (22), it is clear that, in strictness, p_n and p are both pressures on the *windward side only* of a surface inclined to the wind. If correct for the pressure on the windward side, then to be correct for the total pressure, it must be assumed that the suction on the leeward side varies with the inclination in the same manner as the pressure on the windward side, which is not true. The neglect of suction on the leeward side confuses some discussions on the subject.

The formula which is considered the most reliable is that of Duchemin, which is empirical, namely:

$$p_n = p\frac{2 \sin \alpha}{1 + \sin^2 \alpha} \tag{23}$$

This formula was confirmed with remarkable closeness by the experiments of Langley,[1] which, however, did not cover all ranges of α.

Lord Rayleigh[2] has given a theoretical formula based on the principles of hydrodynamics, which is as follows:

$$p_n = p\frac{(4 + \pi) \sin \alpha}{4 + \pi \sin \alpha} \tag{24}$$

[1] LANGLEY, S. P., "Experiments in Aerodynamics," published by the Smithsonian Institute, Washington, D. C., 1891.

[2] RAYLEIGH, "Scientific Papers," vol. I, pp. 291 and 293, Cambridge, England, 1899.

Figure 33 shows the relation between the four formulae, Eqs. (22), (23), (24), and (22a). The following table gives the ratio p_n/p by each formula for various values of α.

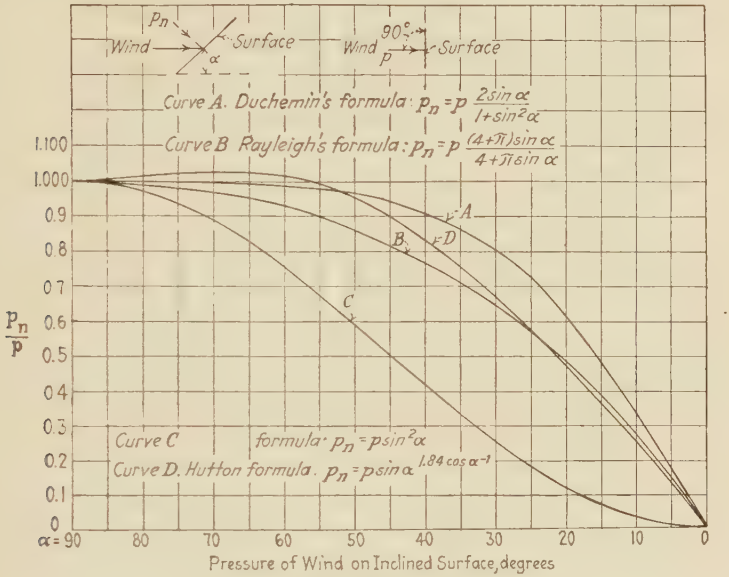


FIG. 33.

WIND PRESSURE ON INCLINED SURFACES

α, degrees	$\frac{p_n}{p}$			
	Eq. (22)	Eq. (23)	Eq. (24)	Eq. (22a)
0	0	0	0	0
10	0.174	0.338	0.273	0.242
20	0.342	0.613	0.482	0.457
30	0.500	0.800	0.640	0.662
40	0.643	0.909	0.763	0.833
50	0.766	0.967	0.852	0.952
60	0.866	0.990	0.922	1.011
70	0.940	0.997	0.966	1.023
80	0.985	0.999	0.987	1.011
90	1.000	1.000	1.000	1.000

Rayleigh's theoretical formula gives the pressure on the windward side only, neglecting the partial vacuum on the leeward side, therefore it can be used only to find the resultant pressure if it is assumed that the suction on the leeward side varies in the same manner as the pressure on the windward side. This may not be true, and depends on the shape of the surface, as will presently be shown.

Some experiments (and Eq. (22*a*) shows a similar result) show a greater normal pressure on an inclined surface than on one normal to the wind,[1] if the inclination is 45° or over. This depends, however, upon the shape of the surface. If a long and narrow rectangular surface is inclined and exposed to the wind, with its *long axis* inclined to the current, there is an excess ($p_n > p$) if α exceeds about 35°, with a maximum excess at about 40°, of about 10 per cent. If the *short axis* is inclined to the current, there is no excess, but there is an excess above Rayleigh's theoretical curve for values of α from 0 to about 40°.[2] This fact is probably due to the effect of the vacuum on the leeward side. It is difficult to see how the pressure on the windward side, for any inclination, can exceed the pressure on a surface normal to the current.

43. Pressure on a Cylinder and on a Sphere.—The pressure on a plane surface depends somewhat upon its shape, but in practice no distinction is made.

If the surface is curved, the pressure will depend upon its curvature, since the pressure on an inclined surface differs from that on a surface normal to the current.

Cylinder.—The pressure on a cylinder, the direction of the wind being perpendicular to a longitudinal axial section, may be found by integration from Eqs. (22), (23), and (24). By Eq. (22) it will be found to be two-thirds of that on an axial section (length times diameter); by Eq. (23) it will be 0.92, and by Eq. (24) it will be 0.58. It is generally taken as two-thirds or three-quarters.

Sphere.—By Eq. (22) the pressure on a sphere will be found to be one-half that on a diametral section, and this value may be assumed without serious error.

The pressure on a cylinder is important to know, as in the case of suspension bridges and suspended wires carried by transmission towers. Since a stranded cable is not always a true smooth cylinder, the actual pressure may be more than two-thirds of that on a diametral section. The formula used may be, if A is the diametral section,

$$P = 0.0023Av^2 \text{ (miles per hour)} \tag{25}$$

The coefficient in this formula will vary with the character of the cable, from, say, 0.002 to 0.003. If the coefficient for a plane surface at right angles to the wind is 0.0033, two-thirds of this, which is the ratio for a cylinder by integrating from Eq. (22), would be exactly 0.0022.

If a stranded cable be hung at an angle α with the horizontal and exposed to a horizontal wind, the pressure normal to the length, according to some reported experiments, will be represented best by Eq. (22) and poorly by Duchemin's formula. The surface of the cable in this case is corrugated, with twisted depressions between the strands, and the

[1] Dines, *Proc. Roy. Soc.*, vol. XLVIII.

[2] Stanton, *Proc. Inst. C. E.*, vol. CLVI, p. 102.

tangential component or drag along the cable, $p \cos \alpha$ in Fig. 32, will probably affect the total normal pressure; or, in other words, p cannot be taken as assumed and resolved as in Fig. 32. Thus the singular result is obtained that if Eq. (22) applies in this case—and it agrees quite closely with some experiments—it is because it is not theoretically accurate.

I think some discrepancies in regard to wind pressure and observed results are because the wind does not actually blow in the direction assumed, or that the suction on the leeward side is not always included, either in the formula or in the tests.

44. Shielding Effect.—If two parallel surfaces are exposed to normal wind pressure, the leeward one will be partially shielded if the distance between the two is small. Stanton found that if two girders are connected by a roadway, their distance apart being equal to the depth of the girder, the pressure on the leeward girder was 15 per cent of that on the windward girder, while if the distance apart was twice as great, the pressure on the leeward girder was 25 per cent of that on the windward girder. In practice, it is sometimes assumed that the pressure on a bridge with two trusses is double that on the vertical projection of one truss, neglecting the shielding effect. The specifications of the A.R.E.A. take the pressure as one and one-half times the vertical projection of the structure. A train on the bridge will of course effectively shield that part of the leeward truss back of the train, and the pressure on the train is of course added to that on the structure. For unloaded bridge it might be better to neglect the shielding effect of one truss or girder on the other.

The full pressure should obviously be taken on the area of windward truss and train, when the structure is loaded. The English Commission on Wind Pressure recommended that on the leeward truss or girder, on the area of vertical projection of metal work below the rails or above the train, a reduced pressure should be taken, 56 pounds being the standard or full pressure, as follows:

1. If the open spaces in the windward truss do not exceed two-thirds the total area of the outline of the truss, one-half the standard, or 28 pounds per square foot.

2. If the open spaces are from two-thirds to three-fourths of the total area, three-fourths the standard, or 42 pounds per square foot.

3. If the open spaces exceed three-fourths the total area, the standard, or 56 pounds per square foot.[1]

By the specifications of the A.R.E.A., the wind pressure is to be taken on the vertical projection of one truss and the floor, and on one-half the vertical projection of the other truss.

45. Effect of Height.—The wind pressure on an unobstructed structure varies with the height above the ground, increasing with the height because the velocity near the ground is reduced by friction. This varia-

[1] See *Railroad Gazette*, p. 537, 1881.

tion should be considered in the design of high towers, such as radio towers, which may be 1,000 feet high. English lighthouse engineers took account of this variation, and Thomas Stevenson, in 1876, determined the velocity at different elevations, and suggested the formula

$$\frac{V}{v} = \sqrt{\frac{H + 72}{h + 72}} \tag{26}$$

V being the velocity at a height H, and v that at a smaller height h.

Professor E. D. Archibald[1] made experiments which suggested the formula

$$\frac{V}{v} = \left(\frac{H}{h}\right)^{1/4} \tag{27}$$

The latest results are those by S. P. Wing,[2] who gives the formula

$$V = (1 + 0.0012H - 0.0000003H^2)V_g \tag{28}$$
$$P = (0.00126H + 1.1b)P_g \tag{29}$$

where V and P are the velocity in miles per hour and the pressure in pounds per square foot, at the height H in feet, and V_g and P_g are the corresponding values near the ground level. These formulae are for a structure which is fully exposed, not sheltered.

46. Effect of Area of Surface.—It seems clear that the average pressure on a large area will be much less than that on a small area. This is because the greatest pressure in a wind current occurs over only a small area, and because the vacuum effect will be less on a large area. Mr. Baker found, at the Forth bridge, that the pressure on the large board was about two-thirds of that on the small board. There was considerable inertia in these pressure boards, and Mr. Baker produced an indication of 65 pounds by a force of 20 pounds, showing that the readings were not always reliable. A pressure exceeding 30 pounds per square foot is considered very rare in the British Isles, and Mr. Baker decided that 56 pounds per square foot had never been shown to have been exerted over a span of 1,700 feet. Mr. Dines found with two of his instruments 11 feet apart, that simultaneous indicated velocities might have a ratio ranging from 0.75 to 1.25, and that one might be rising while the other was falling. The extreme pressures in a cyclone often occur over a very narrow swath.

47. Complexity of Air Currents.—The popular conception of an air current, as used in engineering, is probably that of a steady stream. Everyone knows, however, that the wind comes in gusts, but few realize its extreme variability. There may be ten or more oscillations in 1 second, and in less than a minute the velocity may vary 50 per cent or more on each side of the average. Professor S. P. Langley,[3] using very light cups and

[1] *Nature*, Apr. 22, 1886.

[2] *Electrician*, London, July 1, 1921.

[3] "The Internal Work of the Wind," *Smithsonian Contributions to Knowledge*, 1893. Everyone who is desirous of understanding this subject thoroughly, should read this beautiful memoir of Professor Langley, and also his other one, "Experiments in Aerodynamics," 1891.

arms, and a chronographic device for registering every revolution, or even a fraction of a revolution, found astonishing variability. In one instance, starting with a velocity of 23 miles an hour, it rose in 10 seconds to 33 miles, dropped in 10 seconds to 23, and rose in 30 seconds to 36 miles. Inside of $1\frac{1}{3}$ minutes it varied from 36 miles to zero, with 3 maxima and 3 minima between. In 15 seconds it dropped from 29 miles per hour to zero.

48. It is obvious that the subject of wind pressure is complex and uncertain. Considering the effects of gusts, of area of surface, of height above the ground, of shape, of shielding, it is not strange that many consider the subject entirely empirical. But no phenomena of nature are empirical; they are subject to law, though the law may be complex, and difficult or impossible to discover. The facts and laws should be studied and have been here presented. If empiricism is now necessary, it may be used scientifically. Recorded data and results of experiments are often contradictory and varying, and *a priori* reasoning from fairly accurate premises is often the most satisfactory way of reaching conclusions.

The proper wind pressure to assume in designing a structure is very uncertain, depending upon the country and climate, the exposure, etc. Recorded velocities of over 75 miles per hour are rarely exceeded, though gusts may register 100 or over. In British India it is said that maximum gusts have reached 165 to 174 miles per hour, and at Mt. Tamalpais, California, 148 miles per hour. In tropical countries, subject to hurricanes, higher pressures and velocities may reasonably be assumed than in the United States, though severe hurricanes have occurred in many states of the Union. In the St. Louis tornado of 1896, it was estimated by Julius Baier[1] that the pressure must have reached at least 60 pounds per square foot over a width of about 301 feet, and on a chimney in excess of 85 to 91 pounds per square foot over an area at least 14 feet wide by 110 feet high, *these results being found by computation from the observed effects.* He also states that there were indications that a pressure of somewhere between 20 and 40 pounds per square foot was quite general over a comparatively wide area. He concluded that the wind pressure on a building should be assumed to be at least 30 pounds per square foot on the exposed surface, with 50 pounds per square foot for several stories near the top. This must, of course, depend upon the exposure. If a building is between other buildings, it may have practically no wind pressure except on parts higher than adjoining buildings. For buildings in cities, partially sheltered, 30 pounds per square foot is probably ample or excessive, while in exposed situations it may be desirable to assume as high as 50 pounds. The reader should read the complete paper of Mr. Baier, above referred to.

[1] *Trans. Am. Soc. C. E.*, vol. XXXVII, p. 235.

For bridges, practice in the United States has for many years settled down to an assumption of 50 pounds per square foot on the unloaded structure and 30 pounds per square foot on the loaded structure, the difference being made because it has been assumed that trains would probably not be running in a storm giving over 30 pounds and that the area of surface of a loaded bridge is greater than that of an unloaded bridge. More recently, for railroad bridges, the wind loads per foot of length have been specified. Some good engineers have expressed the opinion that a bridge of over 200-foot span would rarely if ever be exposed to a wind pressure of over 30 pounds per square foot of the entire area.

For transmission or radio towers, a variation of pressure with height should be assumed.

49. Other Lateral Forces.—In addition to wind pressure and centrifugal force, a railroad bridge is exposed to lateral forces due to vibration and "nosing," which may be considerable. The 1925 specifications of the A.R.E.A. provide for this as follows:

> The lateral force to provide for the effect of the sway of the engines and train, in addition to the wind loads specified in Sec. 32, shall be a moving load equal to 5 per cent of the specified live load on one track, but not more than 400 pounds per linear foot, applied at the base of rail.

This provision is unreasonable. It specifies a lateral force extending over the whole length of the train. Now since this force arises from the internal action of the train, it must be balanced; that is, if some wheels of the train are pushing to the right on the rails, other wheels must be pushing to the left with equal force. If all wheels of the train were pushing on the rails toward the right, the track would be pushing on the train to the left, and the train would be moving laterally through the air. The lateral force can act over only a comparatively short length, which should be specified; and there must be an equal lateral force, acting on the bridge, in the other direction, somewhere along the length of the train.

The wind pressure may act *upward* on a structure. In one storm, a locomotive was lifted off the rails; in another, a piano was lifted, carried 270 feet through the air, and deposited without injury. In some tornadoes, a lifting force has been calculated equal to 85 pounds per square foot, based on observed effects.[1] Suction on the leeward side, and lifting effect, should not be lost sight of. Roofs and other structures should be anchored to their supports if they are likely to be subjected to heavy winds. This is frequently not done. It has been suggested that an upward force of 10 pounds per square foot should be provided for on roofs, over and above the weight of the roof itself.[2]

[1] *Trans. Am. Soc. C. E.*, vol. IX, p. 396, 1880.

[2] *Trans. Am. Soc. C. E.*, vol. XXXVII, p. 292, 1897.

The possible effect of earthquakes on buildings may have to be considered under certain circumstances. It involves a horizontal force which has not yet been discussed, but which will be considered in the chapter on Buildings, in a subsequent volume of this work.

50. Liquid Pressure.—The pressure of any perfect fluid against any plane surface, whether the fluid is at rest or in motion, is normal to the surface. The pressure of a liquid is that due to its hydrostatic head, or the pressure per square unit is wh, if w is the weight of the liquid per cubic unit, and h the head, or the depth below the surface if the liquid is at rest. The total pressure on any plane is the area multiplied by the unit pressure at its center of gravity. The vertical component of the pressure will be the vertical projection of the area multiplied by the unit pressure at its center of gravity, and similarly for the horizontal component; these may be found without finding the resultant pressure on the area.

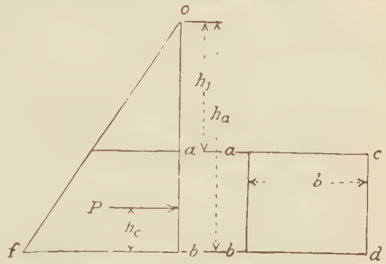


FIG. 34.

The point of application of the resultant pressure on any plane surface, called the *center of pressure*, may be found by taking moments. Suppose *abcd* to represent a rectangle with upper and lower sides horizontal (Fig. 34), the surface being at O. The intensity of pressure at a will be $wh_1 = ae$, and that at b will be $wh_2 = bf$. The total pressure will be represented by the area *abfe* multiplied by b, or

$$P = \frac{wb(h_1 + h_2)}{2}(h_2 - h_1) \tag{30}$$

which is the same as the area $b(h_2 - h_1)$ multiplied by the unit pressure at the center of gravity, or $\dfrac{w(h_2 - h_1)}{2}$. To find the center of pressure, take moments about b, separating the area *abfe* into a rectangle and a triangle. Then the distance h_c of the center of pressure above the base will be given by

$$Ph_c = \frac{wbh_1(h_2 - h_1)^2}{2} + \frac{wb(h_2 - h_1)^2}{2} \cdot \frac{h_2 - h_1}{3}$$

or

$$h_c = \frac{(h_2 - h_1)(h_2 + 2h_1)}{3(h_1 + h_2)} \tag{31}$$

If a liquid is viscous, the pressure which it exerts on a plane surface need not be exactly normal; but if it is not, the liquid will flow, and if it attains a state of equilibrium, with its upper surface horizontal, the pressure will then be normal.

CHAPTER IV

STRESSES IN TRUSSES WITH PARALLEL CHORDS AND SINGLE-WEB SYSTEMS

1. Definition of the Term "Truss."—A truss is a framework composed of bars connected together at their ends by frictionless pins. It follows that there is *no moment* in any bar at the hinge or joint, and this means that the stress in any straight bar, if the loads are applied at the joints, can be only a direct stress, either tension or compression, acting along a line joining the hinges. The stress in any bar must therefore act in the straight line through the hinges at its ends. If the bar is straight, it is exposed to pure tension or compression, though the column action may produce bending; if the bar is curved, it is like a bow, and the direct stress will add to the curvature. If a load is applied to a bar between the joints, it is transferred to the joints by the bar acting as a beam; but this beam action is in addition to, and independent of, the action of the bar as a truss member. The unit stresses due to the beam action must be added to those due to the direct stress.

If the pins are not frictionless, or if there are no pins, but riveted or rigid connections instead, so-called secondary or bending stresses are caused, which are very complicated, and which will be considered in another volume.

In all trusses here considered, the joints will be considered to have frictionless pins.

A bridge or roof truss has two main parts, the *chords* and the *web*. The chords extend from end to end, and may be straight, curved, or broken. In this chapter only trusses with straight parallel chords are considered. The web consists of the bars which connect the chords and may be arranged in many ways, as will be seen. In Fig. 35*a*, the top chord is cc', the bottom chord bb' and the other members are the web members.

In Fig. 35*f* the bars *cb* are considered as web members.

2. General Method of Computing Stresses in Trusses.—The methods of computing the stresses in the members of a truss, with bars assumed hinged at their ends, whether the chords are parallel or not, may all be described as *methods of sections;* that is to say, the truss is considered to be completely separated into two parts by an imaginary section, plane or curved, the portion on one side of this section is considered to be removed, and the equilibrium of the other portion is supposed to be established by

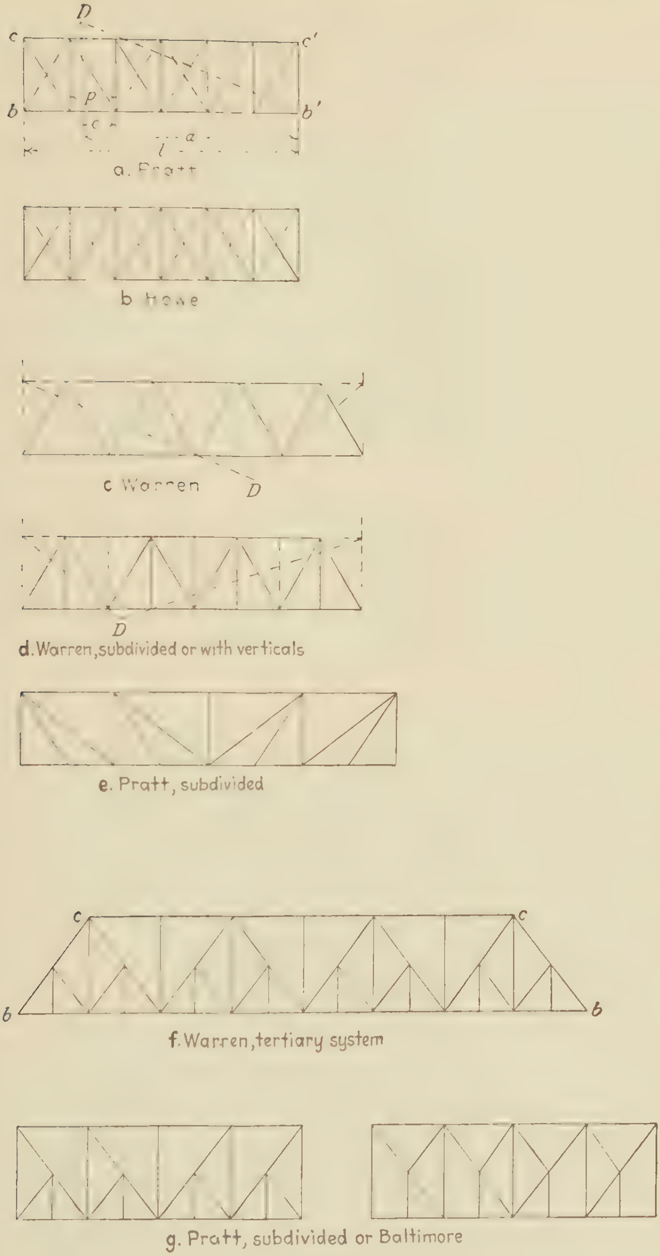

D
c
c′
b
p
b′
c
a
l
a. Pratt
b Howe
c Warren
D
D
d. Warren, subdivided or with verticals
e. Pratt, subdivided
c
c
b
b
f. Warren, tertiary system
g. Pratt, subdivided or Baltimore

Fig. 35.

replacing the stresses in the bars cut, these stresses being the same as outer forces acting on the portion remaining. Thus if we imagine the truss in Fig. 36 cut by the section AB, and the part to the right removed, and if to the left-hand portion (Fig. 37), the forces S_1, S_2, S_3 are applied, equal to the actual stresses acting on the bars cut, this portion of the frame will clearly be in equilibrium under the forces R_1, P_1, P_2, S_1, S_2, S_3, which constitute a system of forces in equilibrium, and therefore fulfil the conditions $\Sigma H = 0$, $\Sigma V = 0$, $\Sigma M = 0$. Of the forces named, R_1, P_1, and P_2 are known loads or reactions, so that the three equations permit the determination of the three unknown stresses.

The outer forces must all be known before the stresses can be found, so that if the section cuts three bars, the three stresses in those bars can in general be found. They can always be found if the three bars cut do not all meet in a point; but if they do all meet in a point, then, since the origin of moments may be taken anywhere in using the equation $\Sigma M = 0$,

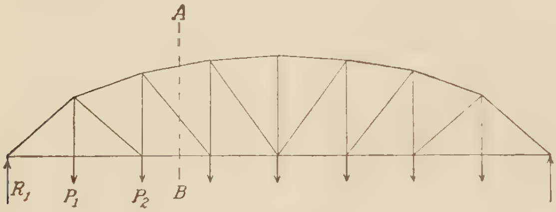


Fig. 36.

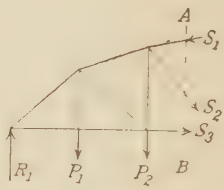


Fig. 37.

if that origin be taken at such point of intersection, the equation does not contain any of the stresses S_1, S_2, or S_3, and there are only two equations, from which the three unknown stresses cannot be determined. If the section cuts more than three bars, there are always more unknown quantities than equations, and, in general, none of the unknown quantities can be found, except under special circumstances. If, for instance, all the bars cut except one were horizontal, the equation $\Sigma V = 0$ would be sufficient for the determination of the stress in that bar; or if all but one passes through a point, the equation $\Sigma M = 0$, taking moments about that point, would give the stress in the bar not passing through it. If the section cuts less than three bars, say two, there are more equations than unknown quantities, and the structure would, in general, be unstable, unless one of the equations is dependent upon the other two; for two of the equations would determine two stresses, and if equilibrium exists, these stresses, so determined, would satisfy the third equation, which must therefore state a relation existing between the outer forces. This reasoning will enable the student to see that a structure which is statically undetermined *with respect to the outer forces* may sometimes be made statically determined by omitting one of the bars, thereby giving an equation of relation between the outer forces alone.

After having taken the section, cutting three bars, the three equations of equilibrium may be applied in different ways to the determination of the stresses. If S_1 and S_3 are horizontal, the equation $\Sigma V = 0$ will give at once the vertical component of S_2, otherwise not. In using the equation $\Sigma M = 0$, the origin of moments may be taken anywhere, the proper choice depending on circumstances. If the origin is taken at the intersection of S_1 and S_2, that equation will give at once S_3, which will be the only unknown quantity, since S_1 and S_2 will have no moment about that point. Similarly S_1 and S_2 may be determined. The stress in any one of the three bars may thus be determined by taking moments about the intersection of the other two, and all three stresses may be found by using the equation $\Sigma M = 0$ three times, about three different origins of moments.

The three unknown stresses may therefore be determined in different ways: (1) by using the three equations $\Sigma H = 0$; $\Sigma V = 0$; $\Sigma M = 0$; (2) by using the equation $\Sigma M = 0$ twice about two different points, and then *either* $\Sigma H = 0$ or $\Sigma V = 0$; and (3) by using the equation $\Sigma M = 0$ three times about three different points. Which of these procedures should be followed depends upon circumstances. If two of the bars cut are parallel, they intersect only at an infinite distance, and the equation $\Sigma M = 0$ about their intersection is not strictly applicable; if applied, it leads to the equation $\Sigma V = 0$, if the axis of V is perpendicular to the two parallel bars. *The rectangular axes of H and V may of course be chosen arbitrarily.*

The method by using $\Sigma M = 0$ for each bar, finding the stress in any bar by taking moments about the intersection of the other two, is called the *method of moments*, because moments alone are used.

If the section cuts two bars only, it is clear the that stress in either may be determined by taking moments about *any point in the line of action of the other*, either within or without the truss.

If the section cuts more than three bars, then if in some other way, by taking other sections, the stresses in all the bars except three may be found, these stresses will be considered as outer forces applied to the cut ends of the bars, and the stresses in the three remaining bars may be found as above explained.

It has been stated that the section could be straight or curved. A particular case arises when it is taken around some one joint of the truss. In this case, after considering the main body of the truss removed, and the unknown stresses applied to the bars cut, there is a system of forces in equilibrium consisting of the load applied at the joint considered, and the stresses in the bars cut. These constitute a system of forces in a plane and *at a point;* hence they must satisfy but *two conditions of equilibrium*, from which the stresses in but two bars can be found. Hence it follows that if a section is taken around a joint, and the stresses in *all but two* of the bars meeting at that joint are known, those two unknown stresses

may be found by considering the equilibrium of the system of forces described, and applying the two equations of equilibrium $\Sigma H = 0$ and $\Sigma V = 0$. When the section is taken around a joint (Fig. 38), the method is known as the *method of joints*. In solving a truss by this method, it is evident that we must begin with a joint *where but two bars meet,* and proceed from this to other joints. The student should study Figs. 58, 59, and 60 in this connection, and see how, by commencing at the abutment, the stresses in all the members of the truss may be worked out by this method.

The method of joints may be used to find the stress in one bar of a truss if there are three bars meeting at a joint and two of them are in a straight line. Thus in Fig. 39, the stress S_3 may be determined without knowing either S_1 or S_2. In applying this method, it must be remembered that the axes of H and V need not be horizontal and vertical, but may be in any two directions at right angles to each other.

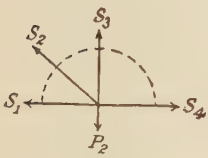


FIG. 38.

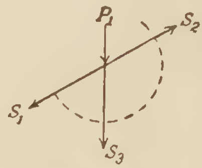


FIG. 39.

In applying any of the methods that have been explained, the first step is to take the section, then consider one side removed and keep consciously in mind which side is removed, then consider the equilibrium of the other side, then see clearly what forces have to be considered, and then apply the equations. Always put arrows on the bars cut, to designate the direction of the stress. It is necessary in the beginning to assume the direction of the stress in each bar. In Fig. 37, S_1 is really compression and S_2 and S_3 tension, and have been so indicated. But if the real character of the stresses is not known, they may be assumed as either tension or compression. It makes no difference how the stress is assumed; if the equations of equilibrium are properly written and solved, a negative result will indicate a stress of *the opposite kind from that assumed.* It is better, however, to adopt a uniform method, namely, to assume tension on every bar in the beginning, that is, to make the arrow point outward from the section. This being done, a negative stress in the result always means compression, and a positive value of the stress always means tension.

3. Tabulation of Methods.—The following tabular statement of the methods of solving trusses embodies the principles that have been explained.

General Methods of Computing Trusses in a Plane.—These methods are all *methods of sections.*

I. The Analytical Method of Joints.—The section is taken about one joint, and the equilibrium of the forces acting at that joint is considered. There are but two statical conditions, $\Sigma H = 0$ and $\Sigma V = 0$. and these enable us to find two unknown stresses, and only two.

II. The Graphical Method of Joints.—This is in principle the same as method I and consists in drawing the polygon of forces for the joint. All the forces but two being known, and the directions of these two also, the polygon may be drawn. This method is often called *Maxwell's method.* It will be fully explained later.

III. The Analytical Method of Sections (not about a single joint). The section cuts three bars not meeting at a point and the stresses in these bars are found from the equations $\Sigma H = 0$, $\Sigma V = 0$, $\Sigma M = 0$; which may be used in different ways:

1. Find the three unknown stresses by using all three equations.
2. Use $\Sigma M = 0$ twice, about two different origins, and then $\Sigma H = 0$.
3. Use $\Sigma M = 0$ twice, about two different origins, and then $\Sigma V = 0$.
4. Use $\Sigma M = 0$ three times, taking the origins at the intersection of each pair of bars to find the stress in the third bar. This is called the *method of moments.*

IV. The Graphical Method of Moments.—This consists in finding graphically the moment of the outer forces about the intersection of two of the bars, and dividing this (graphically) by the distance from this point to the third bar.

4. Illustration. Index Stresses.—In applying these methods, the one best suited to the case should be used, varying the method for differ-

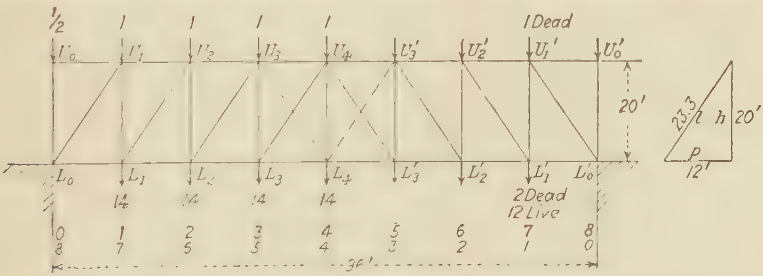


FIG. 40.

ent bars, instead of using the same method for the entire truss. Method II, however, when used, is employed for the entire structure.

In Fig. 40, let

Live load = 2,000 pounds per foot per truss = 12 tons per panel.

Dead load = 500 pounds per foot per truss = 3 tons per panel.

Let the dead load on an upper chord joint be 1 ton, and on a lower chord joint 2 tons.

1. The maximum chord stresses, which will occur for full loading, may be found by the method of moments; thus, omitting the load on U_o, which

is carried direct to the abutment and does not affect the truss, the end reaction is $\frac{1}{2} \times 7 \times 15 = 52.5$.

$$U_0U_1: 0$$

$$L_0L_1 \text{ or } U_1U_2: \frac{52.5 \times 12}{20} = 31.5T$$

$$L_1L_2 \text{ or } U_2U_3: \frac{52.5 \times 24 - 15 \times 12}{20} = 54$$

$$L_2L_3 \text{ or } U_3U_4: \frac{52.5 \times 36 - 15(12 + 24)}{20} = 67.5$$

$$L_3L_4: \frac{52.5 \times 48 - 15\,(12 + 24 + 36)}{20} = 72$$

2. Another method, based on the fact that the moment curve for a uniform load is a parabola is this:

$$L_3L_4 = \frac{2{,}500 \times 96 \times 96}{8 \times 20} = 72$$

Then

$$L_2L_3 \text{ or } U_3U_4 = \frac{15}{16} \times 72 = 67.5$$

$$L_1L_2 \text{ or } U_2U_3 = \frac{3}{4} \times 72 = 54$$

$$L_0L_1 \text{ or } U_1U_2 = \frac{7}{16} \times 72 = 31.5$$

3. A third method consists in first finding the web stresses for full loading (which are not the maximum web stresses), and writing on each web piece, whether vertical or inclined, a member representing the *vertical component* of its stress, found by taking vertical sections through the diagonals, and inclined sections through the verticals and using the equation $\Sigma V = 0$, applying it to the outer forces on the left of the section, *i.e.*, subtracting the loads from the reaction. Thus, in Fig. 41, the reaction being 52.5, the number written on the end diagonal is 52.5; on the vertical U_1L_1, $52.5 - 1 = 51.5$; and so on to L_3U_4. In the panel L_4L_3' the shear will be negative, and if the bar U_4L_3' is in action U_4L_4 will carry only the load at L_4 or 14, as shown.

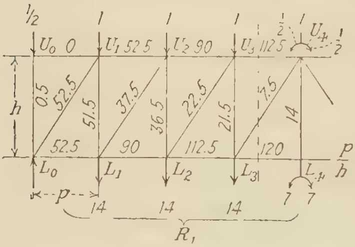


FIG. 41.

Instead of first finding the reaction, beginning at the end and subtracting loads, it is easier to begin at the middle. No matter what the loading on a truss, if we know the load at which the shear changes from + to −, we may begin at that load, which may be considered split into two parts, one part going to the left abutment and the other part to the right abutment. The part going to the left will be the shear just to the left of the load; and the left reaction will be that part plus all the other loads

between it and the left abutment. In the case of a symmetrical truss under full uniform loading, or under symmetrical loading, the load which splits will always be the load at the center, because both reactions are equal. We may in this case, therefore, always begin with the center load, consider one-half of it as going to the left, and add loads one by one to get the numbers to write on the web members, in this case beginning at L_4, writing 14 on U_4L_4 because $U_4L'_3$ is in action, then at U_4 adding the load at U_4, making 15 at the center, and writing $^{15}/_2 = 7.5$ on U_4L_3, and so on by addition. The reader should realize that this is in principle merely taking sections and applying $\Sigma V = 0$.

The reader should also perceive that a truss or beam *carries* a load in two senses, the first being that it *bears* or *supports* it, and the second that it *transfers* it, as a wagon transfers it, from one point to another, namely, from its point of application to the two abutments. In this last sense, *every* load on the structure is divided into two parts, one going to each abutment. Part of a load on the right of the center goes to the left abutment, considered alone, causing + shear to the left of itself, and − shear to the right of itself. But the actual shear at any point is the resultant of the shears due to the individual loads, and hence it is perfectly correct to consider all of the loads to the left of the load where the shear changes sign, together with part of the latter, as going to the left abutment. The load at L_4, then, goes up the vertical, is joined at U_4 by the load there, one-half the sum goes down U_4L_3, and so passes to the left abutment, picking up each load that it comes to, as a train picks up passengers. This is correct for any truss which is symmetrical and symmetrically loaded, provided the chords are parallel and at right angles to the loads. If the chords are not parallel, one or both carry part of the shear, which cannot therefore be followed through the web by adding loads. If the truss or loads are not symmetrical, it would be necessary first to find the loads where the shear changes sign, which can be done only by first finding the reaction, and then subtracting loads, so that in this case it is just as well to begin at the reaction and subtract loads.

All this is merely writing on each web piece the shear in it, the real stress being this shear multiplied by l/h, l being the length of the piece and h its vertical projection.

Starting at the center, the check on the addition is that the number written on the end web piece must equal the reaction; thus, in Fig. 41, $52.5 = 3.5 \times 15 = R_l$. This is the value of the reaction omitting the load on U_0 and L_0. The real pressure of the truss on its supports must include these two loads. The load at U_0 will be dead only in this case. The load at L_0 will depend upon the design of the floor. If the stringers which carry the load rest directly upon the abutment, the load at L_0 will be only the dead weight of the truss carried to that joint, or one-half the weight of all bars meeting at that point; if the stringers rest on an end

floor beam joining the points L_0 in the two trusses, the load will be the dead load from the truss plus half a panel of live load and dead floor load.

The chord stresses are next found by the method of joints, using $\Sigma H = 0$; thus, Figs. 42 and 43,

$$L_0L_1 = 52.5\frac{p}{h};\ L_1L_2 = (52.5 + 37.5)\frac{p}{h} = 90\frac{p}{h}.$$

The numbers written on the chords are such that they must be multiplied by p/h to give the real stress, and they are written down at once by addition. Starting the web with the load which splits, this method therefore involves merely addition. It is well, however, to check one chord, say, the one nearest the center, by moments; thus, as found above, $L_3L_4 = 72$,

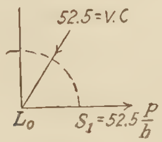


Fig. 42.

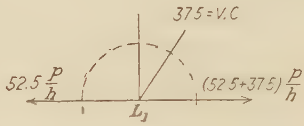


Fig. 43.

while in Fig. 41 it is $120 \times {}^{12}\!/_{20} = 72$, which checks; or since the reaction is $3.5P$ (if P = panel load)

$$L_3L_4 = \frac{3.5P \times 4p - P(1 + 2 + 3)p}{h} = \frac{8Pp}{h} = 120\frac{p}{h} = 72.$$

The distance p is the horizontal projection of a diagonal. It may (Fig. 40) or it may not (Fig. 35f) be the panel length; or, if α is the angle that a diagonal makes with the vertical, $l/h = \sec\alpha$, and $p/h = \tan\alpha$.

This method, which is the simplest for a uniform load, may be called the method by *index stresses*, since numbers are written on the bars which are not actual stresses, but must be multiplied by constants (l/h for the diagonals and p/h for the chords) to give actual stresses. For the verticals the numbers are actual stresses. If the diagonals have different slopes the multiplier is numerically different for the different inclinations, and the index numbers for the chords cannot be obtained by merely adding those for the diagonals, but must take account of the slope of the diagonals, as will be seen later. The method does not apply where either chord is curved.

The stresses indicated in Fig. 41, when multiplied by the proper constant, are those for full loading; that is, they are maximum stresses for the chords, but not for the diagonals, since the latter occur for partial loading.

Note the method of lettering the truss, U_0 being the first upper chord joint, U_1 the second, and similarly for lower chord joints.

5. Maximum Web Stresses.—These occur for partial loading, since the vertical component in any web piece equals the shear, and will be a maximum when the shear is a maximum. In Chap. XI of "Strength of Materials" it has been shown that the maximum shear in any panel, for

a uniform load, will occur when the load extends from one end of the span up to a "neutral point" in the panel. It was there shown how to find the true maximum positive and negative shears for a uniformly distributed load extending from one end of the span to the neutral point. In a truss with parallel chords, the neutral point in any panel, being the point at which a concentrated load must lie in order to cause no shear in the panel, may be easily constructed as follows, if desired. In Fig. 35a, to find the neutral point in the second panel, for example, produce either chord to meet the verticals through the points of support, and from b and b' draw straight lines through the other chord panel points of this panel, to meet at D; D will be vertically over the neutral point in the panel, for

$$\frac{a}{l} = \frac{c}{p}$$

which is the condition for the neutral point. A similar construction in Fig. 35c gives the neutral point in any lower chord panel, as shown. In the following, however, it will be assumed that all panel points to the right of the panel are fully loaded with live load for maximum positive shear, and all joints on the left for maximum negative shear, all other joints having no live load. This assumption, as shown in the chapter referred to, gives slightly greater stresses than result if the neutral point is used. Then in Fig. 40, the maximum vertical components (max $+ S$) are

$$L_0U_1 = 52.5$$

$$U_1L_1 = 51.5$$

$$L_1U_2 = \frac{1+2+3+4+5+6}{8} \cdot 15 - \frac{1}{8} \cdot 3 = 39$$

This may also be found from the stress already found for full loading (37.5) by *removing* the live load from L_1 and therefore *adding* the negative shear produced by that load alone, or

$$L_1U_2 = 37.5 + \frac{1}{8} \cdot 12 = 39$$

Similarly

$$U_2L_2 = 39 - 1 = 38 \text{ (by method of joints at } U_2\text{)}$$

$$L_2U_3 = \frac{1+2+3+4+5}{8} \cdot 15 - \frac{1+2}{8} \cdot 3 = 27$$

or

$$L_2U_3 = 22.5 + \frac{1+2}{8} \cdot 12 = 27$$

$$U_3L_3 = 27 - 1 = 26$$

$$L_3U_4 = \frac{1+2+3+4}{8} \cdot 15 - \frac{1+2+3}{8} \cdot 3 = 16.5$$

or

$$L_3U_4 = 7.5 + \frac{1+2+3}{8} \cdot 12 = 16.5$$

To aid in making the additions, it is desirable to write, below the truss, as in Fig. 40, the lines of numbers shown.

Besides finding maximum positive shear to the left of the center, the maximum negative shear must also be found. This may be done just as well by finding maximum *positive* shears in panels to the *right* of the center. The method of procedure is therefore to begin at the left and proceed to the right, finding maximum positive shear in each panel, as far as there can be positive shear. Thus, in Fig. 40, the maximum positive shears to the right of the center (= maximum negative shears to left of the center) are

$$\text{panel } L_4L_3' = \frac{1+2+3}{8}\cdot 15 - \frac{1+2+3+4}{8}\cdot 3 = 7.5$$

or

$$= -7.5 + \frac{1+2+3+4}{8}\cdot 12 = 7.5$$

$$\text{panel } L_3'L_2' = \frac{1+2}{8}\cdot 15 - \frac{1+2+3+4+5}{8}\cdot 3 = 0$$

or

$$= -22.5 + \frac{1+2+3+4+5}{8}\cdot 12 = 0$$

Hence there can be positive and negative shear only in the two central panels. In panel L_3L_4 the maximum $+S$ is 16.5, the maximum $-S$ is 7.5 (numerically); in panel L_4L_3' the maximum $-S$ is 16.5, the maximum $+S$ is 7.5.

An important principle is the following:

In any panel of a parallel-chord bridge, the shear is positive, for any given loading, if for that loading the (positive) moment at the right end of the panel is greater than the moment at the left, and vice versa. This follows from the fact that in any beam $S = dM/dx$. This principle often makes it easy to see which diagonal in a panel will be in action under a given loading.

It must be remembered that all stresses in a truss are considered to be direct tension or compression. There is no flexure, and therefore no shear. In a truss with parallel chords the web carries all the transverse shear because, the chords having only direct stress, no shear is carried by them. If the joints of the chord are rigid, it can carry shear and moment (secondary tresses), and the web will not carry all the shear. In many cases (as in portal bracing) it is necessary to take account of the rigidity of some members, and of the shear in them.

6. When a Truss Is Statically Determined with Regard to the Inner Forces.—We have seen (Chap. II) that a framed structure may be statically determined or not with reference to the outer forces. For the computation of the inner forces or stresses, the outer forces must be assumed known. The question then arises, when is it possible to find the stresses in the bars by statics alone (statically determined as regards the stresses).

This is clearly a question of algebra, as regards the number of equations and of unknown quantities.

If n = the number of bars, this is the number of unknown quantities. How many statical equations are there? There are two at each joint, expressing the equilibrium of that joint. Hence there are $2m$ equations, if m = number of joints. It might be concluded that $2m = n$ in order that the stresses should be statically determined. But this is not so. There are two other questions to be asked and answered:

1. Are these equations all independent; *i.e.*, do they tell us $2m$ independent facts, none of which follow from any or all of the others?

2. Do these $2m$ facts all relate to the unknown quantities, or do they include facts about other things? They are clearly all independent, since no one of them follows or can be deduced solely from the others. But with reference to the second question, these $2m$ equations tell us that every joint is in equilibrium; therefore they tell us that the frame as a whole is in equilibrium; therefore they tell us that the outer forces, independent of the stresses, are in equilibrium; therefore they tell us three independent facts that have nothing whatever to do with the stresses, namely, the three conditions of equilibrium of outer forces in a plane (we are here considering only a frame which, with the outer forces, lies in a plane). Hence there are really but $2m - 3$ independent equations between the unknown quantities, and the structure is statically determined with regard to the inner forces if

$$n = 2m - 3 \tag{1}$$

This equation is satisfied for $m = 3$, that is, for a triangle. If to this triangle a new joint is added, two new bars must be added, and these may connect the new joint with any old joint. Any truss composed of triangles, built up by making one side of an added triangle coincide with an existing side, will be statically determined. Any truss in which any bar can be solved by taking a section through it and only two other bars is statically determined; but a truss may be statically determined when this is not possible.

Equation (1) is a *necessary* condition; but it is not *sufficient*. A truss may fulfil it, and yet be statically undetermined. Thus, the truss in Fig. 40 is statically determined; but if a diagonal be removed, and another added in a panel which already has one, the numbers of bars and joints are unchanged, but the truss is statically undetermined. *It is clearly a question, not only of the number of bars, but of their arrangement also.* Also, in counting bars, if there are two bars such that only one can be in action at the same time, the two count as one; and similarly for a greater number.

7. Before proceeding farther, the reader should accustom himself to understand clearly the *kind* of stress produced in a diagonal of a

parallel chord truss by positive and by negative shear. Consider a panel of such a truss (Fig. 44). If the shear in that panel, that is, the resultant of all the vertical forces to the left of that panel (the chords being horizontal), is positive, it means that this resultant acts upward. The part of the truss to the right of the section being removed, the stress in the diagonal, which is the only bar cut that has a vertical component, must be downward, and hence, if bar a is that diagonal, its stress is tension. If, instead of bar a, the diagonal were b, its stress would be

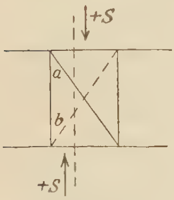


FIG. 44.

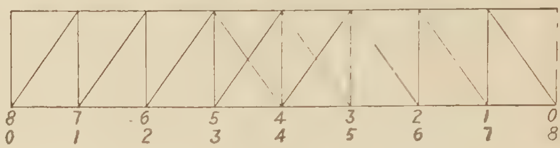


FIG. 45.

compression. Thus, *positive shear in a panel causes tension in bars* ╲ *and compression in bars* ╱, *and negative shear causes compression in bars* ╲ *and tension in bars* ╱.

The chords being horizontal, the diagonals in a panel carry the entire shear in the panel. If there is but one diagonal, it carries all the shear; *i.e.*, its vertical component equals the shear, and its character depends upon whether the shear is positive or negative, and the direction of the diagonal, as above shown. This is not true if either of the chords is inclined to the horizontal, though many students who begin by the study of trusses with parallel chords persist in thinking that the diagonal carries the shear in any truss. Clearly, the shear is carried by (balanced by) all the bars which have vertical components.

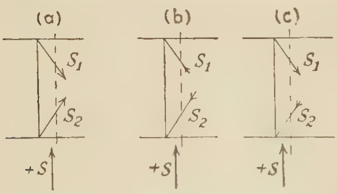


FIG. 46.

The reader should also accustom himself to *see* the character of stress in a diagonal for either positive or negative shear. Positive shear means that the resultant outer force on the left acts upward, and that on the right downward, and it is easy to see that this action must cause tension in a and compression in b.

If there are two diagonals in a panel, as in the fourth panel of Fig. 45, the character of the shear gives no indication, necessarily, of the character of stress in a diagonal. If the shear is positive, one diagonal may be in tension and the other in compression; both may be in tension, or both in compression. All that is necessary is that the *algebraic sum of the two vertical components must be equal and opposite to the shear.* Thus (Fig. 46),

In (*a*), V. C. tension in S_1 − V. C. tension in $S_2 = S$

In (*b*), V. C. compression in S_2 − V. C. compression in $S_1 = S$

In (*c*), V. C. tension in S_1 + V. C. compression in $S_2 = S$

all of which expressions are really identical. All that can be said is that a positive shear *tends* to make S_1 tension and S_2 compression, but it all depends upon whether, in the entirely unloaded structure, there are any *initial* stresses S_1 and S_2. If there are such initial stresses, they must be equal and both of the same kind, for their resultant vertical component, there being no external loads, must be zero. The effect of a positive shear, then, will be to *increase* the tension S_1 and to *decrease* the tension S_2, by perhaps approximately equal amounts; in other words, the shear may be carried approximately equally by the two bars, causing tension S_1 and compression S_2. The actual division of the shear between the two diagonals is uncertain and depends upon their areas. The structure in such a case is generally statically undetermined.

8. Counters.—If the bars of the truss are only those shown by full lines in Fig. 40, and if the shear in a panel may be positive or negative, the diagonal in that panel must of course be so designed that it is able to carry both compression and tension. If, by its construction it can carry only one kind of stress, a bar sloping in the other direction must be put in, able to carry the *same* kind of stress but the *opposite* kind of shear. Such a bar is called a *counter;* the first bar, carrying the kind of shear which is larger, being the *main diagonal.* Thus, it was found that in the truss (Fig. 40) there could be both positive and negative shear in the two center panels; hence counters are needed in those panels, as shown by the dotted lines, if the main diagonals are made of flat or round bars, which can take tension only. The counters are also able to take only tension. Of a main diagonal and its counter, only one is in action under a given load, depending on the character of the shear, if there are no initial stresses.

Counters are unnecessary if the diagonals in panel in which there may be both kinds of shear are so constructed that they can carry either tension or compression, but it is very common to make diagonals of flat bars, which would bend under a compressive force. Similarly, a piece which can stand compression, like a wooden post, may simply abut at its ends against a block or against the other parts of the frame, without being held or fastened in a manner that would enable it to resist tension; in which case a counter will be required if the shear would tend to expose the main diagonal to tension. Thus counters carry the same kind of stress as their main diagonals.

The reader should also remember, from the chapter on shears and moments, that, in a bridge, the shear on the left of the center is predominantly positive; that is, that at any point to the left of the center, the maximum positive shear is numerically greater than the maximum nega-

tive shear; that for a certain distance from the end there can be only positive shear; and that for the remaining distance to the center there may be either kind, depending upon the loading, and similarly on the right of the center. Hence counters, if needed, are required in only a few panels near the center.

9. Camber.—Trusses are generally built so that under no load at all they would curve upward. The amount of the upward deflection d at the center under no load is called the *camber* (see Fig. 47). The objects of this are so that under the dead and live loads it may not appear to sag, and the roadway or track may be nearly level, and so that under those loads its form may be as it is assumed in the calculations. The camber is made such that the loads will not bring the bridge much, if any, below the horizontal. This camber is given by making each panel of the top chord slightly longer than a panel of the bottom chord. If the truss is to

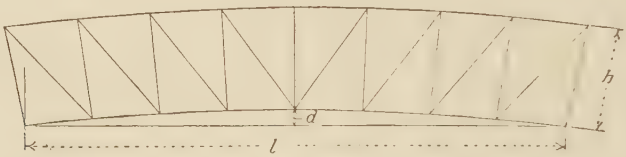


FIG. 47.

be level under the maximum load, the amount of central deflection produced by that load must be computed, and the camber made equal to this. The excess of length of the top panels e is computed thus: Assuming that each chord is an arc of a circle with a radius of bottom chord r, the verticals being in the direction of radii of this circle, and the height of truss h, then

$$2r - d : \frac{l}{2} :: \frac{l}{2} : d$$

$$\text{camber} = d = (\text{very nearly}) \ \frac{l^2}{8r} : r = \frac{l^2}{8d}$$

since d is very small compared to r.

$$h : e :: r : p$$

also

$$e = \frac{hp}{r} = \frac{8phd}{l^2} \qquad (2)$$

The length of the verticals is not changed by the camber; the length of a diagonal may be taken as the side of a right triangle of which one side is h, and the other

$$\frac{1}{2}(2p + e) = p + \frac{e}{2}$$

If the bars of a truss are all hinged at their ends, and if each bar has exactly the proper length to conform to the truss with the desired camber,

then, when the pins are put in at the joints, the truss will take the desired camber, and there will be no stress in any bar under no load, and only direct stress under loading.

But the bars of a truss are not generally all connected by pins. The top chord is generally made of one rigid piece from end to end, though the sectional area varies in the different panels. The web members may be connected to the chord at the joints by pins, but the chord itself is continuous. In this case it must be bent in order to produce the camber, and there will be a bending moment in the chord, producing tension at the top and compression at the bottom, under no load at all; and consequently stresses in the other members under no load. These are called *initial stresses*.

10. Counters May Make Truss Statically Undetermined.—The condition of things in the panels in which there are counters is statically undetermined unless the adjustment is such that if the truss had no load whatever, not even the dead load, that is, if it were lying flat upon the ground, there is no stress in either diagonal. If the truss were so lying on the ground, and if the diagonals, which are tension members only, were provided with turnbuckles, by which the length could be changed, it is clear that these turnbuckles might be tightened to any desired extent, even to rupture. If such a stress is put into either diagonal, since the shear in the panel is zero, an equal stress must be caused in the other diagonal. (This statement is not exactly true if one or both chords are continuous, *i.e.*, do not have hinges at the joints, because in that case the camber results in some shear in the continuous chords and a corresponding stress in the diagonals. It is, however, approximately true.) These stresses in the diagonals are accompanied by compression in both chords.

If the truss is so adjusted that when there is no load there is no stress in the diagonals, which could be true if all joints were hinged, then any shear in a frame is carried entirely by one diagonal, and the stresses are statically determined. But if there are initial stresses in the diagonals, then either diagonal, even if a round rod, is able to carry a compression due to the loads, to the amount of the initial tension. Assuming that a shear is carried equally by each diagonal, then if the positive shear were $+S$, and if there were in each diagonal an initial shear (vertical component of stress) of S_1, a part of S would be carried by each diagonal until the stress in the counter were reduced to zero, and the balance would be carried by the main diagonal, the resultant shear in which would be S. If the initial shear in each diagonal is equal to or less than $\frac{1}{2}S$, the resulting shear S is carried entirely by one diagonal; if the initial stress is greater than $\frac{1}{2}S$, then the application of the shear S still leaves both diagonals in action, and the stress in one of them is larger than S.

Counters are often, or generally, made adjustable, with turnbuckles. It is not necessary to have main diagonals also adjustable, since a stress

in one necessarily puts an equal stress into the other. Clearly, the maximum stress that it may be proper to put in for cambering is such that when the maximum $+S$ is acting, the counter has just zero stress, or only a small stress; and when the maximum $-S$ is acting, the same is true of the main diagonal.

The same effect that is produced by a turnbuckle is produced by inaccuracies of workmanship. Unless each diagonal has precisely the proper length to fit the unloaded truss, there will be initial stresses, and these may be considerable. If the truss is cambered, and has continuous chords, so that initial stresses are necessary to give it the camber, then there must be initial stresses.

These are all objections to the use of counters, and in the minds of many engineers justify the opinion, which the writer holds, that it is better not to use counters, in general, but to make the main diagonals able to carry any kind of shear that they may be exposed to, even though this involves a reversal of stress. The main argument for counters is that their use obviates such reversal (see Chap. XIX).

If the counters are so adjusted that there is no initial stress, which can be done only when there is no camber, or when, if cambered, all bars are hinged at both ends, then when one is in action, the other is not; the two count as but one bar, and there is no statical indetermination because of them (see Art. 6). If there is initial stress, it is necessary to assume that any shear is carried equally by both, until one becomes slack, or to use more elaborate methods of computing.

11. Stress Sheet.—A stress sheet is a diagram of the frame, on each member of which is marked the stress under a given load, or the maximum stress, or the index stress. Frequently the dead stress, the maximum live stress, and the maximum total stress are all indicated, or the index stresses for full loading.

12. Loadings to Be Considered.—It has been explained in Chap. III that, for railroad bridges, the usual specifications call for the use of a series of concentrated loads, followed by a uniform load. For many other structures actual concentrated loads should be used. For some structures a uniformly distributed live load may properly be assumed, covering the portion of the structure necessary to produce the maximum stress in the member considered. The dead load is generally considered uniformly distributed. In some cases, also, a single concentrated load, or two such loads, must be considered; as, for instance, a highway bridge in which a single heavy car or auto truck is assumed, together with a uniform load on the portion not occupied by the car. It has also been shown that a railroad train may be represented approximately by a uniform load and one or two concentrated loads representing the excess load of the drivers; and also that the effect of impact may be allowed for by increasing the load of the locomotive by an empirical excess.

In the following pages, therefore, the various forms of trusses will be considered, and it will be shown how to determine the stress in any member, and its maximum stress, for the following loadings: (*a*) a series of concentrated loads, and (*b*) a uniformly distributed load, with one or two concentrated loads.

The loads on a frame, so far as its action as a frame is concerned, are all concentrated at the joints. If there is a load on any member between its ends, it is carried to the ends by the member acting as a beam, and the flexural stress intensity is to be added to the intensity due to the direct stress in the member as a bar of the frame, as already stated. The effect of camber will not be considered.

The dead load on any joint is to be taken as one-half the weight of all the members including the floor members and the dead weight upon them, which meet at that joint. Hence the actual dead loads at the joints will, in general, all differ from each other, except that at symmetrical joints the loads will be the same. This degree of refinement, however, is not used. The dead load on a bridge is generally assumed to be uniformly distributed over its entire length, though it is often divided between the upper and lower chords; the weight of the floor is carried on the chord to which it is applied, and the weight of the truss equally divided between the top and bottom chords, though this is clearly not precisely correct. The loaded chord, however, *i.e.*, the chord which carries the floor, always carries a greater proportion of the dead load than the unloaded chord. In some cases, especially in large structures, it may be desirable, before the design is completed, to compute and use the actual dead load at each joint.

EXAMPLES OF TRUSSES WITH PARALLEL CHORDS

13. The different forms of trusses will now be described, and the manner of finding the stresses illustrated. For simplicity, the same span and panel length will be used in most cases, namely, 10 panels at 20 feet, or a span of 200 feet, and it will be assumed to be a single-track railroad bridge. The height will be assumed as 25 feet.

The dead weight of steel will be found by the formula

$$W = \frac{l^2 + 45l}{0.089} = \frac{49,000}{0.089} = 550,000 \text{ pounds}$$

Deduct about 3 per cent for shoes,	17,000
	533,000
Add for floor above stringers, 500 pounds per foot	100,000
	633 000

or 316,500 pounds per truss.

This will be assumed distributed as follows:

On loaded chord: 215,000 = 21,500 pounds per panel, say, 11 tons

On unloaded chord: 101,500 = 10,150 pounds per panel, say, 5 tons

For actual design the accurate figures should be used, to the nearest 100 pounds, but for our purposes this would only add an unnecessary complication.

For the live load, the E-60 loading will be assumed, with one or two locomotive excesses. This gives for the uniform load 30 tons per panel of each truss.

The question of the locomotive excess and its determination, if it is used, has been fully discussed in Chap. III. It will here be assumed as 20 tons per truss.

Generally, loads and stresses are computed in "kips," a kip being 1,000 pounds. Hence the loads to be assumed are for one truss:

$$\text{On unloaded chord} \quad 10 \text{ kips dead}$$
$$\text{On loaded chord} \begin{cases} 22 \text{ kips dead} \\ 60 \text{ kips live uniform} \\ 40 \text{ kips live excess} \end{cases}$$

The load at the abutment joint is that of one-half a panel.

The distance between corresponding drivers of the two locomotives is 56 feet; hence the two excesses will be assumed to be 60 feet, or 3 panels, apart.

Remember that we are computing these trusses on the incorrect assumption that one joint may be fully loaded and the adjoining joint entirely without live load.

All bridges will for the present be assumed as "square," that is, with abutments at right angles to the trusses.

1. THE PRATT TRUSS

14. Characteristics.—The Pratt truss is shown in Fig. 48. It has rectangular panels, and the diagonals slope upward toward the ends, or are predominantly in tension, and the verticals in compression. Generally the diagonals can take tension only, and counters are needed in some panels, though this is not always true. The material may be of steel, or of a combination of wood and steel.

15. Computation.—Let the bridge be a *through* bridge, or with the floor on the bottom chord. The diagram for deadloads and the index stresses are shown in Fig. 48a, with only one typical joint load indicated.

The check is: $R = 4.5 \times 32 + 5 = 149$

$$\frac{1}{8}\frac{wl^2}{h} = \frac{1}{8} \times \frac{32}{20} \times \frac{200 \times 200}{h} = \frac{8,000}{h} = 400\frac{p}{h}$$

The live index stresses for full live load are shown in Fig. 48b, and may be checked in the same manner.

The writer generally writes p/h and l/h to one side of his diagram, as shown, so that he may not neglect to multiply the index figures by these ratios.

For the excess loads, the maximum stress in U_3L_4, for instance, will be when one excess is at L_4 and the other at L_3. The index figures for all the chord excess stresses will be:

$$U_0U_1: 40 \times \frac{9+6}{10} \times \frac{p}{h} = 60\frac{p}{h}: \text{index} = 60$$

$$U_1U_2: 40 \times \frac{8+5}{10} \times \frac{2p}{h} = 104\frac{p}{h}: \text{index} = 104$$

$$U_2U_3: 40 \times \frac{7+4}{10} \times \frac{3p}{h} = 132\frac{p}{h}: \text{index} = 132$$

$$U_3U_4: 40 \times \frac{9}{10} \times \frac{4p}{h} = 144\frac{p}{h}: \text{index} = 144$$

$$U_4U_5: 40 \times \frac{7}{10} \times \frac{5p}{h} = 140\frac{p}{h}: \text{index} = 140$$

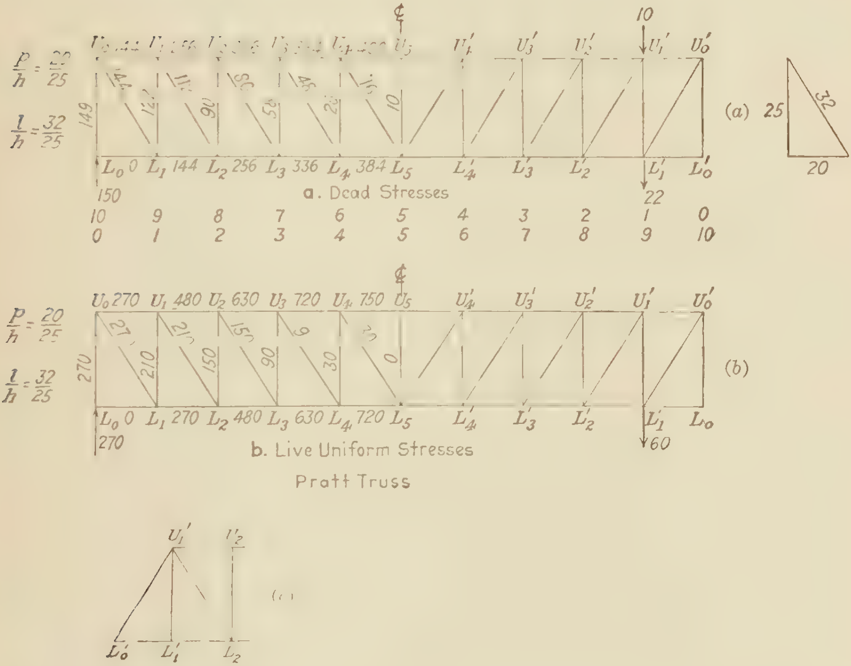


FIG. 48.

These assume the first excess at the first loaded joint and the other three panels behind. Of course, the drivers are not at the very head of the train, so that these figures are not accurate. If accuracy is desired, the actual wheel loads must be used.

For the web members, the index numbers given in Fig. 48*b* are not for the uniform loading giving maximum stresses, except for U_0L_0 and for U_0L_1. For the maximum in U_3L_4, for instance, the live load must be

removed from L_1, L_2, and L_3, and the excess load placed at L_4 and L_3. The index figures for all the maximum live web stresses will be:

$$U_0L_0 \text{ and } U_0L_1\colon 270 + \frac{15}{10} \times 40 = 330$$

$$U_1L_1 \text{ and } U_1L_2\colon 210 + \frac{1}{10} \times 60 + \frac{13}{10} \times 40 = 268$$

or

$$\frac{36}{10} \times 60 + \frac{13}{10} \times 40 = 268$$

$$U_2L_2 \text{ and } U_2L_3\colon 150 + \frac{1+2}{10} \times 60 + \frac{11}{10} \times 40 = 212$$

or

$$\frac{28}{10} \times 60 + \frac{11}{10} \times 40 = 212$$

$$U_3L_3 \text{ or } U_3L_4 : 90 + \frac{1+2+3}{10} \times 60 + \frac{9}{10} \times 40 = 162$$

or

$$\frac{21}{10} \times 60 + \frac{9}{10} \times 40 = 162$$

$$U_4L_4 \text{ or } U_4L_5 : 30 + \frac{10}{10} \times 60 + \frac{7}{10} \times 40 = 118$$

or

$$\frac{15}{10} \times 60 + \frac{7}{10} \times 40 = 118$$

$$U_5L_5 \text{ or } U_5L_4' : -30 + \frac{15}{10} \times 60 + \frac{5}{10} \times 40 = 80$$

or

$$\frac{10}{10} \times 60 + \frac{5}{10} \times 40 = 80$$

Clearly a counter is needed in this panel if L_5U_4' cannot take compression, since the maximum $+S = 80 - 16$ (dead) $= 64$.

$$U_4'L_3'\colon -90 + \frac{21}{10} \times 60 + \frac{3}{10} \times 40 = 48$$

or

$$\frac{6}{10} \times 60 + \frac{3}{10} \times 40 = 48$$

Since the dead shear in this panel is -48, while the maximum live shear is $+48$, the sum is zero, and no counter would be needed if the live shear stated could not be exceeded. But impact must be added to this live shear, so that a counter is needed.

$$U_3'L_2'\colon -150 + \frac{28}{10} \times 60 + \frac{2}{10} \times 40 = 26$$

or

$$\frac{3}{10} \times 60 + \frac{2}{10} \times 40 = 26$$

The dead negative shear in this panel is 80. The impact, by Eq. (2) of Chap. III is $\frac{300}{300 + 40} \times 26 = 22.9$ Hence the total live shear is 48.9, and no counter is needed. Counters are therefore needed in the four central panels for the loads assumed.

It is good practice and advisable to use counters in some panels where the shear is not reversed, on the principle that the factor of safety should be the same as in other parts of the truss; also to allow for an increase in the live loads, and for the fact that the real impact is quite uncertain. Any possible reversal of shear should be provided for. Suppose the ultimate strength of the material to be 60,000 pounds per square inch, and suppose that in a diagonal the dead stress is 6,000, and the live and impact 10,000 pounds per square inch, making a total of 16,000, as often specified. Now the dead stress is constant; hence the live stress and impact must be multiplied by $^{54}/_{10}$ in order that the total may reach the ultimate. In order that there may be the same factor regarding reversal of stress, there should be a counter in a panel to the right of the center unless the negative dead shear equals or exceeds $^{54}/_{10}$ times the maximum positive live shear and impact. On this theory there should be a counter in panel $L_3'L_2$. If the elatsic limit is 32,000 pounds, and this is taken as a criterion, the live stress must be multiplied by $^{26}/_{10}$ in order to reach the elastic limit; this criterion is perhaps more reasonable than the first, and according to it also there should be a counter in panel $L_3'L_2'$.

For $U_2'L_1'$, maximum $+S = \frac{1}{10} \times 100 = 10$.

The impact is $^{300}/_{320} \times 10 = 9.4$, making a total of 19.4.

The dead negative shear is 112; hence according to neither of the above theories is a counter needed.

On the whole, in this truss, it is desirable to have counters in the six central panels. In the panels where the loads assumed do not give a reversal of stress, the counter may be proportioned for a unit stress equal to the elastic limit (or the ultimate) after increasing the live load plus impact in the proportion determined by the main diagonal in the same panel (see also Chap. XIX).

Thus in panel L_3L_2 we have for the main diagonal

Maximum live $+S$ (see U_2L_3)	= 212
Impact = $212 \times {}^{300}/_{400}$	= 145
Total live + impact	= 357
Dead positive shear	= 80
Total	= 437

This total stresses the main diagonal to 16,000 pounds; hence the dead stress is 16,000 $\times\, ^{80}/_{437} = 2{,}929$ pounds and the live is 13,071 pounds. Using the criterion of the elastic limit, assumed as 32,000, the factor of safety of the live load is $\frac{32{,}000 - 2{,}929}{13{,}071} =$ 2.22; as found above for the counter the live + impact is 48.9, and $48.9 \times 2.22 =$ 108.6. Subtracting the dead negative shear of 80, we have 28,600 pounds, which multiplied by $l/h = 1.28$ gives a stress in the counter of 36,600 pounds, which is to be used with 32,000 pounds unit stress, requiring 1.15 square inches, or a rod $1\frac{1}{8}$ inches square.

This method of computation suggested for the counters is not common, but it has force. If used, it should be applied to all the counters if it requires larger sections than by using the figured stresses and the specified unit stress. The writer believes that in fact most counters have not the same factor of safety as the rest of the tension bars; or, in other words, that there is not the same factor of safety against reversal of stress as there is against non-reversal.

In computing impact by the above method it is clearly proper to take the "loaded length" up to the last joint fully loaded; if the accurate method for a uniform load were used, it would be up to the neutral point; if actual loads are used, it would be up to the first load.

The above computations suffice to find the maximum stress in any bar. For instance, take U_3U_4:

Dead		$= 384$
Live:	$720 + 144$	$= 864$
Impact:	$864\,{}^{300}\!/_{500}$	$= 518.4$
Total..........		$1766.4 \times 0.8 = 1{,}413.12 = 1{,}413{,}120$ pounds

For a web piece, say, U_2L_3, we have

Dead		$= 80$
Live		$= 212$
Impact:	$212\,{}^{300}\!/_{440}$	$= 144.5$
Total..........		$436.5 \times 1.28 \times 1{,}000 = 558{,}720$ pounds.

16. The Pratt Truss under Actual Loads. *Diagonals.*—The maximum stress in any diagonal is the maximum shear times l/h. All that is necessary is, therefore, to find the maximum and minimum shear in any panel, as explained in Chap. XI of "Strength of Materials."

Chords.—For any chord member, taking a vertical section, the stress will be the moment at one of the opposite joints divided by h. The only question is, which joint to take. If there is no counter in the panel, this question does not arise. Thus, for U_1U_2, the maximum stress will be the maximum moment about L_2 divided by h, since there is no counter in the panel U_1U_2. For U_3U_4, however, the origin of moments will be L_3 or L_4 depending on which diagonal is in action. The following principle enables this question to be answered:

Considering the part of the truss to the left of the center, if for any given loading the positive moment about the joint at the right end of the panel is greater than that about the joint at the left, the shear in the panel is positive and the tension diagonal ╲ *is in action.*

This follows from the fact that $dM/dx = S$. Now the maximum M at the right of any panel (nearest the center of the span) will always in practice be greater than the maximum M at the left, and will consequently be greater than the moment at the left for the loading giving maximum M at the right. Hence, when the maximum M occurs at any joint, the shear in the panel to the left of that joint will be positive, and the ╲ diagonal will be in action. *Hence find maximum M at L_1, L_2, L_3, L_4, L_5 and divide by h, and the results will be the maximum stresses in the upper chord.*

For the lower chord, the maximum stress in L_3L_4, for instance, will be found from the maximum M at L_3 if U_3L_4 is in action, and from the

maximum M at L_4 if U_4L_3 is in action. When the maximum M at L_4 occurs, we have seen that diagonal $\diagdown$ is in action; when maximum M at L_3 occurs, it is uncertain which diagonal is in action. Obviously, the problem is to determine the maximum M at L_3 which is consistent with $\diagdown$ in action, and the maximum M at L_4 which is consistent with / in action, and to take the greater of these; this divided by h, will give maximum in L_3L_4. *This difficulty may be avoided, and it will always be on the safe side, and near enough, to take maximum M at L_4 and divide by h, for maximum stress in L_3L_4. In other words, take maximum lower chord stress equal to the maximum stress in top chord of the same panel, in panels where there are counters.*

Verticals.—The maximum stress in a vertical will always occur for the loading giving maximum stress in the diagonal which meets it at the unloaded chord. There is no live load at this joint. The vertical stress can always best be found by taking $\Sigma V = 0$ at that joint.

17. Comparison of Methods.—Computing the stresses in U_3U_4 and U_2L_3 for the E-60 loading we find:

U_3U_4: live index figure 861.4
U_2L_3: live index figure 213.7

The agreement with the previous results is suprisingly close, and indicates the accuracy of the method of using excesses; this is notwithstanding the fact that the second locomotive drivers are near L_4 in computing the chord, and the first locomotive to the left, whereas in our method we had assumed one excess at L_4 and the other to the right.

18. Inclined End Posts.—The Pratt truss is scarcely even built as shown in Fig. 48, but the end is usually made as shown in Fig. 48c. This leads to economy. The stress in $U'_1L'_1$ is the maximum panel load, that in $L'_0L'_1$ = that in L_1L_2, that in L'_0U' = that in U_0L_1; the other stresses are unchanged. The effect of the change is thus:

1. To replace U_0U_1 by a tension piece $L'_0L'_1$ having the same stress, and thus less material.

2. To discard L_0L_1, which had no stress, but would require some material, since, though the stress is zero from the vertical loads, the truss could not be built without it.

3. To discard the heavy end post U_0L_0.

4. To replace the tension bar U_0L_1 by the compression piece $L'_0U'_1$ with the same stress, involving some extra material.

5. To replace the post U_1L_1 by the tension bar $U'_1L'_1$, having, for spans of five or more panels, a smaller stress. It is clear that a considerable saving of material is effected.

19. Compression in Bottom Chords.—The horizontal longitudinal force due to the application of brakes, or to the tractive effort, causes horizontal reactions, which may be in either direction. Also, expansion

and contraction due to changes of temperature cause horizontal frictional forces. One end of a truss should be placed upon rollers and the other fixed. If the rollers were without friction, temperature changes would cause no stresses in the truss, and the horizontal longitudinal forces would all be resisted at the fixed end. But rollers are not without friction, and often they become rusted, or obstructed by dirt and cinders, so that they do not roll at all. Hence there are likely to be horizontal outer forces at each end, which may act in either direction. If directed outward, they add to the tension in the bottom chord; if directed inward they subtract from it. The Pratt truss without inclined end posts should therefore always have the member L_0L_1 a stiff piece, able to carry compression. And even with inclined end posts, the horizontal force due to expansion may exceed the tension in L_0L_1 due to the dead load, so that it is better even in such case to make L_0L_2 able to carry compression. This force should be computed as accurately as practicable, as it may be so large as to require increased area in L_0L_2, though generally it will be less than the dead tension.

Also, the wind pressure perpendicular to the truss acts on the lateral system at the bottom chord as on a horizontal truss, and according to the direction of the wind may cause either tension or compression throughout the length of that chord. These stresses should be computed, as explained in the chapter on Lateral and Sway Bracing, in the next volume, and added to those due to the vertical loads.

20. Deck Pratt Trusses.—If the floor is at the top, the truss, instead of resting on the abutment at L_0, may be supported at U_0, the bridge seat being carried to that level. This adds masonry, but bars U_0L_0 and L_0L_1 are omitted. Such a truss may, however, be supported at L_0, and may be constructed in all respects like Fig. 48 except that the stresses will be different; or it may have an inclined end post, as in Fig. 48c. In the latter case the end stringer may be supported at U_1, at one end and on the abutment at the other, or there may be a column L_0U_0 used, with a floor beam above it to carry one end of the stringer, and the piece U_0U_1 has nominally no stress, but simply braces the top of the column L_0U_0.

2. THE HOWE TRUSS

21. Characteristics.—The Howe truss (Fig. 40) differs from the Pratt truss in having the diagonals slope the other way. Thus the diagonals are predominantly in compression and the verticals in tension. Since a compression piece always weighs more than a tension piece having the same length and stress, this form of truss should never be built of steel.

But the Howe truss has a more restricted definition. It is a truss in which all the members except the verticals are of wood, and, moreover, in which the details are characteristic. The diagonals merely abut at their

ends against triangular wooden or metal blocks which are let into the chords; hence the diagonals can carry no tension and, if loosened by a tendency to put tension into them, will move out of place, though sometimes there are means applied to prevent this. The typical details are described in the next volume. If a truss has diagonals sloping as in a Howe truss it is sometimes said to have "Howe truss bracing," but it is not a Howe truss unless it has the typical material and details.

Both chords of a Howe truss are of wood, generally of several sticks side by side. The vertical tension rods pass between these sticks and are held on the top of the top chord and on the bottom of the bottom chord by nuts bearing against iron straps extending across the entire width of the chord. The bottom chord is like the top chord, except in dimensions and in the splices, which of course have to be made differently.

The Howe truss was invented in 1840 by William Howe.

Although counters are needed only in certain panels near the center, the typical Howe truss has them in every panel. The main diagonals consist of two sticks and the counters of single sticks passing between the two that compose the main diagonal. This use of counters is necessary in order to give the camber.

22. Computation.—The computation of the Howe truss is exactly similar to that for the Pratt truss, and need not be elaborated here. The student should work one truss out completely.

As the true Howe truss is of wood, with wooden stringers and floor beams, it is usually impracticable to make the panels 20 feet long, as the stringers and floor beams cannot be designed for such dimensions. More often the ties rest directly upon the chords, a fact which makes each such chord member a beam as well as a truss member, and would lead to very large sizes for such long panels. Hence the panels of true Howe trusses are generally made 10 or 12 feet long, or even less. The Howe truss is not suited for very long spans, and is seldom used for spans over 150 feet, though it has been used for railroad spans of 250 feet.

3. THE TRIANGULAR OR WARREN TRUSS

23. Characteristics.—The triangular truss has all web members inclined at an equal angle with the vertical, and may be deck or through. A through truss is shown in Fig. 49. There are no counters, but the stress in some web members near the center will be reversed if the shear can change sign. The true Warren truss, first used about 1850 by an English engineer, Warren, had all the web members inclined at 45° with the vertical, but the term is often applied to trusses in which the angle is different.

24. Computation.—The index figures for dead loads and for full uniform live loads are shown in the figure.

For the diagonals, the first excess should of course be placed at the joint just to the right. For the upper chords, since the origin of moments is at a bottom chord joint, one excess should be placed there and the other three panels distant, on the longer segment of the span. For the lower chords, the origin of moments is at an upper chord joint, and the excess must be placed at the lower chord joint to one side or the other of the origin. It is easy to show that it should be at the joint on the longer of

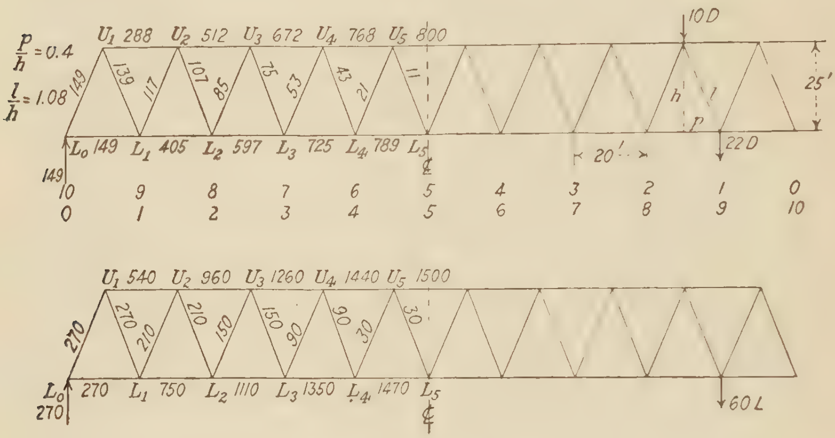


Fig. 49.—Triangular truss.

the two segments into which the origin divides the span. Consider the moment at any distance x from the left support: The question is, then, whether a load will cause a greater moment at this point when placed a given distance a to the left or the same distance to the right of this point. If the load is to the left (Fig. 50) the moment at x will be

$$M_1 = P\frac{x - a}{l}(l - x)$$

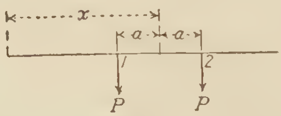


Fig. 50.

If the load is to the right, the moment at x will be

$$M_2 = P\frac{l - x - a}{l} \cdot x$$

M_2 is greater than M_1, if $l - x > x$.

That is, the load must be in the longer segment. If there are two excesses both must be in this segment.

Three typical stresses are found as follows:

$$U_2U_3\text{: dead, } 512\frac{p}{h}$$

$$\text{live, } 960\frac{p}{h} + \frac{13}{10} \times 40 \times \frac{4p}{h} = 1{,}168\frac{p}{h}$$

$$\text{impact, } 1{,}168\frac{p}{h} \times \frac{300}{460} = 762\frac{p}{h}$$

$$\text{total} = 2{,}442\frac{p}{h}.$$

$$L_2L_3\text{: dead, } 597\frac{p}{h}$$

$$\text{live, } 1{,}110 + \frac{11}{10} \times 40 \times \frac{5p}{h} = 1{,}330\frac{p}{h}$$

$$\text{impact, } 1{,}330\frac{p}{h} \times \frac{300}{440} = 907\frac{p}{h}$$

$$\text{total} = 2{,}834\frac{p}{h}.$$

$$U_3L_3\text{: dead, } 75\frac{l}{h}$$

$$\text{live, } 150 + \frac{1+2}{10} \times 60 + \frac{7+4}{10} \times 40 = 212\frac{l}{h}$$

$$\text{impact, } 212\frac{l}{h} \times \frac{300}{440} = 144.5\frac{l}{h}$$

$$\text{total} = 431.5\frac{l}{h}.$$

25. Warren Truss for Concentrated Loads. *Diagonals.*—For these it is only necessary to find the maximum shear of each kind in each panel.

For the upper chord of through span or lower chord of deck, it is only necessary to find the maximum moment at each joint of the loaded chord.

For the lower chord of through span or upper chord of deck, the origin of moments is opposite the center of a loaded panel, that is, not at a joint or panel point of the loaded chord. Since the moment line is always a straight line between adjoining concentrated loads, it is, for any given loading, straight between two adjacent joints of the loaded chord. It follows that if, for the loading causing maximum M at any joint such as L_3, we find the corresponding M at L_2, the M about U_3 will be the average of these two. In the same way, if for the loading giving maximum M at L_2 we find the corresponding M at L_3, the M at U_3 will be the average of these. Certainly it will be near enough to take the average of these two averages.

Hence, when finding maximum M at any joint of the loaded chord, the corresponding M (for the same loading) at the two adjacent joints should

be found. This will enable us to find the maximum stresses in the bars of the unloaded chord with no difficulty.

Generally it is quite accurate enough to take maximum M at $U_3 = \frac{1}{2}$ (maximum M at L_2 + maximum M at L_3). This is a little on the safe side; for if (Fig. 50*a*) a and b are any two consecutive joints on the loaded chord, ac = maximum M at a, bd = maximum M at b, $ae = M$ at a for loading giving bd, $bf = M$ at b for loading giving ac, it is clear that for loading giving bd, the M at a is less than ac, or is ae, and the moment at any point between a and b is represented by the line ed; similarly for loading giving maximum M at a, the moment is represented by cf. The line cd can never represent the moment between a and b, and the moment half way between a and b must always be less than one-half $(ac + bd)$.

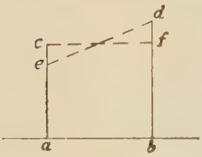


Fig. 50*a*.

4. THE WARREN TRUSS WITH SECONDARY VERTICALS OR SUBDIVIDED PANELS (FIG. 35*d*)

26. Reasons for Subdividing Panels.—Figure 49 shows clearly a difficulty with the triangular or Warren truss. It has to do with economy of material. The material in the chords is inversely as the height, since the stress is a fixed moment divided by the height. If the span is increased, the height should be increased in proportion; but if this is done, either the panels become very long or the diagonals very steep. In Fig. 49 the horizontal projection of a diagonal is only 10 feet; that is, each diagonal carries its load only 10 feet toward its destination, the abutment. If a load is at the center, half of it goes to each abutment. It is the web members that do this carrying. Each inclined web piece carries it a certain distance; each vertical brings it down so that another inclined piece may carry it further. The chords do nothing to carry the load in this sense; they are simply necessary for the inclined web pieces to push or pull against. Now it is not economical for a load to travel on such a zigzag route as is made necessary by short panels. Clearly there is an economical angle for the web members. But as the span is increased, if the height is increased and the economical angle of diagonals preserved, the panels will become too long to be economical, as the floor beams and stringers will become very heavy. Hence it is desirable to subdivide a long panel into two, by making a new joint at its center. We have seen in Art. 6 that for each new joint there should be two new bars; one of these is supplied by cutting the old chord bar in two, and the third is supplied if the new joint is connected to any old joint, preferably to the one just above (or below) it. Thus is developed the Warren truss with secondary verticals.

27. Stresses.—Each vertical carries only a panel load; each diagonal carries the shear in its panel; each chord has a stress equal to the moment at the opposite panel point divided by the height. The slight indetermination existing in the Warren truss is here avoided, since there is a

joint in the loaded chord at or opposite to each origin of moments for chord stress.

Sometimes in this truss, verticals are added at the joints of the loaded chord as well as at the joints of the unloaded chord, as shown by the dotted lines in Fig. 35*d*. These only support the unloaded chord, and so have no live-load stress.

It is unnecessary to work out the index stresses for this truss, as it is so similar to the previous one, but the reader should work one out for himself.

5. PRATT TRUSS WITH SUBDIVIDED PANELS

28. This truss is shown in Fig. 35*e*. The supporting members added run in this case diagonally to either joint opposite the new joint, preferably to the one nearer the abutment, in order not to carry the load toward

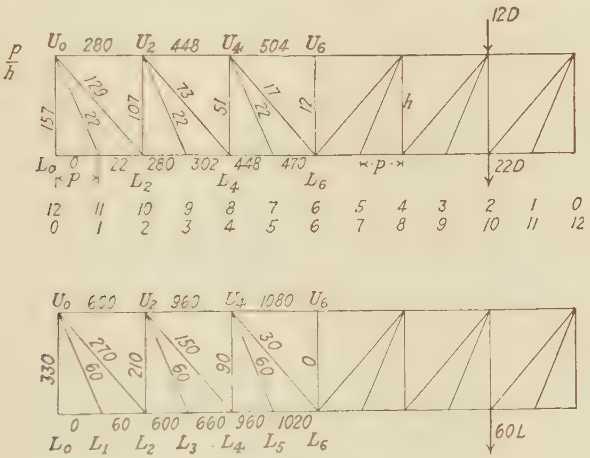


FIG. 51.

the center and then back again. The index figures for a truss of this type are given in Fig. 51. Here 12 panels are used instead of 10, and the panel loads are considered as indicated in the figures, with two $40k$ excesses as before.

The dead-load reaction should be $5\frac{1}{2} \times 22 + 3 \times 12 = 157$, which checks. The dead load on U_0 is 6. The first upper chord stress is 22 $p/h + 129 \times 2p/h$, or the index is 280. The central upper chord stress should be $1/h(157 \times 6p - 6 \times 6p - 12 \times 6p - 22 \times 15p) = 504p/h$, which checks. The first figures written should be on the center vertical, and on the secondary suspenders which carry the loads 22. These loads are then treated as acting at the joints to which these bars carry them.

The maximum in U_2L_4 will be (index figure)

$$U_2L_4 = 73 + 150 + \tfrac{13}{12} \times 40 + \tfrac{6}{12} \times 60 + \text{impact}$$

The maximum in U_2U_4 will be (index figure)

$$U_2U_4 = 448 + 960 + {}^{13}\!/_{12} \times 40 \times 4 + \text{impact}$$

If the diagonals, as in this case, do not all have the same slope, care must be taken, in getting chord-index figures, not to add web-index figures, but to take account of the different slopes.

29. Baltimore Truss.[1] —A more usual form for this truss is that shown in Fig. 55, sometimes called the Baltimore truss. In making a new joint in the center of a long panel, the new joint may be connected with any old joint, or it may be connected with another new joint in another bar,

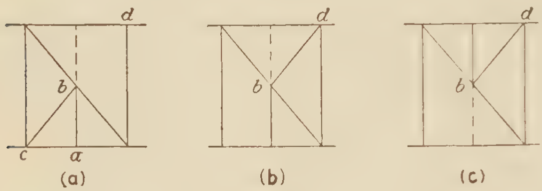


Fig. 52.

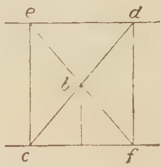


Fig. 53.

and the latter connected with any old joint. Thus the new joint a (Fig. 52a) is connected to the new joint b, and the latter to either old joint c or old joint d, *but not to both* unless the truss is so constructed that two bars shall count as one, by providing that both cannot act at the same time. Thus the construction as in Fig. 53 would be statically indeterminate, unless, for instance, bars be and bc could not be in action at the same time.

An objection to the truss shown in Fig. 51 is that the unloaded chord members, such as U_2U_4, may be very long. In Fig. 52 they are shortened by adding the short dotted verticals. This is particularly important in through bridges, in which a long column like U_2U_4 would have a considerable bending moment due to its own weight, which would cause a deflection that is objectionable in the case of a column, though much less so in the case of a tie. It is necessary to analyze the situation at joint b before this truss can be understood. Consider Fig. 52b and joint b,

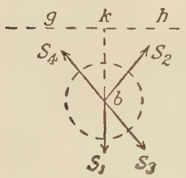


Fig. 54.

shown in Fig. 54. Taking a section around the joint, the four forces are in equilibrium. If S_1 acts in the middle of the panel gh, or bisects the angle between S_2 and S_4, taking moments about g and resolving S_2 into vertical and horizontal components at h,

$$\text{V. C. of tension in } S_2 = \frac{1}{2} \text{ tension in } S_1$$

[1] The rectangular truss with subdivided panels is sometimes called the *Baltimore truss* if the chords are parallel; while the same truss with curved upper chord is often called the *Pettit truss*. There is no virtue in these names.

The stress S_2 is of the same kind as S_1; more generally, S_2 is tension if the force at b acts down; this force might be a compression in kb. Taking moments about h, and resolving S_3 and S_4 at g,

$$\text{V. C. tension } S_4 - \text{V. C. tension } S_3 = \frac{1}{2} \text{ tension } S_1$$

That is, if S_1 acts down, it makes the tension S_4 greater than the tension S_3 by a vertical component $\frac{1}{2} S_1$. If k is not in the middle of gh,

$$\text{V. C. } S_2 = S_1 \frac{gk}{gh}$$

and

$$\text{V. C. } S_4 - \text{V. C. } S_3 = S_1 \frac{kh}{gh}$$

The condition is the same as if there were a separate truss ghb, loaded at b with S_1, superposed upon the original Pratt truss, and the stresses added. A similar situation exists at joint b in Fig. 52a. The reader should visualize these results, and not simply regard them as equations.

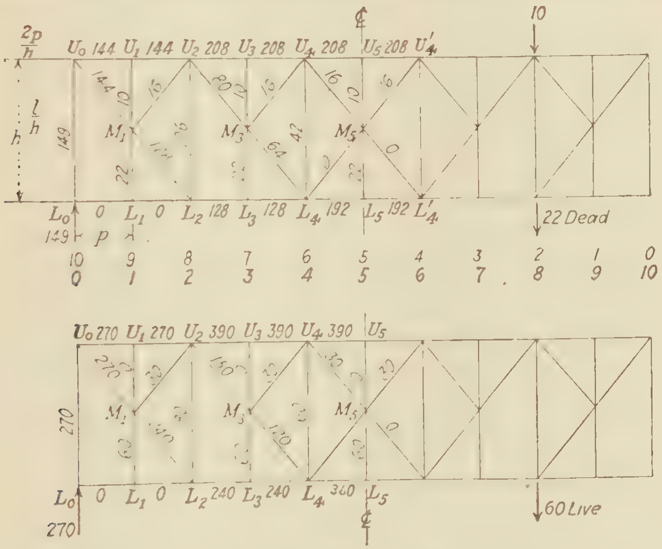


FIG. 55.—Baltimore truss.

30. A complete Baltimore truss is shown in Fig. 55. It has not only the loaded chord subdivided, but the unloaded chord as well. The lower half of the intermediate verticals (such as U_1L_1) is a tension piece and the upper half a compression piece. Thus there is no single piece U_1L_1, but two pieces U_1M_1 and M_1L_1.

In order to avoid statical indetermination, we decide that all the diagonals in the large panel L_4L_4' shall be tension bars, incapable of taking compression, and with no initial stresses. It should be observed that in Fig. 52a the secondary diagonal cb is a compression piece, while in

Fig. 52*b* the diagonal *bd* is a tension piece. Hence in the present truss if L_4M_5 and L'_4M_5 were stiff members, able to take compression, the load of 32 coming to M_5 could either go up U_4M_5 and U'_4M_5 in tension, or go down L_4M_5 and L'_4M_5 in compression, or be divided among all four in some manner that could not be determined by statics. But by the construction assumed, the two lower diagonals must have zero stress for this symmetrical loading. Likewise if there are any counters such as L_2M_3, they must be made as tension pieces.

In writing the index stresses for dead load, begin with the secondary verticals like U_1M_1, all of which have 10; then those below, like M_1L_1, all of which have 22; then the secondary short diagonals like M_1U_2, all of which have $\frac{1}{2}(10 + 22) = 16$; then proceed with the remaining web, and finally the chord. To check (which should always be done):

$$R = 32 \times 4\frac{1}{2} + 5 = 149$$

$$U_4U_5 = \frac{1}{h}(149 \times 5p - 5 \times 5p - 32 \times 10p + 16 \times p) = 416\frac{p}{h} = 208 \cdot \frac{2p}{h}$$

The section is taken just to the left of U_5L_5, and the center of moments at L_5. The section cuts the diagonal U_4M_5 which has a vertical component at M_5 equal to 16, acting at L'_4, and a lever arm p. The reader should draw a figure showing this section considered, and the forces acting upon it.

The index figures for the live panel load of 60 are found in a similar way.

The next step is to find how many counters are needed. To find whether L_2M_3 is needed, find maximum $-S$ in the panel, and we will do this independent of the live-load stresses just found.

Maximum $-S$ in $L_2L_3 = \frac{3}{10} \times 60 + \frac{2}{10} \times 40 - 80$, which is negative; hence there is no negative shear, and no counters are needed. The maximum stress in each bar will now be given, not including impact.

$$U_0U_2\text{:} \left(144 + 270 + \frac{9 + 6}{10} \times 40\right)\frac{2p}{h}$$

$$U_2U_4\text{:} \left(208 + 390 + \frac{7 + 4}{10} \times 40 \times 2\right)\frac{2p}{h}$$

$$U_4U_4\text{:} \left(208 + 390 + \frac{5 + 2}{10} \times 40 \times 3\right)\frac{2p}{h}$$

For U_0U_2 it is clear that $\frac{9 + 6}{10} \times 40$ is the maximum shear in U_0M_1 due to the excesses; hence its horizontal component is the maximum stress they cause in U_0U_2. For U_2U_4, $\frac{7 + 4}{10} \times 40$ is the maximum shear they cause in U_2M_3, and when in this position (at L_3 and L'_4), they cause the same shear in U_0L_2; hence twice the horizontal component is the

maximum stress they cause on U_2U_4. The same result is reached by taking moments; for taking a section cutting U_2U_4 between L_2 and L_3, the center of moments for U_2U_4 is at L_4. If the excesses are at L_4 and L_3', the left reaction is $\frac{6+3}{10} \times 40$, and its moment is $\frac{6+3}{10} \times 40 \times 4p$, and the stress in U_2U_4 is this divided by h; while if the excesses are at L_3 and L_4' the left reaction is $\frac{7+4}{10} \times 40$, and the moment about L_4 is $\frac{7+4}{10} \times 40 \times 4p$. The reader must not be confused here by the fact that the center of moments (L_2) is to the right of the first load, which is at L_3, for the section is taken to the left of L_3, and only forces to the left of the *section* (not the center of moments) are to be considered.

In the same way, for U_4U_4' the excesses are at L_5 and L_2'. To continue:

$$L_0L_2: 0$$

$$L_2L_4: \left(128 + 240 + \frac{8+5}{10} \times 40\right)\frac{2p}{h}$$

$$L_4L_4': \left(192 + 360 + \frac{6+3}{10} \times 40 \times 2\right)\frac{2p}{h}$$

$$U_1M_1, \text{ etc.}: 10$$

$$M_1L_1, \text{ etc.}: 22 + 60 + 40$$

$$M_1U_2 \text{ and } M_3U_4: \frac{1}{2}\left(60 + 40 + 32\right) = 66$$

$$U_0M_1: \left(144 + 270 + \frac{9+6}{10} \times 40\right)\frac{l}{h}$$

$$U_2M_3: \left(80 + 150 + \frac{7+4}{10} \times 40\right)\frac{l}{h}$$

$$U_4M_5: \left(16 + 30 + \frac{5+2}{10} \times 40\right)\frac{l}{h}$$

$$U_0L_0: 149 + 270 + \frac{9+6}{10} \times 40$$

U_2L_2: with respect to this, the question is whether the maximum would occur with the maximum in U_2M_3 or with the maximum in U_2M_1. Taking a diagonal section through L_2L_3, U_2L_2, and U_1U_2, the maximum positive shear is

$$\frac{28}{10} \times (32 + 60) + \frac{8}{10} \times 10 + \frac{7+4}{10} \times 40 - \frac{1}{10} \times 32 - \frac{2}{10} \times 22 = 302$$

This acts upward on the left of the section, and to it must be added the vertical component of $U_2M_1 = 16$, which also acts up, making a total of 318 in U_2L_2 (see Fig. 56). To put the live load on L_2 would merely diminish the shear, without adding anything to U_2M_1; but to put it on

L_1 and L_2 would add $\frac{1}{2}(60 + 40) = 50$ to U_2M_1. In this case the stress in U_2L_2 is, with excesses on L_7 and L_4',

$$\frac{28}{10} \times 92 + \frac{8}{10} \times 10 + \frac{11}{10} \times 40 - \frac{1}{10}(92) - \frac{2}{10} \times 82 + 46 = 330$$

Hence maximum in U_2L_2 is 330.

U_4L_4: treating this in the same manner, with load up to L_5,

$$\frac{15}{10} \times 92 + \frac{6}{10} \times 10 + \frac{7}{10} \times 40 - \frac{6}{10} \times 32 - \frac{4}{10} \times 22 + 16 = 160$$

With load to L_3, and excesses on L_5 and L_2',

$$\frac{15}{10} \times 92 + \frac{6}{10} \times 10 + \frac{7}{10} \times 40 - \frac{6}{10} \times 32 - \frac{3}{10} + 60 - \frac{4}{10} \times 82 + 46 = 148$$

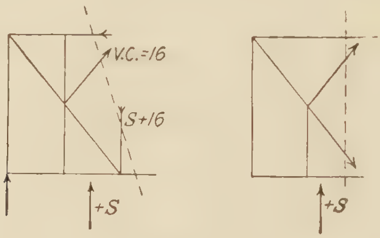


FIG. 56.

Hence the maximum is 160, for load up to L_5.

M_1L_2:

Load to L_2,

$$\frac{36}{10} \times 92 + \frac{13}{10} \times 40 - \frac{1}{10} \times 32 + 16 = 396\frac{l}{h}$$

Load to L_1,

$$\frac{36}{10} \times 92 + \frac{6}{10} \times 40 - \frac{1}{10} \times 132 + 36 = 408\frac{l}{h} \text{ (maximum)}$$

M_3L_4:

Load to L_4,

$$\frac{21}{10} \times 92 + \frac{9}{10} \times 40 - \frac{6}{10} \times 32 + 16 = 226\frac{l}{h}$$

Load to L_3,

$$\frac{21}{10} \times 92 + \frac{4}{10} \times 40 - \frac{6}{10} \times 32 - \frac{3}{10}100 + 66 = 226\frac{l}{h} \text{ (maximum)}$$

M_5L_4':

Load to L_4',

$$\frac{10}{10} \times 92 + \frac{5}{10} \times 40 - \frac{15}{10} \times 32 + 16 = 80\frac{l}{h} \text{ (maximum)}$$

Load to L_5,

$$\frac{10}{10} \times 92 + \frac{2}{10} \times 40 - \frac{15}{10} \times 32 - \frac{5}{10} \times 100 + 66 = 68\frac{l}{h}$$

31. The reason why the maximum occurs in M_1L_2 and in U_2L_2 for a load extending to the left of L_2 is obvious. The short tension diagonals like M_1U_2 force a part of the load at L_1 going to the left abutment to pursue a roundabout course, going to U_2, then down to L_2 and then back to the left abutment. If a load is to go to the left, it is generally uneconomical to make it go to the right and back again. Of the load at L_1, nine-tenths must go to the left abutment: in this truss one-half goes directly, by way of U_0M_1 and U_0L_0; the remaining four-tenths goes to U_2,

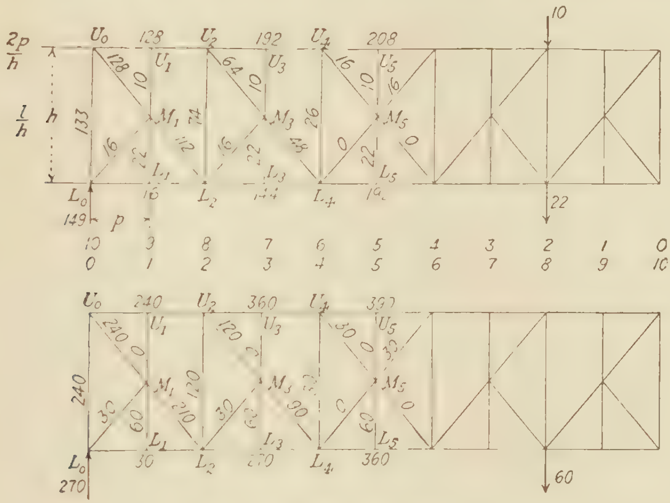


FIG. 57.—Baltimore truss.

L_2, and then to U_0 and L_0. Hence, it appears more economical to design this truss with short compression diagonals rather than with short tension diagonals, as shown in Fig. 57, in which the bars L_0M_1 etc., are struts. Here the loads going to either abutment go always in the proper direction. A load at L_1 goes one-half through M_1L_0 to L_0, four-tenths through M_1U_0, and one-tenth through M_1L_2 to the right abutment. The stress in U_0M_1 is equal to that in M_1L_2 plus a vertical component equal to half the load at L_1.

The following are the maximum stresses or index stresses for all the bars in this truss, not including impact, the usual index figures being shown on the drawings. The reader should compare with Fig. 55.

To find whether M_3U_4 is needed:

$$\text{maximum } -S \text{ in } L_3L_4 = \frac{6}{10} \times 92 + \frac{3}{10} \times 40 - \frac{21}{10} \times 32 = 0$$

Hence it is not needed. If put in, L_2M_3 should be a tension piece. If not put in, L_2M_3 must be a compression piece. We will assume that there is no M_3U_4.

$$U_0U_2\text{: }\left(128 + 240 + \frac{13}{10} \times 40\right)\frac{2p}{h}$$

$$U_2U_4\text{: }\left(192 + 360 + \frac{9}{10} \times 40 \times 2\right)\frac{2p}{h}$$

$$U_4U_5\text{: }\left(208 + 390 + \frac{7}{10} \times 40 \times 3\right)\frac{2p}{h}$$

$$L_0L_2\text{: }\left(16 + 30 + \frac{1}{2} \cdot 40\right)\frac{2p}{h}$$

$$L_2L_4\text{: }\left(144 + 270 + \frac{11}{10} \times 40 + 20\right)\frac{2p}{h}$$

$$L_4L_5\text{: }\left(192 + 360 + \frac{9}{10} \times 40 \times 2\right)\frac{2p}{h}$$

U_1M_1, etc., 10

M_1L_1, etc., 122

M_1L_0, etc., $66\frac{l}{h}$

$$M_1L_2\text{: }\left(112 + 210 + \frac{13}{10} \times 40 + \frac{1}{10} \times 60\right)\frac{l}{h}$$

$$M_3L_4\text{: }\left(48 + 90 + \frac{9}{13} \times 40 + \frac{6}{10} \times 60\right)\frac{l}{h}$$

$$U_4M_5 = U_4'M_5\text{: }\left(16 + 30 + \frac{7}{13} \times 40 + \frac{10}{10} \times 60\right)\frac{l}{h}$$

$$U_0M_1\text{: }\left(128 + 240 + \frac{13}{10} \times 40\right)\frac{l}{h}$$

assuming that excess loads may be anywhere, not necessarily with the first one at the very head of the live load:

$$U_2M_3\text{: }\left[64 + \left(\frac{28}{10} - \frac{1}{2}\right)60 + \frac{9}{10} \times 40\right]\frac{l}{h}$$

With excess loads on points 7 and 4, the shear is $^{11}\!/_{10} \times 40$, and the vertical component in U_2M_3 is $(^{11}\!/_{10} - \frac{1}{2})40$; with loads on points 6 and 3, the V. C. in U_2M_3 is $^{9}\!/_{10} \times 40$, or larger. The maximum positive live shear in panel L_2L_3 is

$$\frac{1 + 2 + \cdots 7}{10} \times 60 = \frac{28}{10} \times 60$$

the V. C. in U_2M_3 is $(^{28}\!/_{10} - \frac{1}{2})60$.

$$M_5L_4' \text{ or } M_5L_4\text{: }\left(0 + \frac{10}{10} \times 60 + \frac{5}{10} \times 40\right)\frac{l}{h} = 80\frac{l}{h}$$

M_5U_4' is always in action, carrying at least one-half a dead panel load, since L_4M_5 and M_5L_4' are tension bars.

$$U_2L_2\text{: maximum V. C. in } U_2M_3 + 10$$
$$U_4L_4\text{: maximum V. C. in } U_4M_5 + 10$$

This truss has been explained in detail, and should be mastered by the reader, because if it is thoroughly understood, other parallel-chord trusses should be easy.

32. Proportions in the Last Example are Poor.—The proportions of the last truss are not good if the height is 25 feet, for the reason that the diagonals are too flat with a height of 25 feet and a horizontal projection of 40 feet. The height should be greater, or the panel length shorter; but the panel length has been kept the same as in the other trusses studied, and is about right. For this reason the stresses have all been kept in terms of p/h or l/h. A height of 25 feet also is about in the right proportion to the span. The economics of trusses will be discussed in Chap. IX.

33. The Baltimore Truss for Actual Wheel Loads. (Fig. 57.)—The maximum in any bar like U_1M_1 is of course the dead load of 10,000 pounds.

The maximum in bars like L_1M_1 is the maximum panel load.

The maximum V. C. in bars like L_0M_1 is half the maximum panel load.

The maximum V. C. in bars like M_1L_2 is the maximum shear in the panel.

The maximum in any vertical is 10 plus the maximum in the diagonal meeting it at the top chord.

To find whether a counter is needed in any double panel to the left of the center, such as L_2L_4, find whether there can be a negative shear in the right half of this double panel, such as L_3L_4; if there can be a negative shear, a counter is needed, and all the diagonals in the double panel should be tension bars if the truss is to be statically determined.

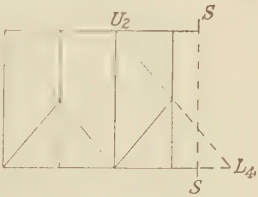


FIG. 58.

In any double panel like U_2L_4, to the left of the center, in which there is no counter, the following stresses occur:

1. The maximum in the *top chord* is the maximum moment at the joint corresponding to L_4, divided by h.

2. The maximum in the *bottom chord* (see Fig. 58) is the maximum moment about the joint corresponding to U_2 *of all the loads and reaction to the left of the section SS*, divided by h. This will not be what is termed the moment at U_2, which is the moment of all the outer forces to the *left of U_2L_2*, while what is desired in this case is this moment *plus the moment of the load on L_3, since this load is to the left of the section.* It is close enough, if not absolutely accurate, to find the maximum moment

at L_2 by the usual method; for this loading find the load on L_3, and add the moment of this load.

3. The maximum in bars like U_2M_3 is a V. C. equal to the maximum $+S$ in L_2L_3 minus one-half the load on L_3; the live load extending to L_3 and the first excess on L_4, because, for a load at L_3 the V. C. of stress is $7/10 - 1/2$, while for a load at L_4 it is six-tenths of the load; and for two excesses, the first at L_3, the V. C. is $11/10 - 1/2$, while for the first is at L_4 it is nine-tenths.

In any double panel at or to the left of the center, like L_4L_4', in which there is a counter, all diagonals being tension bars, it is desirable to study the situation in such a panel, to see when certain diagonals will be in action, and consequently where to take the origin of moments (see Fig. 59). All diagonals being tension bars:

Bar *bd* may be acting (1) as a secondary, carrying only one-half the load at *d*, or (2) as a main diagonal.

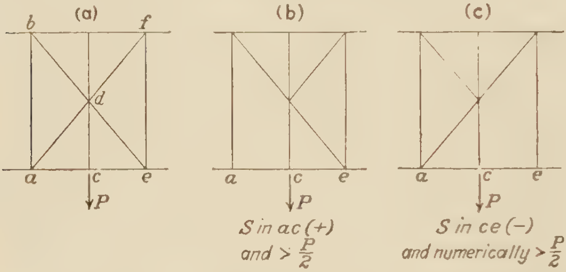


FIG. 59.

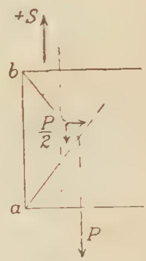


FIG. 60.

Bar *df* may be acting (1) as a secondary, carrying only one-half the load at *d*, or (2) as a counter.

Bar *ad* is either out of action, or acting as a counter.

Bar *de* is either out of action, or acting as a main diagonal.

In other words, the bars *in action* are either as shown in Figs. 59*b* or 59*c*. In condition *b* the shear in *ac* is positive; but the converse is not true; that is to say, it does not follow that if the shear in *ac* is positive, the situation will be *b*, for the shear in *ac* might be positive; yet if less than $P/2$, condition *c* would exist (see Fig. 60). Hence, *condition b requires that the shear in ac is positive and greater than $P/2$.* Similarly, *condition c requires that the shear in ce is negative and numerically greater than $P/2$.*

If these facts are fully grasped, it will be easy to see that the following conclusions are true:

1. Maximum in the top chord:

If condition *b* of Fig. 59 exists,

$$\text{maximum in top chord} = \frac{\text{maximum } M \text{ at } e \text{ (nearest center)} + Pp}{h}$$

If condition c of Fig. 59 exists,

$$\text{maximum in top chord} = \frac{\text{maximum } M \text{ at } a \text{ (nearest end)} + Pp}{h}$$

In each case P is the load at c when the maximum M occurs at e or a.

2. Maximum in the bottom chord:

If condition b exists,

$$\text{maximum in bottom chord} = \frac{\text{maximum } M \text{ at } a \text{ (nearest end)}}{h}$$

If condition c exists,

$$\text{maximum in bottom chord} = \frac{\text{maximum } M \text{ at } e \text{ (nearest center)}}{h}$$

Since the maximum M at e always exceeds numerically the maximum M at a, it would be inaccurate, but always on the safe side, to use the last expression; but we have seen that the loading giving maximum M at e would not in general give condition c. In that case, the moment at e to be taken is the maximum which is consistent with condition c.

3. Maximum V. C. in bd = maximum $+S$ in ac (if $>P/2$) or

$$\frac{\text{maximum joint load}}{2}$$

whichever of these is the greater.

4. Maximum V. C. in df = maximum $-S$ in ce (if numerically $>P/2$) or

$$\frac{\text{maximum joint load}}{2}$$

whichever of these is the greater.

5. Maximum V. C. in ad, numerically = maximum $-S$ in ac + ½ (corresponding load at c).

6. Maximum V. C. in de = maximum $+S$ in ce + ½ (corresponding load at c).

THE WARREN TRUSS WITH TERTIARY SYSTEM

34. As the Warren truss may have the panels subdivided by verticals connecting the new joints to the opposite chord joints, so the latter truss may have its panels once again divided, as in Fig. 61, by connecting the new joint a to another new joint b, and the latter to an old joint c. A truss of this kind consisting of but one large triangle, with four panels, is called an A-truss. In Fig. 61 the index figures for assumed dead loads are given. There are no counters in this truss or in the Warren truss with verticals only, and it should cause the reader no difficulty if the previous trusses have been mastered.

The maximum in any upper chord is the maximum M at the lower chord joint opposite, divided by h.

The maximum in any lower chord, as L_2L_4, is found by taking a section through L_3L_4, and moments about U_2; hence it will be the maximum M at L_2 plus the corresponding load at L_3 times p, divided by h. The excesses will be at L_3 and L_6, and if each excess is called E, the stress due to the two will be

$$\left(\frac{9+6}{12} \times E \times 2p + E \times p\right) \div h$$

The lower part of each long diagonal will have a V. C. equal to the maximum S on the section cutting it and the two chords.

Each short diagonal will have a V. C. equal to half the maximum load carried to the joint at the upper end.

The stresses in the verticals are obvious.

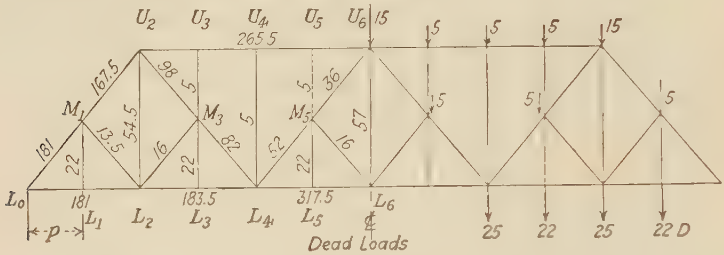


FIG. 61.

The upper part of the long diagonals will have maximum stresses for uniform live panel loads of 60, and excesses E, as follows (vertical components):

$$U_2M_1: \frac{55}{12} \times 60 + \frac{10+7}{12}E$$

$$U_2M_3: \left(\frac{45}{12} - \frac{1}{2}\right)60 + \frac{8+5}{12} \cdot E$$

$$U_6M_5: \frac{21}{12} \times 60 + \frac{6+3}{12}E$$

For the actual concentrated loads, U_2M_3, for instance, will have a V. C. equal to maximum $+S$ in L_2L_3 minus one-half the corresponding load on L_3; or, by moving the loads back to the right a little, it may be possible to make S in $L_2L_3 - \frac{1}{2}$ load on L_3 greater than with maximum S in L_2L_3.

This truss may obviously be built either through or deck.

35. The Warren truss may have its panels subdivided by the system shown in Fig. 62 instead of by secondary verticals. By considering joints a and b, it is at once seen that the stress in ab, with the (equal) loads at a and b, determine the stresses in ac and bc, and that the latter must be of the same kind and magnitude; hence ac and bc each carry

half the load at c. As compared with the use of secondary verticals, this system would require more material in the three bars ab, ac, and bc, than would be required for one vertical, but this system would brace

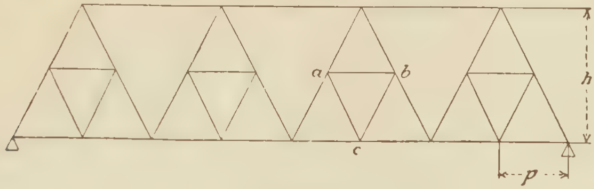


Fig. 62.

the long diagonals of the Warren truss, which would be an advantage, and a saving of material in the struts.

THE FINK AND BOLLMAN TRUSSES

36. The Fink truss, invented by Albert Fink, in 1852, is shown in Fig. 63. This truss was formerly used for quite large spans, as in the bridge across the Ohio at Louisville, now replaced by a modern structure. The

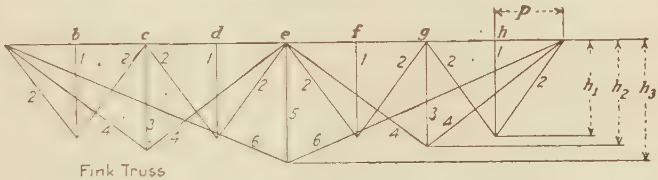


Fig. 63.

heights of all the verticals may be the same, instead of unequal as in the figure. The stresses are easily found as follows:

In bars such as 1, the maximum stress is the maximum panel load.

In bars 2, the V. C. is one-half a panel load plus half the load at the bottom of 1.

In bars 3, the stress for any loading is the load at the top joint plus the V. C. of the two diagonals 2, meeting at that joint. For uniform load and excess, the stress will be two panel uniform loads plus one excess. For the actual wheel loads, let the influence line be drawn, as in Fig. 63a. This line has the same general form (not dimensions) as the

Fig. 63a.

influence line for the moment at c on a simple beam of span ae supported at a and e; hence the loading that will give maximum stress in 3 will be the same that would cause maximum M at c on a beam ae, which can be found as already shown. This being done, the loads at the joints can be accurately found, and

$$\text{maximum stress in } 3 = \text{load at } c + \frac{1}{2}\,(\text{load at } b + \text{load at } d)$$

The bars 4 will have V. C. equal to half the stress in bar 3 plus one-half the load at the joint at the bottom of 3.

For bar 5, the influence line will be a triangle with vertex over *e*. For uniform load and excesses,

$$\text{maximum in } 5 = 4 \text{ panel uniform loads} + 1\frac{1}{4} \text{ excess loads.}$$

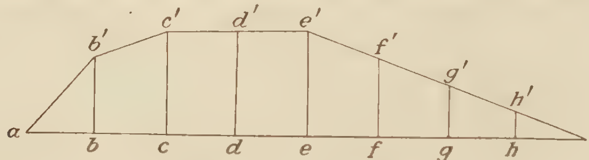
For actual loads,

maximum in 5 = load at *e* + three-fourths loads at *d* and *f* + one-half loads at *c* and *g* + one-fourth loads at *b* and *h*

the loads at all points except *e* to include loads at the bottom of the verticals as well as at the top.

In bar 6, the V. C. will be half the stress in 5 plus half the load at the bottom joint.

The stress in the top chord will be the H. C. of stresses in bars 2, 4, and 6. For the uniform load this stress will be the same throughout, since at *c* and *e* the diagonals on each side have equal H. C. which balance. The excess stresses will not be the same throughout. Thus for the two



Influence line for stress in bc

Fig. 64.

excesses at different points, the excess stresses in the chord will be as follows: For excess E at b and e:

$$\text{stress in } ac = E\left(\frac{1}{2}\cdot\frac{p}{h_1} + \frac{1}{4}\cdot\frac{2p}{h_2} + \frac{5}{8}\cdot\frac{4p}{h_3}\right)$$

$$ce = E\left(\frac{1}{4}\cdot\frac{2p}{h_2} + \frac{5}{8}\cdot\frac{4p}{h_3}\right)$$

For excess at c and f:

$$\text{stress in } ac = E\left(\frac{1}{2}\cdot\frac{2p}{h_2} + \frac{5}{8}\cdot\frac{4p}{h_3}\right)$$

$$ce = \text{same}$$

For excess at d and g:

$$\text{stress in } ac = E\left(\frac{1}{4}\cdot\frac{2p}{h_2} + \frac{5}{8}\cdot\frac{4p}{h_3}\right)$$

$$ce = E\left(\frac{1}{4}\cdot\frac{2p}{h_2} + \frac{5}{8}\cdot\frac{4p}{h_3} + \frac{1}{2}\frac{p}{h_1}\right)$$

For excess at e and h:

$$\text{stress in } ae = E\left(\frac{5}{8}\cdot\frac{4p}{h_3}\right)$$

To find maximum chord stresses for actual wheel loads, no formula can be given. It must be done by trial and common sense. The influence

line may be used with advantage. Figure 64 is the influence line for the stress in ac, the ordinates being as follows, for uniform height h of all verticals:

$$hh' = \frac{1}{8}\cdot\frac{4p}{h} = \frac{1}{2}\cdot\frac{p}{h};\ gg' = \frac{p}{h};\ ff' = 1.5\frac{p}{h};\ ee' = 2\frac{p}{h}$$

$$dd' = \frac{1}{4}\cdot\frac{2p}{h} + \frac{3}{8}\cdot\frac{4p}{h} = 2\frac{p}{h};\ cc' = \frac{1}{2}\cdot\frac{2p}{h} + \frac{1}{4}\cdot\frac{4p}{h} = 2\frac{p}{h}$$

$$bb' = \frac{1}{2}\cdot\frac{p}{h} + \frac{1}{4}\cdot\frac{2p}{h} + \frac{1}{8}\cdot\frac{4p}{h} = 1.5\frac{p}{h}$$

This influence line may be used, by usual methods, to find the loading giving maximum stress, and to find the stress itself. Chord section ce may be treated similarly.

The Fink truss has the peculiarity that, for a uniform load, all bars have maximum stress for full loading, and that there is no reversal of stress, no counters, no lower chord, and that all minimum stresses are those due to the dead load alone. It may be built as a through bridge, still without lower chord, with floor beams suspended from bottom of verticals or riveted to them, the verticals at b and d in this case merely supporting the upper chord.

On the other hand, this truss is uneconomical of material because of the flat inclination of the long diagonals and the fact that the stress in the top chord is nearly constant throughout. The long diagonals would sag badly if unsupported, and should be supported on the verticals which they cross. The loads do not travel to the abutments in direct lines, but make long detours. Of the load at b, seven-eighths goes to the left abutment: but only one-half goes there directly, through the short diagonal; one-quarter goes to the next diagonal, from c, and one-eighth through the longest diagonal, from e; so that one-quarter of the load at c goes to e and from there is distributed to the abutments. The bridge is not now built.

37. The Bollman truss, invented by Wendell Bollman about 1850, is shown in Fig. 65. Each vertical bears a panel load, and for any given

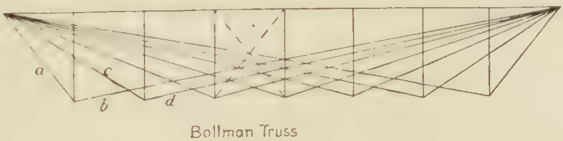


FIG. 65.

loading the stress in the top chord is uniform from end to end. This truss is not now built, and is very objectionable on account of the long diagonals. It is really nothing more than a series of trusses each consisting of a single triangle and vertical (King-post trusses), all superposed and with a common top chord. All the King-post trusses are unsym-

metrical except one. As built many years ago, this truss often had adjustable diagonals in each panel except the end ones, as indicated by the dotted lines. These trusses, both the Fink and the Bollman, generally had compression pieces of cast iron. A Bollman truss across the Potomac at Harper's Ferry was a good example of that type of structure, and was not replaced till within the writer's memory.

38. The King-post Truss.—This is the simplest form of truss (Fig. 66) and may be either through, as shown, or inverted as a deck span. It can, of course, be used only for short spans. There is but one joint carrying live load, and all stresses are a maximum for maximum joint load. The stress in the vertical is P, in either diagonal $P/2 \cdot l/h$, and in the chord $P/2 \cdot b/h$. It is usually symmetrical, but might be unsymmetrical, in which case the stresses would of course be changed.

The chord is almost always continuous, although as a through span, the chord being in tension, there could be a pin connection at the center.

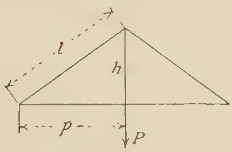


Fig. 66.

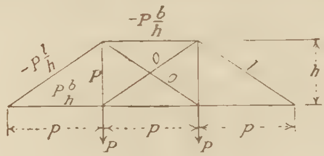


Fig. 67.

The fact that the span is short and the chord generally continuous, especially as a deck span, makes the chord able to carry a moment which may be considerable in proportion to the direct stress, particularly if the depth h is small and the deflection as a truss correspondingly large. Such a structure is often really a trussed beam rather than a truss, and should be computed as a trussed beam, which is a statically undetermined structure, and which is discussed in the next volume. For actual wheel loads, if the structure is considered a truss, find maximum P, and all maximum stresses from this.

39. Queen-post Truss.—This is shown in Fig. 67. Like the King-post truss, if the bottom chord is continuous, this truss is really a trussed beam, and is treated in the next volume. If considered as a truss, the stresses are as shown in the figure for full uniform load. The center diagonals should be tension bars, and the V. C. of stress is the maximum shear in the center panel.

For actual wheel loads, the end and center diagonals are found from the maximum shears. The end bottom chord is the horizontal component of the end diagonal. When the maximum M occurs at L_1, the moment at L_2 will be smaller; hence the shear in the center panel will be negative and diagonal ╱ in action; hence the origin of moments for the top chord will be at L_1, and its maximum stress will be the maximum M at L_1 divided by h. This will also be the loading giving maximum stress

in left end diagonal, and hence giving maximum + shear in the left panel. Hence, maximum in vertical will be either this shear, or the maximum load on L_1 consistent with diagonal $\diagdown$ being in action, whichever is the greater. The maximum in the center panel of the lower chord will be the maximum M at L_1 consistent with diagonal $\diagdown$ being in action, divided by h.

40. The K-truss.—This truss is shown in Fig. 68. It is clearly statically determined; in the figure shown, the number of joints m is 16, and $2m - 3 = 29$, which is the number of bars. Designating the bars by numbers, the stresses may be found as follows:

Bar 1, by a vertical section, from the shear.

Bar 2, by a vertical section, from moment about a.

Bar 3, the load at b.

Bar 4, by an inclined section, from the shear.

Bars 5 and 6, by a section around their intersection, where there are two equations to determine these unknown quantities.

Bar 7, equal to 2.

Bar 8, by method of joints at a.

Bars 9 and 10, by method of joints at c.

Bars 11 and 12, by method of joints at d.

Bars 13 and 14, by method of joints at e.

Bar 15, by method of joints at top or bottom.

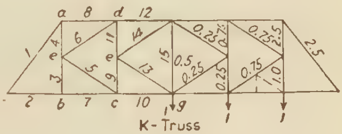


Fig. 68.

Assuming a load 1 at each lower joint of the truss, the vertical components on the web pieces will be as indicated on the right half of the figure.

This truss may have subdivided panels, as indicated by the dotted lines in one panel. This does not make the structure statically undetermined.

This truss has the advantage that the vertical posts are braced at midheight, but the diagonals are flatter. For very long spans, where the height should be very large, for economy, without making the panels too long, this truss is an excellent form, as in this case the diagonals will not be too flat. In a long span, this form of truss would have the chords inclined; it was so used in the second (or existing) Quebec bridge, where it was developed and used for the first time by Phelps Johnson of Montreal, President of the Dominion Bridge Company and by G. H. Duggan, Chief Engineer.[1] The writer does not know any case of its use for a bridge with parallel chords.

If the loads are vertical, the maximum stresses are easily found. In that case the only load at e is vertical; consequently the horizontal components in 5 and 6 are equal, and as these bars have equal slopes, their vertical components and their total stresses are equal, and each carries half the shear in the panel. Hence, maximum stresses are as follows:

[1] See *Trans. Can. Soc. C. E.*, 1918.

In 1, maximum V. C. is maximum shear in panel.

In 5 and 6, maximum V. C. is one-half maximum shear in panel.

In 13 and 14, maximum V. C. is one-half maximum shear in panel.

In 3, maximum stress is maximum panel load.

In 4, maximum stress is V. C. of maximum in 1.

In 11, maximum stress is V. C. of maximum in 6.

In 9, maximum stress is maximum difference between V. C. in 5 and load at lower joint.

In 15, maximum stress is maximum sum of V. C. in two diagonals and panel load.

In 2 and 7 maximum stress is maximum H. C. of 1.

Since bars like 5 and 6 have equal horizontal components, one tension and one compression, it follows that upper and lower chords in any panel have equal and opposite stresses.

SKEW OR UNSYMMETRICAL TRUSSES

41. Trusses are frequently unsymmetrical about the center of the span. This generally arises when there is a skew, that is, when the trusses are not at right angles to the abutments. This generally occurs where a railway or highway crosses another not at right angles. If the abutments are parallel to the lower highway, they will

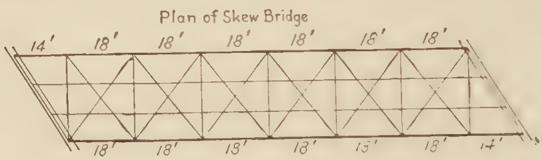


FIG. 69.

not be at right angles to the trusses. A skew bridge may always be avoided, and the abutments located not parallel to the lower highway, but this makes the clear span larger.

It is desirable that floor beams should be at right angles to trusses, so that connections will be convenient and stringers at right angles to floor beams, so that in skew bridges one end panel is often longer than the other. Figures 69, 70, and 71 show plans

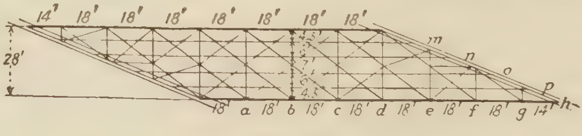


FIG. 70.

of skew bridges, the last two for double-track bridges. The diagonal lines represent the lower lateral bracing, which has not yet been discussed. In Fig. 70 there are three trusses, and in the others, two. Sometimes the abutments are not parallel, causing further complication. If the skew is great, several of the floor beams at each end of the span may rest at one end on the abutment, as shown in the figure.

The difficulties in computation which arise from an unsymmetrical arrangement are mainly with regard to the loads; the stresses may easily be found by the usual

methods. The loads on the end panels are not the same at the two ends of a truss, and are not easily found with accuracy. One wheel may rest on a stringer and the other wheel or the same axle may rest on the abutment. If exactness is desired, the dead load on each panel point may be found with accuracy after the structure is designed, and for the live load the influence line for the stress in any bar may be drawn, and the stress found from it, but it will be necessary to draw the influence line for a load *on each rail*. If a uniform live load is assumed, each stringer must be considered by itself at the ends of the span.

In Fig. 71 let us assume, for illustration, a live load of 3,600 pounds per foot *per track* (a light load for a railroad bridge), and let the total dead load of floor be assumed as 300 pounds per foot *per stringer* including the stringers, and let the floor beams be assumed to weigh 200 pounds per foot of their length.

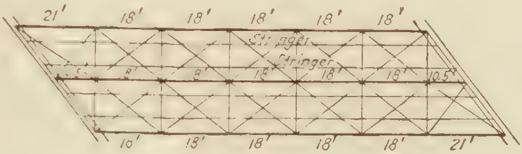


Fig. 71.

Of course, the floor beams do not all weigh the same, or the stringers either, but for simplicity let us assume that they do. Then the panel loads on the lower truss will be as follows:

On a: live, 3,600 × 18 = 64,800 pounds.
dead, 600 × 18 + 200 × 14 = 13,600 pounds from floor.

To this is to be added the dead load of the truss itself, and the lateral bracing, which we will assume as 600 pounds per pound per truss, equally divided between the two chords, or 300 × 18 = 5,400 pounds on each joint. Hence we have, at *a*:

Lower chord joint,	live,	64,800 pounds
	dead,	19,000 pounds
Upper chord joint,	live,	0
	dead,	5,400 pounds

On b and c: The loads are the same as on *a*.

On d: suppose the end stringers to project 3 feet beyond the line joining the end joints of the trusses; then we shall have the lengths of floor beams as follows: at *d*, 28 feet; at *e*, 21 feet; at *f*, 14 feet; at *g*, 7 feet if the floor-beam bearings are in the line joining the end joints of trusses; and for end stringer lengths we shall have, approximately:

$$m: \frac{4.5}{28} \times 68 + 3 = 11 + 3 = 14$$

$$n: \frac{10.5}{28} \times 68 + 3 - 18 = 10.5$$

$$o: \frac{17.5}{28} \times 68 + 3 - 36 = 9.5$$

$$p: \frac{23.5}{28} \times 68 + 3 - 54 = 6.0$$

Floor beam *d* is therefore loaded as shown in Fig. 72. The live load at the left stringer is $1{,}800 \times \frac{18 + 14}{2} = 28{,}800$ and the dead load $300 \times \frac{18 + 14}{2} = 4{,}800$.

Hence on joint d we have, approximately, on lower chord,

$$\text{live: } \frac{1}{28}[4.5 \times 28{,}800 + (10.5 + 17.5 + 23.5)32{,}400] = 64{,}200 \text{ pounds}$$

dead: $\frac{1}{28}[4.5 \times 4{,}800 + 51.5 \times 5{,}400] + 14 \times 200 + 5{,}400 = 18{,}900$ pounds and on top chord, 5,400 dead.

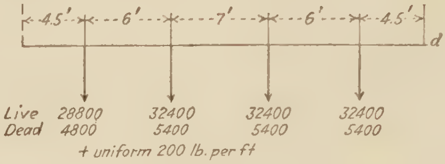


FIG. 72.

In a similar manner we shall find the following:

On e: lower chord: live 45,900; dead 15,200
upper chord: dead 5,400
On f: lower chord: live 28,000; dead 10,800
upper chord: dead 5,400
On g: lower chord: live 7,700; dead 6,800
upper chord: dead 4,800

The truss is then supposed loaded as shown in Fig. 73.

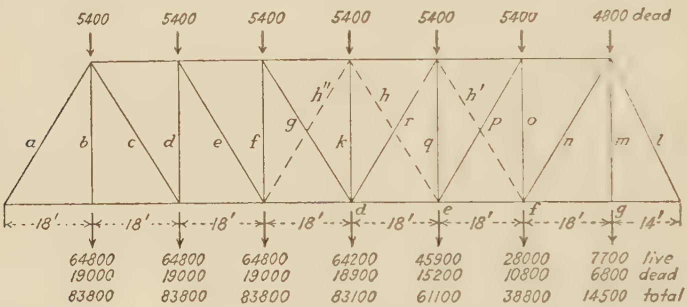


FIG. 73.

This loading (full) is unsymmetrical; hence the left reaction must be found and the shears obtained by subtraction. Writing the index figures on the web members, those for the chords are found by addition. For the maximum web stresses, if we designate the index number for the above full loading by the letter of the bar, we shall have (index numbers):

Maximum $(a) = a$; maximum $(b) = b$

$$\text{Maximum } (c) = c + \frac{18}{140} \cdot 64{,}800$$

$$\text{Maximum } (d) = d + \frac{18 + 36}{140} \cdot 64{,}800$$

$$\text{Maximum } (e) = e + \frac{18 + 36}{140} \cdot 64{,}800$$

$$\text{Maximum } (f) = f + \frac{18 + 36 + 54}{140} \cdot 64{,}800$$

$$\text{Maximum } (g) = g + \frac{18 + 36 + 54}{140} \cdot 64{,}800$$

$$\text{Maximum } (h) \;\; = -r + \frac{18 + 36 + 54}{140}\,64{,}800 + \frac{72}{140}\cdot 64{,}200$$

$$\text{Maximum } (h') \;\; = -p + \frac{18 + 36 + 54}{140}\,64{,}800 + \frac{72}{140}\cdot 64{,}200 + \frac{90}{140}\,.45{,}900$$

$$\text{Maximum } (h'') = -g + \frac{14}{140}\cdot 7{,}700 + \frac{18 + 14}{140}\cdot 28{,}000 + \frac{50}{140}\cdot 45{,}900 + \frac{68}{140}\cdot 64{,}200$$

Maximum (k) = either maximum (h) + 5,400, or maximum (h'') + 5,400

$$\text{Maximum } (n) \;\; = n + \frac{14}{140}\cdot 7{,}700$$

and so on.

Or the web stresses for full loading need not be found, but the maximum stresses at once; thus

$$\text{Maximum } (e) = \frac{1}{140}[14 \times 19{,}300 + 32 \times 44{,}200 + 50 \times 66{,}500 + 68 \times 88{,}500 + 86 \times 89{,}200 - (18 + 36)24{,}400]$$

This example has been illustrated in detail, that the reader may see the principles involved. In practice, exactness in a case of this kind is neither necessary nor attainable.

41. Stresses in Trusses by Influence Lines.[1]—The stress in any bar of a truss may be found by drawing the influence line for that stress. The stress for a uniform load will be obtained from the area between the influence line and the axis; that for a system of concentrated loads by multiplying each load by the ordinate to the influence line at the point where the load acts, and adding the results. This method, however, is seldom as convenient as those that have been explained.

42. Verticals. Influence Lines if There are Counters.—If a structure has two bars whose relation to each other is that when one is in action the other is not in action, the influence lines for these bars, and for others that are affected, differ from the usual influence line. Consider the main diagonal $a'b$ and its counter ab' (Fig. 74). The influence line $LgkR$ is not, strictly speaking, the influence line for the vertical component of the stress in $a'b$; that is, each ordinate does not show the effect of a unit load on $a'b$, because when a unit load is to the left of the neutral point, the counter is in action and there is no stress in $a'b$, or the true influence line is $LhkR$. Similarly, for the counter the influence line is $LghR$. The line $LgkR$ is the influence line for $a'b$ only on the assumption that the counter does not act, or it is the influence line for ab' assuming that $a'b$ does not act. This is the line that must be used, because for any given loading either the main diagonal or the counter is in action, and all loads on the span produce stress in it, since they do not act singly; and (this is the important point) the section taken is the same in either case. Really, the influence line is that for the shear in the panel, which we may use understandingly.

The matter is entirely different in the case of the vertical bb'. Here there are four possibilities:

1. Main diagonals $a'b$ and $b'c$ in action.
2. Counters ab' and bc' in action.
3. Main diagonal $a'b$ and counter bc' in action.
4. Main diagonal $b'c$ and counter ab' in action.

For the first case the section would be parallel to the main diagonals, and the influence line for the vertical would be $L'k'n'R'$. For the second case the section would be

[1] See the writer's article in *Trans. Am. Soc. C. E.*, 1887.

sloping the other way, and the influence line for the vertical would be $LgkR$. In the third case, the section would be around the joint b', and only a load at that joint would cause stress in the vertical; in the fourth case the section would be around the joint b. Obviously, for any given loading, the stress in the bar cannot be found until we find which diagonals are in action. The proper procedure is to assume certain diagonals in action, for instance, the main diagonals, to find the loading giving maximum stress

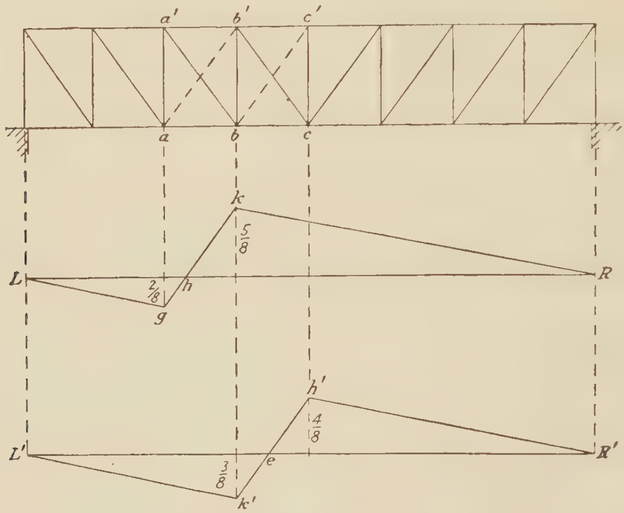


FIG. 74.

and that maximum stress, and then to see if for that loading those diagonals are in action. If they are, we have the maximum stress for that condition; if they are not, we must find the maximum stress *consistent* with those bars being in action.

In this truss case 4 is impossible because it means positive shear in panel bc and negative shear in ab. Case 1 when the vertical is to the left of the center will give a larger stress in it than case 2. Case 3 gives the stress equal to maximum panel load, which for a deck bridge may be large. Thus there are for this truss only two cases to consider.

CHAPTER V

TRUSSES WITH PARALLEL CHORDS AND MULTIPLE-WEB SYSTEMS

THE WHIPPLE TRUSS

1. Object.—We have seen that, in order to subdivide a long panel, the new joint may be connected with any old joint or with a new joint, and the latter connected with an old joint. This process may be carried farther. The new joint may be connected with another new joint, this with a third new joint, and so on, connecting the last new joint with an old joint. Such a system will be statically determined. Thus in Fig. 75 the full lines represent a Pratt truss of six panels; new joint a may be connected to new joint b, this with c, and so on till g is connected to old joint h. The vertical at e is a single secondary like one on the Warren truss with verticals. The above truss is statically determined, but if a is connected with old joint k the truss is statically undetermined, with one superfluous bar. It really consists of two separate trusses superposed, the full-line system of web and the dotted system of web, with chords and end posts in common. The bar cl is a counter to de.

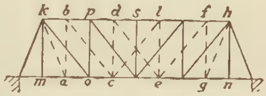


FIG. 75.

This truss is called the Whipple truss, and was invented in 1852 by Squire Whipple. It attains the object of subdividing the long panels of the Pratt truss, but at the disadvantage of making the truss indeterminate. In order to find the stresses it is necessary to use the theory of superfluous bars, which is very complicated and also likely to be inaccurate, on account of variations in the lengths of the bars from the theoretically correct lengths, or else to make assumptions. The latter course is usually followed.

The assumption commonly made is that each web system acts independent of the other, and carries the loads on the joints of its own system, the bars belonging to both systems carrying the stresses of both. So far this is simple. But there are joints, and loads at them, belonging to both systems, such as k and h; m and n also really belong to both systems, since the loads at these points are carried by the verticals to k and h just the same as if the loads acted there originally. What shall be done with the loads at such joints, of which there are but these four in this truss? Such a load may be considered as carried entirely by either system, in such a way as to produce maximum stress in any bar considered, or it

may be considered as carried one-half by each system. The last assumption will generally be made in what follows.

But the case is not quite so simple as this, for the load (shear) carried by the bar ak is brought to the joint k just as is the load carried by mk, and when it arrives there, it may be distributed on the two systems. In other words, eight-tenths of the load at a certainly goes to the left abutment, but instead of the shear in ka being eight-tenths, the entire load at a or even more (including some that comes from c) may go through ak, and some of it go back through ko, diminishing the tension in the latter bar. It is also easy to see, even supposing there were no loads on k, m, h, or n, that the two-web systems cannot be truly independent, for the assumption that the full-line system acts by itself means that the chord bars kp and ps are straight, with angles at p, s, etc.; while the assumption that the dotted system acts by itself means that the chord bars kb and bd are straight, with angles at b, d, etc., and these results are contradictory. It is therefore obvious that the two-web systems must affect each other. There would be stresses in both even if there were loads only on one. Nevertheless, we shall compute the truss as though they acted independently. Another method would be to assume that the shear in any panel is equally divided between the two diagonals cut by a vertical section.

2. Example (Fig. 76).—The typical panel loads are indicated on one joint, dead, live uniform, and excess, in kips. The index figures on the diagram are for the dead and uniform live loads, without excesses. The chord-index figures are to be multiplied by p/h, the web figures by l/h or l_1/h. Here again, as in the case of the Warren truss with tertiary system, if the height is 25 feet, the long diagonals are too flat, and the height should be greater or panel length less.

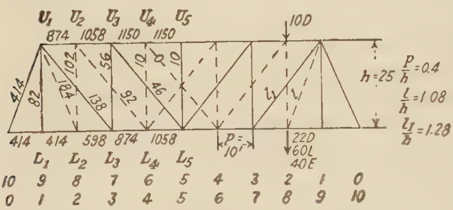


Fig. 76.

If excess loads are used for the chords, the excess stresses are, with two excesses spaced three panels apart:

$$U_1U_2: \frac{8+5}{10} \cdot E \cdot \frac{p}{h} + \frac{8}{10} E \frac{p}{h} + \frac{5}{10} E \cdot \frac{2p}{h}$$

$$U_2U_3: \text{excesses on 7 and 4: } \frac{11}{10} \cdot E \cdot \frac{p}{h} + \frac{7}{10} E \cdot \frac{2p}{h} + \frac{4}{10} E \cdot \frac{(2+1)p}{h}$$

$$U_3U_4: \text{excesses on 6 and 3: } \frac{9}{10} E \cdot \frac{p}{h} + \frac{6}{10} E \frac{(2+1p)}{h} + \frac{3}{10} E \frac{(2+2)p}{h}$$

$$U_4U_5: \text{excesses on 5 and 2: } \frac{7}{10} E \frac{p}{h} + \frac{5}{10} E \frac{(2+2)p}{h} + \frac{2}{10} E \frac{(2+2+1)p}{h}$$

The foregoing expressions are obtained by following the stresses through the web. It is simpler to use the straight method of moments. Thus for U_4U_5 the origins of moments are for the heavy system, L_5, and for the dotted system, L_4'. With excesses at 5 and 2, for the former we have $5/10E \cdot 5p/h$; for the latter, $2/10E \cdot 6p/h$, the sum of which agrees with the first result.

Lower chord excess stresses may be found in the same way; thus, for L_3L_4, origin of moments for heavy system is at U_3, and for light system at U_2. Placing the excesses at L_2 and L_5, the stress in L_3L_4 is

$$\frac{8}{10}E \cdot \frac{2p}{h} + \frac{5}{10}E \cdot \frac{3p}{h}, \text{ and this is the maximum.}$$

For the web members, the maximum total stresses are:

U_1L_1: maximum panel load

U_1L_0: $\left(414 + \frac{15}{10}E\right)\frac{l}{h}$

U_1L_2: This may be taken as

$$\frac{8 + 6 + 4 + 2 + 0.5}{10} \cdot 92 + \frac{8}{10} \cdot 40 - \frac{0.5}{10} \cdot 32 = 219$$

or, if loads at indeterminate joints are treated so as to obtain maximum possible stresses, so far as they are concerned,

$$\frac{8 + 6 + 4 + 2 + 1}{10} \cdot 92 + \frac{8}{10} \cdot 40 = 225.2$$

It is better not to find web stresses in this truss by starting with full loading and removing live loads to the left, but to find directly the maximum shear in the bar, as above.

It is unnecessary to find the remaining stresses here.

3. Remarks.—The Whipple truss was generally built with inclined end posts, but sometimes with vertical end posts. Its use has been practically abandoned in this country, as it offers no advantages that cannot be obtained by the use of more statically determined forms. Some 40 or 50 years ago it was much used, and when combined with typical details bore more specific names, such as the Linville truss, etc.

4. The Double Warren Truss.—Figure 77 shows a double Warren truss, consisting of two superposed Warren trusses which are assumed to act separately. This is taken as a deck bridge, and the typical joint loads are shown, with the index stresses for dead and uniform live loads.

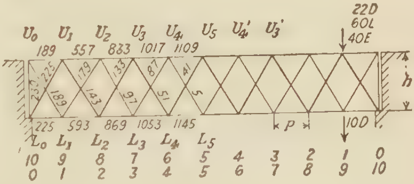


FIG. 77.

To find the excess stresses in any top chord piece, such as U_2U_3, the origin of moments is at L_2 for the heavy system, and at L_3 for the light system. With excesses at U_2 and U_5 the stress will be, calling each excess load E,

$$\frac{2}{10}E \cdot \frac{7p}{h} + \frac{5}{10}E \cdot \frac{2p}{h} = \frac{24}{10}E \cdot \frac{p}{h}$$

With excesses at U_3 and U_4', the stress will be

$$\frac{7}{10}E \cdot \frac{2p}{h} + \frac{4}{10}E \cdot \frac{3p}{h} = \frac{26}{10}E \cdot \frac{p}{h}, \text{ or larger than before.}$$

For a bottom chord piece, such as L_3L_4, the origin of moments will be at U_4 (light system) or U_3 (heavy system). With loads at U_4 and U_3', the stress will be

$$\frac{6}{10}E\cdot\frac{4p}{h}+\frac{3}{10}E\cdot\frac{3p}{h}=\frac{33}{10}E\cdot\frac{p}{h}$$

With loads at U_3 and U_4' the stress will be

$$\frac{7}{10}E\cdot\frac{3p}{h}+\frac{4}{10}E\cdot\frac{4p}{h}=\frac{37}{10}E\cdot\frac{p}{h}, \text{ or larger than before.}$$

For a web piece such as L_2U_3, the joints on the right up to U_3 must be loaded, with excess loads on U_3 and U_4, and we shall have, for the numerical loads shown,

$$\text{maximum } +S=\frac{1+3+5+7}{10}\cdot 82+\frac{7}{10}\cdot 40+\frac{2+4+6}{10}\cdot 10-\frac{1}{10}\cdot 22-\frac{2}{10}\cdot 10=167$$

$$\text{maximum } -S=\frac{1}{10}\cdot 82+\frac{1}{10}\cdot 40+\frac{2}{10}\cdot 10-\frac{1+3+5+7}{10}\cdot 22-\frac{2+4+6}{10}\cdot 10=-33$$

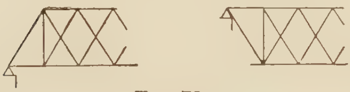


Fig. 78.

There is no negative shear; the maximum compression in the bar is $167l/h$, and the minimum compression is $33p/h$; there is never tension for these loads.

Other web bars may be computed similarly.

The double Warren truss has been much used. The ends are sometimes made as shown in Fig. 78.

TREBLE AND QUADRUPLE SYSTEM WARREN TRUSSES

5. Quadruple System.—This form is quite common, though not built now as often as it was a generation ago. It is shown in Fig. 79. As shown in Art. 1, each additional system results in one additional redundant member, making the truss more statically undetermined. The four systems of the truss shown are distinguished, one light, one heavy, one dotted, one dot and dash. Two only of these are symmetrical. Beginning with these, starting at the center, the index figures for full loading (live and uniform dead) are written on the webs and end post. For the heavy

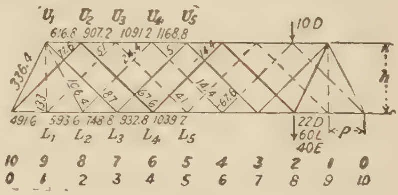


Fig. 79.

system the left reaction is $\frac{2+6}{10}\cdot 82+\frac{4+8}{10}\cdot 10=77.6$. The two dead loads at the top of end posts are not regarded as belonging to any one system, but to be carried directly down the end posts without affecting the web between, and they are written first on the end posts.

It is obvious that in the heavy system the load at 6 splits, 67.6 going to the left and 14.4 to the right, causing negative shears in the bars. The index figures on the web pieces are given with their proper sign, signifying the kind of shear. The dot-and-dash system is the heavy system reversed, and the figures may be written at once if the figures on the heavy system are carried through from end to end. In any vertical section of the truss, the total shear will be found balanced by the stresses in diagonals cut. The end reaction must be $4\frac{1}{2} \times 92 = 414$, which equals $336.4 + 77.6$. After the chord-index figures are put on, a section should always be taken just to one side of the center, to see if $\Sigma H = 0$, which it does here. The first upper chord index is $336.4 + 106.4 + 87 \times 2 = 616.8$; and so on. If the index figures on the web are given signs corresponding to the shear in the bars, the figures for the chords may be written mechanically by following the signs; thus, for L_4L_5, $932.8 + 67.6 \times 2 - 14.4 \times 2 = 1{,}039.2$. The reader should draw a diagram of a joint and prove the truth of the above statement, then adopt the rule to write the figures on the web with the proper signs, and then write the chord figures mechanically, without considering each joint in detail with reference to the directions of the forces.

The *maximum positive* shear in a web piece, such as U_3L_5 will be $\frac{1+5}{10}\cdot 82 + \frac{3}{10}\cdot 10 + \frac{5}{10}\cdot 40 - \frac{1}{10}\cdot 22 - \frac{3}{10}\cdot 10 = 67$. Here the loads of 10 at the hip joints (top of end posts) are not considered, as already explained. The *maximum negative shear* will be $\frac{1}{10}\cdot 82 + \frac{3}{10}\cdot 10 - \frac{6}{10}\cdot 22 - \frac{3}{10}\cdot 10 = -5$; which, being negative, shows that there is no negative shear. The stress in this bar is always tension; its maximum value is $67l/h$, and its minimum value $5l/h$. In some bars, such as U_1L_1, L_0U_2, U_1L_2, and U_0L_3 the minimum stress is that due to the dead load alone, since the live load cannot produce a stress of the opposite kind.

6. Treble System.—This form (Fig. 80) is not common, but the writer has met with several of them in his practice. The top chord joints of a through truss of this kind are over the centers of the bottom chord panels, and *vice versa* for deck bridges. The horizontal projection or "run" of a diagonal (except of two at each end) is one and one-half panel lengths; and of the two at each end, half a panel length. These facts must not be forgotten in writing index figures on the chords; otherwise this truss involves nothing new.

Fig. 80.

OTHER FORMS OF TRUSS

7. Double Warren with Secondary Verticals.—If the panels of a double Warren truss are too long, secondary verticals may be added, running to the intersection of two diagonals, as shown in Fig. 81.

We have seen that any double-system truss is statically undetermined, with one superfluous bar. We ascertained this by considering the effect of adding a new joint to a statically determined truss, and so building up another system. The same result is found by counting bars and joints; thus the truss in Fig. 81, if there were no secondary verticals, would have 12 joints, and 22 bars, instead of $2 \times 12 - 3 = 21$, which it should have, to be statically determined. Note here that the diagonals are not supposed to be connected at their intersections. As a matter of fact, they generally are so connected. Each such connection forms another joint, and adds two bars, by dividing the two diagonals, and so does not add to the degree of indetermination. Thus, if so connected, there are 22 joints and 42 bars, or still one too many. The addition of the secondary verticals does, however, add to the uncertainty of *calculation*, by introducing joints that are common to both systems. While in the ordinary double Warren truss with end verticals there are no uncertain joints belonging to both systems, if verticals are added every other joint becomes uncertain. The loads on these verticals must be considered as going partly into each system, and there is no apparent reason why they should not be considered as divided equally. Hence the index figures will be as shown in the figure. The reaction is $5 \times 82 + 2 \times 10 = 430$, which checks with $256 + 174 = 430$. The section at the middle checks as to the horizontal forces. The maximum in U_2M_3 will be

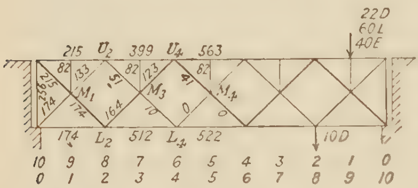


FIG. 81.

$$\text{maximum } +S \text{ in } U_2M_3 = \frac{4}{10} \cdot 82 + \frac{1}{2} \cdot \frac{1+3+5+7}{10} \cdot 82 + \frac{1}{2} \cdot \frac{7}{10} \cdot 40 + \frac{4}{10} \cdot 40 + \frac{8}{10} \cdot 10 - \frac{2}{10} \cdot 22 - \frac{1}{2} \cdot \frac{1}{10} \cdot 22 = 130.9$$

$$\text{maximum } -S \text{ in } U_2M_3 = \frac{2}{10} \cdot 82 + \frac{2}{10} \cdot 40 + \frac{1}{2} \cdot \frac{1}{10} \cdot 82 - \frac{4}{10} \cdot 22 - \frac{1}{2} \cdot \frac{16}{10} \cdot 22 - \frac{8}{10} \cdot 10 = -5.9$$

Hence there is no negative shear; the maximum tension is $130.9l/h$, and the minimum tension is $5.9l/h$.

Of course, a larger stress may be found if the loads on the uncertain joints are differently distributed.

The writer does not believe this truss advisable. He believes in making frames as nearly statically determined as practicable, with due regard to all circumstances. Trusses such as this are seldom if ever built now in this country, but have been considerably in favor in the past.

The long bridge of the Pennsylvania Railroad over the Susquehanna near Harrisburg was of this type until replaced by a stone arch.

8. Small trusses are frequently built with double Warren web systems (Fig. 82). There are many of these on the elevated railways of New York. Here the panels are short and the ties rest directly on the top chords. Such trusses are often used also in the portal and transverse frames of bridges (Fig. 83). Here it is sufficient to assume the transverse shear equally divided between the two diagonals cut by a transverse section. The chord stress is then found by taking moments about the

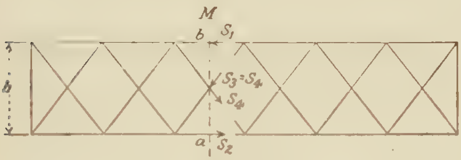


FIG. 82.

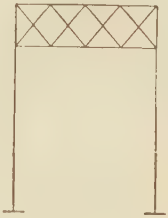


FIG. 83.

point a or b of a section through the intersection of the diagonals. The two diagonals will have equal and opposite moments about this point, and the chord stresses will be $S_1 = M_a/h$; $S_2 = M_b/h$, if M_a and M_b are the moments about a and b respectively of the outer forces on either side of the section. If there are longitudinal forces acting, as there are in a portal frame, these moments need not be equal.

9. Quadruple-system Warren Truss with End Verticals.—There are several ways of arranging the ends of such trusses, as shown in Fig. 84. The second method results in a bending moment in the end post, because the stresses will not be equal in the two diagonals which meet at its center. This is prevented by inserting the bar ab, which carries to b any unbalanced horizontal component at a. All this, of course, increases the indetermination.

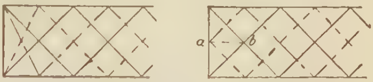


FIG. 84.

10. Lattice Trusses.—Trusses have frequently been built, both of wood and of metal, with a great number of web systems, sometimes with panels not over 2 or 3 feet long. If of metal, the web members were flat bars, riveted together at all intersections and stiffened by stiff verticals running from chord to chord. The floor beams in through bridges could be attached to these verticals, or otherwise supported on the bottom chord. In deck bridges the floor was supported on the top chord. In this country, the Town lattice truss, invented in 1920, was made entirely of plank, the web members connected at intersections by wooden pins, and the chords consisting of several layers of planks, between which the web

members passed and extended some distance beyond the chords to give area enough to prevent shearing out of the wooden pins by which they were connected to the chords. Each chord frequently consisted of two horizontal lines of plank several feet apart vertically. Figure 85 shows a view of such a bridge. Even in recent years wooden bridges have been built of this type for railroad service, and many of them are still in use in New England.

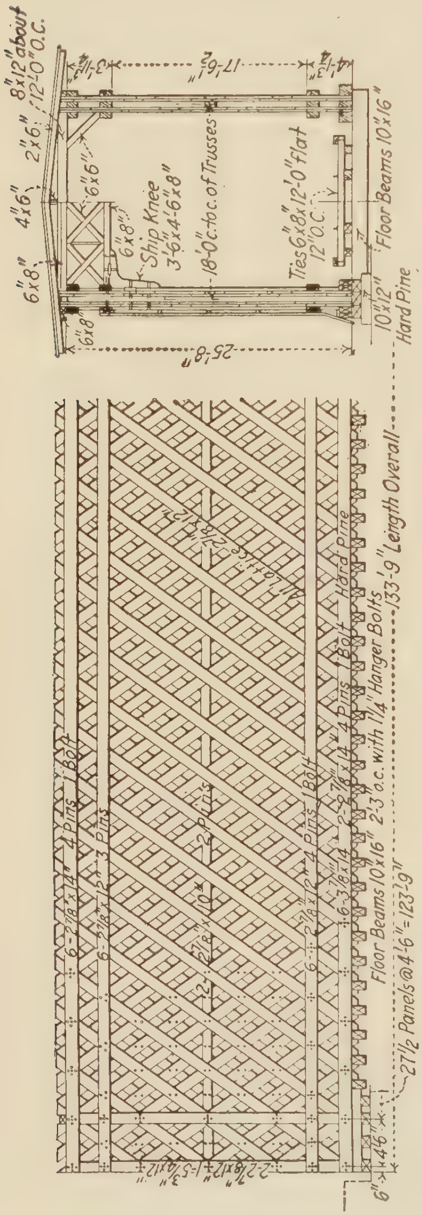


Fig. 85.

Lattice bridges of iron have mostly been used in Europe. They were an early development of the plate girder, and were built when bridge works were not well equipped to construct large and heavy members such as are used in modern trusses. The bridge over the Weichsel at Dirschau, with six spans of about 430 feet each, is typical of this kind of structure, and had the longest spans built of this type. There were three continuous spans, each over two openings. The distance between web intersections on the chords was not over 3 feet. The only way to compute these webs is to assume the shear on a vertical section uniformly distributed over all the web members cut. Such bridges are, happily, no longer built.

The structural engineer will sometimes have to deal with lattice bridges if his practice involves reporting on existing structures, and he must be able to form some idea of the stresses in them. The stress in the web may be found by dividing the total shear on a section by the number of web members cut. The compression pieces are very indeterminate,

being riveted at each intersection and to the vertical stiffeners, and must be considered somewhat like the web of a plate girder. The chord stress is found by dividing the moment by the distance between chords. If the chords are double, and if the depth between centers of inside chords is h_1 and between centers of outside chords h_2, and if they have the same area, then if T is the stress in either outside chord, that in either inside chord will be Th_1/h_2, and the external moment M equals $Th_2 + Th_1^2/h_2$.

11. The Post Truss.—Among bridges still occasionally found, but no longer built, is the truss invented in 1865 by S. S. Post. It is shown in Fig. 86. It was based on the idea that the web struts should be inclined, as well as the web ties, in order to carry the load toward the abutments. The inclination of the posts was $\tan^{-1} \frac{1}{3}$, and of the diagonals 45°, the height being one and one-half panel lengths, and the top chord joints

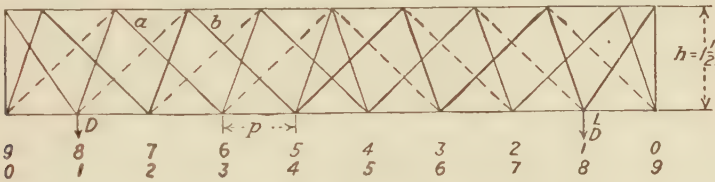


FIG. 86.—Post truss.

over the middle of the bottom chord panels. Counters were carried to the end, and also were inclined at 45°. Two posts came together at the middle of the top chord. If the height was not one and one-half panel lengths, the design was sometimes modified and the inclinations changed.

This truss is very indeterminate, as the two systems of webs cannot be separated, but come together at the center. We may assume, as an extreme,

$$\text{Maximum shear in } a = \frac{1 + 2 + 3 + 4 + 6}{9}(L + D) + \frac{6}{9}E - \frac{1}{9}D$$

$$\text{Maximum shear in } b = \frac{1 + 2 + 3 + 4 + 5}{9}(L + D) + \frac{5}{9}E - \frac{2}{9}D$$

There is nothing to recommend this truss that cannot be obtained in a better and more economical way, and it is not now built, though it was built quite extensively between 1870 and 1880.

12. Computation of Multiple-system Trusses for Concentrated Loads.—There are two ways in which the stresses in trusses having multiple systems may be found for a set of concentrated loads: (*a*) by influence lines and (*b*) by using panel concentrations. Both rest on assumptions and involve uncertainties regarding loads which act at joints common to several web systems.

a. By Influence Lines.—If the influence line for a bar is drawn, the stress may be found for a series of concentrated loads by the formula $S = \Sigma Py$, or by multiplying each load by the ordinate to the influence line at the point where the load acts, and

adding the results. The loading giving maximum stress can be found only by inspection and trial, bearing in mind the principles already explained. In Fig. 87 is shown a Whipple truss, and influence lines for stresses in bars a and b.

For bar a, when a unit load is at 4, the stress is $\frac{4}{8} \times 4p/h = \frac{16}{8}p/h$. For a load at 5 (upper line of numbers), the origin is at 5, and the stress is $\frac{5}{8} \times 3p/h = \frac{15}{8}p/h$. The ordinate at v may be laid off as 16, and that at u as 15, to some scale, all ordinates to be multiplied by $p/8h$ to obtain stress. Joining u and v with A and B, all ordinates will be on these lines, AuB referring to the light web system and AvB to the heavy. Thus for a load at 3, the stress will be $\frac{3}{8} \times 3p/h$, the origin being at 5. From u and v to A and B the influence line will zigzag between AuB and AvB, except that there is an uncertainty regarding points 1 and 7. If a load at point 1 is assumed to be entirely on the light system, the stress will be $\frac{1}{8} \times 3p/h$, and the point of the influence line will be on the line uB; if the load is assumed on the heavy system,

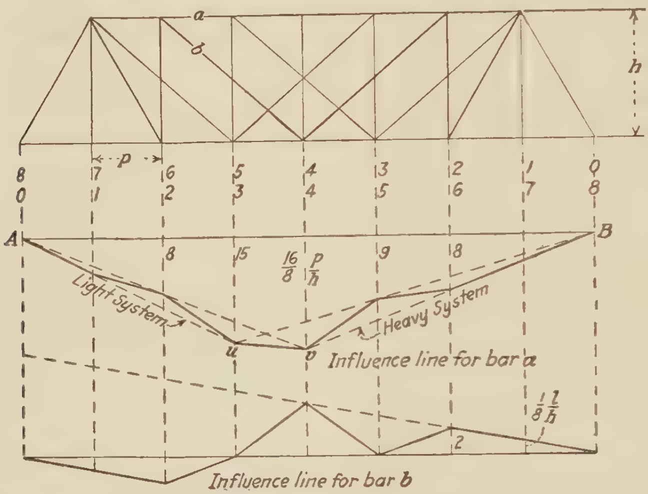


Fig. 87.

the point will be on vB; if it is assumed half on each, it will be midway between. In the figure, the most unfavorable case or largest stress is assumed.

The influence line for bar b is similarly drawn, no stress being assumed when the load is on the light system, and the joints next to the end being assumed to belong to the heavy system in computing a bar of that system, though this last assumption is illogical, and it would be better to assume half and half.

To find the loading causing the maximum stress in a, it is evident that the heaviest loads must be as near point 4 as possible, with probably one at that point. The skeleton of the truss and the influence line being drawn to scale, and the loads being plotted to the same scale on tracing cloth, the loading causing maximum stress may be found by trial, placing the system of loads with one of the heaviest at point 4 and finding (graphically) ΣPy, then trying another load at point 4, and so on until the maximum is found. The same method may be used for all the bars of the truss, and it is equally accurate for square or skew bridges, the influence line being drawn in such a way as to take account of the actual arrangement of floor beams and stringers at the ends. This method is no doubt the most accurate of any which can be devised, and with practice it is rapid in execution.

b. By Finding Panel Concentrations.—This method is best illustrated by an actual example. It is to be remarked, however, that it is used only in computing the web

members. It is not generally necessary to compute the chords for the actual loads, and when it is, the method by influence lines is to be preferred. Let us consider the truss shown in Fig. 88, and the system of loads shown below it. The floor beams in the lower figure are numbered, and they must be considered as movable with the loads, so that they may be placed anywhere on the truss. We first imagine the load so placed that a floor beam may come beneath the second driver, and we then find the load coming on each floor beam; these are called the panel concentrations. Then suppose we want the live stress in a when this second driver is beneath it; this stress is

$$\frac{5}{6} \times 17.4 + \frac{3}{6} \times 10.8 + \frac{1}{6} \times 16.53$$

The dead load may be found separately and its stresses throughout the truss computed and added. The above is not necessarily the maximum stress in a, and we might therefore calculate another system of panel concentrations, starting with the first driver over the floor beam, and perhaps still a third system with the third driver over a floor beam; these once computed, the corresponding stress in a, with the said

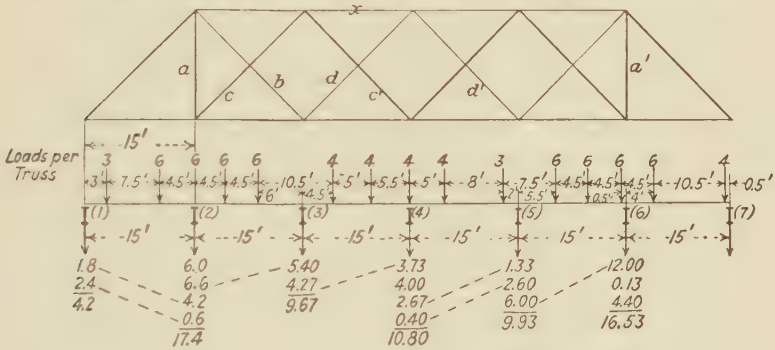


FIG. 88.

first or third driver just below it, could be found in the same way as shown above. When the second numbered floor beam is taken as the one below bar a, the loads on the odd numbered floor beams in the lower figure are at joints of the light system and are assumed to cause no stress in bar a. Now it is obvious, on inspection, that if the third driver is placed over the second floor beam, the load on that floor beam will be about the same as shown in the figure, while the loads on the fourth and sixth floor beams will be reduced, so that a loading with the third driver at a floor beam need not be considered. With the first driver at the second floor beam, the load on this floor beam would be decreased, but that on the sixth increased, while that on the fourth would not be materially changed. The panel concentrations for this loading should be worked out, and the stress compared with that found above. For the above load concentrations we have, also, the following:

Stress in b $= \frac{4}{6} \times 17.4 + \frac{2}{6} \times 10.8$ V. C. live alone
Stress in c and c' $= \frac{3}{6} \times 17.4 + \frac{1}{6} \times 10.8$ V. C. live alone
Stress in d and d' $= \frac{2}{6} \times 17.4$ V. C. live alone, positive shear

If there were any load on the floor beam to the left of point 1, this load would be considered, giving a negative shear for c and d.

This method, therefore, consists in finding panel concentrations, and then, beginning at the left-hand end of the span, finding the web stresses by backing the engine gradually off the bridge. The dead stresses may be found separately, and added to

the live, or the panel concentrations may include the dead loads, and the total stresses found at once; in this case, of course, there are subtractive terms for the loads to the left of the bar considered. It is possible, in double systems, that the maximum stress in a bar may occur when no load lies at the first joint, but the process just explained gives results that are practically correct.

13. Unsymmetrical and Skew Trusses with Double Systems. *Actual Loads.*—The method by use of influence lines may here be used without modification; the influence line is of course not the same, at one end, as it would be if the bridge were square or symmetrical. The method by panel concentrations may also be used without difficulty; it is only necessary, for each bar considered, to see that the panel concentrations are such as would actually exist for the load system in the position assumed. The end concentrations will therefore be different for different bars. It is also to be remarked that in unsymmetrical and skew trusses the chord stresses are not symmetrical; that is, the stresses in the two end pieces of bottom (or top) chord are not the same; the stresses in diagonals are also not symmetrical. Hence in all structures of this kind, the stresses throughout the truss must be computed; that is to say, it is not sufficient to compute the chord and web stresses on one side of the middle, but the stress in each separate bar must be found. Thus, in the truss of Fig. 88, if unsymmetrical or skew, the stresses in a and a' will not be the same.

Uniform Loads.—The method explained on a previous page, for simple trusses, is equally applicable here. For a given load per foot on each track, with known lengths of floor beams and stringers, and methods of supporting them, the student will have no difficulty in computing the live and dead loads on each joint, and the maximum and minimum stresses resulting therefrom.

CHAPTER VI

THE ANALYTICAL METHOD OF JOINTS

1. The methods used in Chaps. IV and V for the computation of trusses with parallel chords have been combined methods, using sometimes the method of joints and sometimes the method of moments, as convenience dictated. An example will now be given of the use of the method of joints for the computation of an entire truss.

Since by this method it is necessary to begin at a joint where only two bars meet, and then proceed from joint to joint, so that the stress in a given bar cannot be found without previously finding the stresses in several, perhaps many, other bars, it is obvious that this method in its entirety, *should be used only when the maximum stresses in all the bars occur for the same loading*, or at least where there are very few cases of loading to be considered. It would be foolish to use it in cases where the maximum stress in each bar occurs under a different loading. Hence it would never be used in railroad or highway bridge trusses.

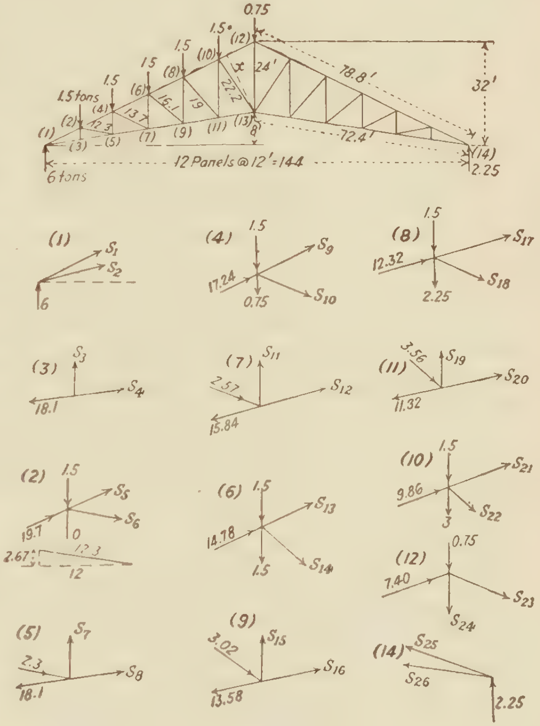


FIG. 89.—Roof truss by method of joints.

For roof trusses, however, there are generally, in fact, few cases of loading, namely, (1) dead load, (2) snow load on one side, (3) wind load on the right, and (4) wind load on the left. From these four loadings, the maximum stress in every bar may generally be obtained, by proper combinations of the results. The case chosen for illustration, therefore, is that of a roof truss of common form (Fig. 89).

2. The method is a very simple one. By it, by taking a section around any joint, the stresses in the bars meeting at that joint may be found, *provided* only two of them are unknown, since there are two equations, $\Sigma H = 0$ and $\Sigma V = 0$. The outer forces (loads and reactions) on the truss must all be first found. Then commence at some joint *where only two bars* meet, since none of the inner forces are known. Having found these two, proceed to another joint, where more than two bars meet, but where all but two of them are bars whose stress has just been found. In using this method, use no general formulae, but proceed numerically in each case, observing the following directions:

1. Indicate by an arrow the *assumed* direction of stress in each bar, always assuming unknown stresses as tension, *i.e.*, acting *from* the section; known stresses are, of course, indicated as they actually are. Thus, in Fig. 90, suppose that the stress in the lower left-hand bar has been found, and is a tension of 10 tons, and that in the left-hand diagonal it has also been found and in compression of 6 tons. Indicate all the stresses as shown.

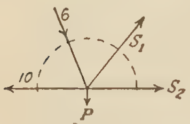


Fig. 90.

2. Assume certain directions as positive, and write out numerically the two conditions, thus, downward being + and to the right +.

$$\Sigma V = 0: \text{(V. C. of 6)} - \text{(V. C of } S_1) + P = 0$$
$$\Sigma H = 0: \text{(H. C. of 6)} + \text{(H. C. of } S_1) + S_2 - 10 = 0$$

3. *Always* write the equations in the form ΣH (or ΣV) $= 0$.

4. Solving these equations, find S_1 and S_2. If S_1 is positive, it means that it acts *in the direction assumed;* if negative, in the opposite direction. If the unknown stresses are always assumed to be tension, then if any stress comes positive, it is tension, and if negative, it is compression. These directions should be strictly observed; otherwise errors will be likely to arise.

3. Example (Fig. 89).—Let the truss be computed for a snow load on one side, the apex loads being as shown, and the reactions having been computed. First number the joints. Then, commencing with joint 1, sketch and compute each:

Joint 1:

$$\Sigma V = 0 \text{ gives } 6 + S_1\frac{32}{78.8} + S_2\frac{8}{72.4} = 0$$

$$\Sigma H = 0 \quad \text{gives} \qquad S_1\frac{72}{78.8} + S_2\frac{72}{72.4} = 0$$

$$54 + \frac{216}{78.8}S_1 = 0;\ S_1 = -19.7 \text{ (compression)}$$

$$13.5 - \frac{54}{72.4}S_2 = 0;\ S_2 = +18.1 \text{ (tension)}$$

Joint 3:

$$S_3 = 0;\ S_4 = +18.1$$

Note that the rectangular axes need not be taken horizontal and vertical, but in any directions; here if we take them parallel and perpendicular to S_4, we have at once $S_3 = 0$; hence $S_4 = +18.1$.

Joint 2:

$$\Sigma V = 0 \text{ gives } 19.7 \times \frac{32}{78.8} + S_5 \frac{32}{78.8} - 1.5 - S_6 \frac{2.67}{12.3} = 0$$

$$\Sigma H = 0 \text{ gives } 19.7 \times \frac{72}{78.8} + S_5 \frac{72}{78.8} \qquad + S_6 \frac{12}{12.3} = 0$$

$$-1.5 - \frac{8}{12.3} S_1 = 0; \; S_6 = -2.3 \text{ (compression)}$$

$$S_5 = -19.7 + \frac{78.8}{72} \times \frac{12}{12.3} \times \frac{12.3 \times 1.5}{8} = -17.24$$

This joint may also be solved by taking axes parallel and perpendicular to S_5: thus, find the distance from joint 13 normal to the top chord, and then resolve S_6 parallel and perpendicular to S_5. The above is simpler in this case.

Joint 5:

$$\Sigma V = 0 \text{ gives } 18.1 \times \frac{8}{72.4} - S_8 \times \frac{8}{72.4} - S_7 + 2.3 \times \frac{2.67}{12.3} = 0$$

$$\Sigma H = 0 \text{ gives } 18.1 \times \frac{72}{72.4} - S_8 \frac{72}{72.4} - 2.3 \times \frac{12}{12.3} = 0$$

$$-9S_7 + 2.3 \times \frac{36}{12.3} = 0$$

$$S_7 = +0.75 \text{ (tension)}$$

$$S_8 = 18.1 - 2.3 \times \frac{12}{12.3} \times \frac{72.4}{72} = 15.84 \text{ (tension)}$$

Joint 4:

$$17.24 \times \frac{32}{78.8} + S_9 \times \frac{32}{78.8} - 2.25 - S_{10} \times \frac{6.67}{13.7} = 0$$

$$17.24 \times \frac{72}{78.8} + S_9 \times \frac{72}{78.8} \qquad + S_{10} \times \frac{12}{13.7} = 0$$

$$-2.25 - S_{10} \times \frac{12}{13.7} = 0$$

$$S_{10} = -2.57 \text{ (compression)}$$

$$S_9 = -17.24 + \frac{78.8}{72} \times \frac{12}{13.7} \times 2.25 \times \frac{13.7}{12} = -14.78 \text{ (compression)}$$

Joint 7:

$$15.84 \times \frac{8}{72.4} - S_{12} \times \frac{8}{72.4} - S_{11} + 2.57 \times \frac{6.67}{13.7} = 0$$

$$15.84 \times \frac{72}{72.4} - S_{12} \times \frac{72}{72.4} \qquad - 2.57 \times \frac{12}{13.7} = 0$$

$$-9S_{11} + 2.57 \times \frac{72}{13.7} = 0$$

$$S_{11} = +1.50 \text{ (tension)}$$

$$S_{12} = 15.84 - 2.57 \times \frac{12}{13.7} \times \frac{72.4}{72} = +13.58 \text{ (tension)}$$

Joint 6:

$$14.78 \times \frac{32}{78.8} + S_{13} \times \frac{32}{78.8} - 3.0 - S_{14} \times \frac{10.67}{16.1} = 0$$

$$14.78 \times \frac{72}{78.8} + S_{13} \times \frac{72}{78.8} \qquad + S_{14} \times \frac{12}{16.1} = 0$$

$$-3 - S_{14} \times \frac{16}{16.1} = 0;\ S_{14} = -3.02 \text{ (compression)}$$

$$S_{13} = -14.78 + \frac{78.8}{72} \times \frac{12}{16.1} \times \frac{3 \times 16.1}{16} = -12.32 \text{ (compression)}$$

Joint 9:

$$13.58 \times \frac{8}{72.4} - S_{16} \times \frac{8}{72.4} - S_{15} + 3.02 \times \frac{10.67}{16.1} = 0$$

$$13.58 \times \frac{72}{72.4} - S_{16} \times \frac{72}{72.4} \qquad - 3.02 \times \frac{12}{16.1} = 0$$

$$-9S_{15} + 3.02 \times \frac{108}{16.1} = 0$$

$$S_{15} = +2.25 \text{ (tension)}$$

$$S_{16} = 13.58 - 3.02 \times \frac{12}{16.1} \times \frac{72.4}{72} = +11.32 \text{ (tension)}$$

Joint 8:

$$12.32 \times \frac{32}{78.8} + S_{17} \times \frac{32}{78.8} - 3.75 - S_{18} \times \frac{14.67}{19} = 0$$

$$12.32 \times \frac{72}{78.8} + S_{17} \times \frac{72}{78.8} \qquad + S_{18} \times \frac{12}{19} = 0$$

$$-3.75 - S_{18} \frac{20}{19} = 0$$

$$S_{18} = -3.56 \text{ (compression)}$$

$$S_{17} = -12.32 + \frac{78.8}{72} \times \frac{12}{19} \times \frac{3.75 \times 19}{20} = -9.86 \text{ (compression)}$$

Joint 11:

$$11.32 \times \frac{8}{72.4} - S_{20} \times \frac{8}{72.4} - S_{19} + 3.56 \times \frac{14.67}{19} = 0$$

$$11.32 \times \frac{72}{72.4} - S_{20} \frac{72}{72.4} \qquad - 3.56 \times \frac{12}{19} = 0$$

$$-9S_{19} + 3.56 \frac{144}{19} = 0$$

$$S_{19} = +3.0 \text{ (tension)}$$

$$S_{20} = 11.32 - 3.56 \times \frac{72.4}{72} \times \frac{12}{19} = +9.06 \text{ (tension)}$$

Joint 10

$$9.86 \times \frac{32}{78.8} + S_{21} \times \frac{32}{78.8} - 4.5 - S_{22} \times \frac{18.67}{22.2} = 0$$

$$9.86 \times \frac{72}{78.8} + S_{21} \times \frac{72}{78.8} \qquad + S_{22} \times \frac{12}{22.2} = 0$$

$$-4.5 - S_{22} \times \frac{24}{22.2} = 0$$

$$S_{22} = -4.16 \text{ (compression)}$$

$$S_{21} = -9.86 + \frac{78.8}{72} \times \frac{12}{22.2} \times \frac{4.5 \times 22.2}{24} = -7.40 \text{ (compression)}$$

Joint 12:

$$7.40 \times \frac{32}{78.8} - 0.75 - S_{24} - S_{23} \times \frac{32}{78.8} = 0$$
$$7.40 \times \frac{72}{78.8} \qquad\qquad + S_{23} \times \frac{72}{78.8} = 0$$

$$S_{23} = -7.40 \text{ (compression)}$$
$$S_{24} = 7.40 \times \frac{64}{78.8} - 0.75 = +5.26 \text{ (tension)}$$

Considering the joint at the right abutment,

$$2.25 + S_{25} \times \frac{32}{78.8} + S_{26} \times \frac{8}{72.4} = 0$$
$$S_{25}\frac{72}{78.8} + S_{26}\frac{72}{72.4} = 0$$

$$2.25 + S_{25}\frac{24}{78.8} = 0;\ S_{25} = -7.4 \text{ (compression)}$$
$$S_{26} - 7.4 \times \frac{72.4}{78.8} = +6.8 \text{ (tension)}$$

In this solution, no computation is necessary except what is shown above. By cancelling and using the slide rule, no scratch paper need be used. Note how, under joint 4, for instance, the *expression* giving the value of a stress is inserted in the equation, instead of the final value of the stress, in order to facilitate cancellation.

This method has obviously the disadvantage that any error is carried on, or is cumulative; and that it is not possible to find the stress in any particular bar except by working from one end up to this bar, finding the stresses in all the intermediate bars, even if these stresses are not desired. It is not, therefore, much used for a complete solution, even in the case of trusses like the one just computed. It has been used, however, in the computations of chord stresses in bridge trusses with parallel chords, in Chap. IV and the student should be prepared to use it whenever occasion demands.

A mixed method is almost always the best, using for each bar the method which is simplest in that case. This example has been worked through in full, by one method, in order that the reader may see all its advantages and disadvantages. The reader should note the essential simplicity of the method. It requires no trigonometry, and in fact no mathematics except arithmetic and the relations of similar triangles.[1]

The graphical method of joints is fully explained in the chapter on Graphical Statics.

[1] The reader who desires to see how grossly this method may be misapplied and complicated by unnecessary mathematics should read DUBOIS, "Stresses in Framed Structures," Chap. XXI, p. 16. See also a paper by the author "On the Value of Mathematics to the Civil Engineer and on the Teaching of That Subject to Civil Engineers," in the *Proc.* of the International Congress of Mathematics at Rome, April, 1908.

CHAPTER VII

THE METHOD OF MOMENTS

1. The methods enumerated under (*a*), (*b*), and (*c*) of the table on pages 74–75 should be thoroughly grasped by the student, for they may often be applicable. They are not, however, systematically applied to the complete solution of any truss, and are used only in special cases of a section. The method (*d*) however, that is, the pure method of moments, sometimes known as Ritter's method, may be, and often is, used for a complete solution. It is very easy of application, and it allows the stress in any bar to be found, independent of that in any other bar, if a section can be taken through the bar cutting two others *which are not parallel*. It would not therefore be used in finding web stresses of parallel chord trusses.

The reader, in using this method, no matter how familiar he is with the subject, should carefully take each of the following steps: (1) see where the section is and keep it in mind; (2) decide which side of the section he will consider; (3) see what loads there are (including reactions) on this side of the section; (4) see where the origin of moments is; (5) write the equation of moments about this origin, being very particular to get the signs correct. The unknown stresses should always be assumed as tension, and if they come out negative, it means that they act in a direction opposite to that assumed. The student should, until he becomes perfectly familiar with the method, always sketch a figure showing the portion of the truss considered, with the outer forces, and inner forces shown by arrows.

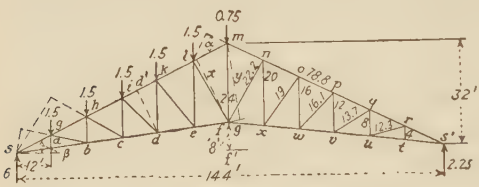


FIG. 91.

2. Example (Fig. 91).—In order to illustrate the use of this method, the same truss which in the previous chapter was computed by the method of joints will be here computed by the method of moments.

First compute the length of all the diagonals, as shown on the right half of the figure. Then compute the perpendicular x (from f on sm) and y (from m on sf produced) by the proportions

$$x:24::72:78.8 \qquad x = 21.93 \text{ (by the slide rule)}$$
$$y:24::72:72.4 \qquad y = 23.87 \text{ (by the slide rule)}$$

The lever arms of the verticals are 12′, 24′, 36′, etc.; of the upper chords they are 3.65′, 7.31′, 10.96′, 14.62′, 18.27′, 21.93′; of the lower chords, 3.98′, 3.98′, 7.96′, 11.93′, 15.91′, 19.89.

For the diagonals the following proportions exist (l being the lever arm):

$$
\begin{aligned}
&gb,\ l{:}24.13{::}\ 3.98{:}12.3;\ l = \ 7.80 \text{ (by the slide rule)}\\
&hc,\ l{:}36.2\ {::}\ 7.96{:}13.7;\ l = 21.03 \text{ (by the slide rule)}\\
&id,\ l{:}48.26{::}11.93{:}16.1;\ l = 35.8 \text{ (by the slide rule)}\\
&ke,\ l{:}60.33{::}15.91{:}19.0;\ l = 50.5 \text{ (by the slide rule)}\\
&lf,\ l{:}72.4\ {::}19.89{:}22.2;\ l = 64.75 \text{ (by the slide rule)}
\end{aligned}
$$

It is now possible to compute all the stresses:

For

$$
\begin{aligned}
&sg,\ 6 \times 12 + S \times 3.65 = 0; && S_{sq} = -19.72 \text{ (compression)}\\
&gh,\ 6 \times 24 - 1.5 \times 12 + S \times 7.31 = 0; && S_{gh} = -17.24 \text{ (compression)}\\
&hi,\ 6 \times 36 - 1.5 \times 36 + S \times 10.96 = 0; && S_{hi} = -14.78 \text{ (compression)}\\
&ik,\ 6 \times 48 - 1.5 \times 72 + S \times 14.62 = 0; && S_{ik} = -12.32 \text{ (compression)}\\
&kl,\ 6 \times 60 - 1.5 \times 120 + S \times 18.27 = 0; && S_{kl} = -\ 9.86 \text{ (compression)}\\
&lm,\ 6 \times 72 - 1.5 \times 180 + S \times 21.93 = 0; && S_{lm} = -\ 7.40 \text{ (compression)}
\end{aligned}
$$

For pieces to the right of the center, consider the right of the section thus:

For rs', $-2.25 \times 12 - S \times 3.65 = 0;\ S_{rs} = -7.40$ (compression)
For qr, pq, op, no, mn, we also obtain -7.4 (compression)

$$
\begin{aligned}
\text{For } &sa,\ 6 \times 12 - S \times 3.98 = && 0;\ S_{sa} = +18.15 \text{ (tension)}\\
&ab,\ 6 \times 12 - S \times 3.98 = && 0;\ S_{ab} = +18.15 \text{ (tension)}\\
&bc,\ 6 \times 24 - 1.5 \times 12 - S \times 7.96 = && 0;\ S_{bc} = +15.84 \text{ (tension)}\\
&cd,\ 6 \times 36 - 1.5 \times 36 - S \times 11.93 = && 0;\ S_{cd} = +13.58 \text{ (tension)}\\
&de,\ 6 \times 48 - 1.5 \times 72 - S \times 15.91 = && 0;\ S_{de} = +11.32 \text{ (tension)}\\
&ef,\ 6 \times 60 - 1.5 \times 120 - S \times 19.89 = && 0;\ S_{ef} = +9.07 \text{ (tension)}\\
&fs',\ -2.25 \times 12 + S \times 3.98 = 0;\ S_{fs}' = +6.8 \text{ (tension)}
\end{aligned}
$$

For the verticals:

$$
\begin{aligned}
&ga,\ S \times 12 = 0;\ S_{ga} = 0\\
&hb;\ 1.5 \times 12 - S \times 24 = 0;\ S_{hb} = 0.75 \text{ (tension)}\\
⁣\ 1.5 \times 36 - S \times 36 = 0;\ S_{ic} = 1.50 \text{ (tension)}\\
&kd;\ 1.5 \times 72 - S \times 48 = 0;\ S_{kd} = 2.25 \text{ (tension)}\\
≤\ 1.5 \times 120 - S \times 60 = 0;\ S_{le} = 3.00 \text{ (tension)}
\end{aligned}
$$

mf; take the section circularly enclosing l and m, with origin at l, and consider the upper portion: then

$$0.75 \times 12 + S \times 12 - 7.4 \times 9.73 = 0;\ S_{mf} = +5.26 \text{ (tension)}$$

In this equation, 7.4 is the compression in mn, and 9.73 is its lever arm about l, obtained from the equation

$$\frac{x}{13.13} = 2\,\frac{32}{78.8} \times \frac{72}{78.8};\ x = 9.73$$

since $\frac{x}{13.13} = \sin \alpha = \sin 2\beta = 2 \sin \beta \cos \beta =$ the expression above.

For the diagonals:

$$
\begin{aligned}
&gb;\ 1.5 \times 12 + S \times 7.80 = 0;\ S_{gb} = -2.31 \text{ (compression)}\\
&hc;\ 1.5 \times 36 + S \times 21.03 = 0;\ S_{hc} = -2.57 \text{ (compression)}\\
&id;\ 1.5 \times 72 + S \times 35.8 = 0;\ S_{id} = -3.02 \text{ (compression)}\\
&ke;\ 1.5 \times 120 + S \times 50.5 = 0;\ S_{ke} = -3.56 \text{ (compression)}\\
&lf;\ 1.5 \times 180 + S \times 64.75 = 0;\ S_{lf} = -4.17 \text{ (compression)}
\end{aligned}
$$

None of the diagonals or verticals to the right of the center has any stress. The above computations therefore completely solve this truss for these loads.

The method of moments may evidently be used when the section cuts more than three bars, provided the stresses in all except three are known.

The method of moments affords the readiest means of determining *the character* of the stress in any bar of a frame, independent of its magnitude. To do this, it is only necessary to determine the character of the resultant moment about the origin of moments, of the outer forces on one side of the section. Frequently this may be done by simple inspection.

The graphical method of moments is fully explained in the chapter on Graphical Statics, so that all the methods classified in the table on pages 74-75 will have been completely treated and illustrated.

CHAPTER VIII

TRUSSES WITH INCLINED OR CURVED CHORDS

1. Object.—It has been seen that, in trusses with parallel chords, (*a*) the chords carry no part of the shear in a panel; (*b*) the chord stress diminishes from a maximum at the center of the span to little (in some cases nothing) at the ends; (*c*) the web stresses are least at the center and greatest at the ends; and (*d*) the duty of transferring the loads from the joints to the abutments is all done by the diagonals.

By giving a chord piece an inclination, its stress may be increased by a relatively small amount while it may yet carry a considerable amount of shear, and correspondingly relieve the diagonals. Also, by making the height of the truss greatest at the center, where the moment is greatest, the stress in the chord is made more uniform from end to end, and this results in a saving of material, because it is practically never possible to design a chord to fit accurately the varying stresses in the different panels. The section in the panel where the stress is greatest must be designed first, with angles and plates, or other shapes, and in the panels toward the end it is only a question of leaving off certain plates if the stress allows, and this can scarcely ever be done so as to fit accurately the varying stress. If the chords are inclined, as will be shown, the maximum chord stress may be made nearly uniform from end to end.

The object of inclining the chords is therefore largely economy of material. The appearance is also improved.

2. When we speak of a curved chord, it is not to be understood that the chord itself is really curved, but only that the joints lie on a curve. The chord itself is polygonal, because in a framed structure each bar is supposed to be straight between the ends, since the stress acts in the straight line between joints. If the chord itself is curved, a bending stress is introduced in addition to the direct stress as a member of the frame.

3. Neutral Point in a Panel.—It has been shown how to find the so-called "neutral point" in a panel, where a single load must be placed in order to cause no shear in that panel, or no stress in the diagonals, of a truss with parallel chords. This is the point up to which a uniform load must extend in order to cause maximum positive or negative shear in the panel, according as it covers the part to the right or the part to the left of this point.

In a bridge with inclined chords, the neutral point is the same if it is defined as the point where a load causes no shear; but it is not the same if it is defined as the point where it causes *no stress in the diagonal.*

In Fig. 92 a truss with curved chords is shown. Let it be desired to find the point at which a load unity (or any load) must lie in the second panel (or any panel) *in order that there may be no stress in the diagonal b* in that panel. Taking a section through that panel, the two chords meet at O, and in order that there may be no stress in bar b, the moment about O of all the loads to the left of the section must be zero. Those

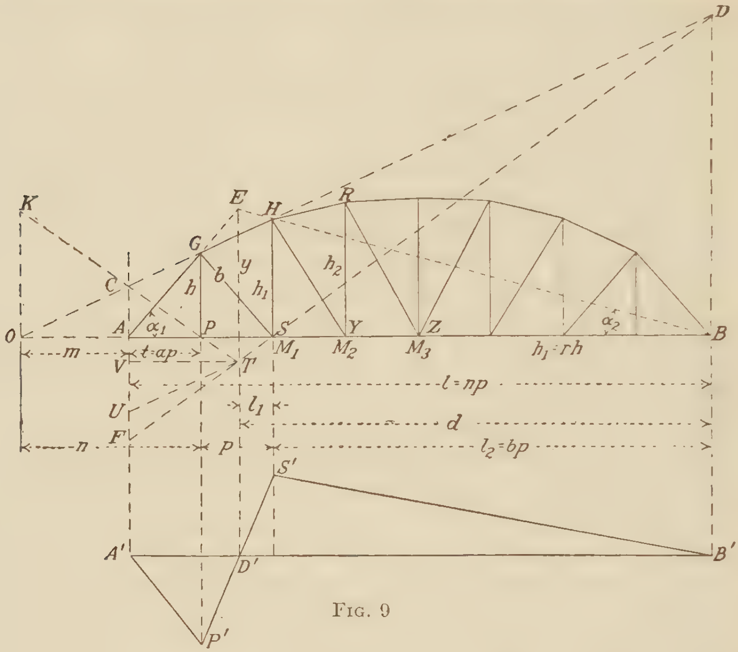


FIG. 9

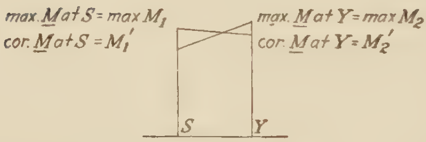


FIG. 92*a*.

loads are the left reaction d/l, and the load at the joint to the left of the load l_1/p. Hence the condition for no stress in b is

$$\frac{d}{l} \cdot m = \frac{l_1}{p} n$$

$$l_1 = \frac{d}{l} \cdot \frac{m}{n} \cdot p \tag{1}$$

A uniform load must extend from the right up to this point in order to produce maximum tension in b, and from the left to this point in order to produce maximum compression.

This point may be easily constructed. It was shown in Art. 5 of Chap. IV that in parallel-chord bridges the neutral point is found by pro-

ducing the top chord to meet the verticals through the supports and through these points drawing lines through the ends of the panel of the bottom chord, and that these lines will meet on the vertical through the neutral point. The same construction holds for curved-chord bridges. Prolong the line of the top (or curved) chord GH, to meet the verticals through the supports at C and D, and through these points draw the lines CP and DS, and these lines meet on the vertical through the point where a load must lie to cause no stress in the bar a.

This is easily seen by resolving the load into two components along the lines CP and DS. If CF represents the load, these components will be CT and TF. Resolve these components at P and S into vertical components CV and VF, and equal and opposite horizontal components VT. Then the load is held in equilibrium by FV at S and VC at P, or VC is the load at the joint P. Similarly, resolve the components CT and TF at C and D into vertical components CU and UF and equal and opposite components TU along CD (TU being parallel to CD). Then the load is held in equilibrium by FU at D and UC at C, since the components along CD balance; that is, FU and UC are the right and left reactions, respectively. From the similar triangles OCK and TCU, and OPK and TCV,

$$\frac{m}{OK} = \frac{TV}{UC}: TV \cdot OK = m \cdot UC = m \times \text{left reaction}$$

$$\frac{n}{OK} = \frac{TV}{VC}: TV \cdot OK = n \cdot VC = m \times UC$$

$$= n \times \text{component at } P\text{: which proves the construction.}$$

Instead of producing the curved chord (in this case the top chord GH) to meet verticals through the supports, lines AG and BH may be drawn, and they will intersect on the vertical through the neutral point. This is evidently the simplest construction. If α_1 and α_2 are the angles that AG and BH respectively make with the horizontal, it is easy to see that

$$l_1 = \frac{p \tan \alpha_1 + h - h_1}{\tan \alpha_1 + \tan \alpha_2} \tag{2}$$

for if $y = DE$,

$$\frac{l_2 + l_1}{l_2} = \frac{y}{h_1}: \frac{p - l_1 + t}{t} = \frac{y}{h}$$

from which equating the values of y,

$$l_1 = \frac{p\dfrac{h}{t} + h - h_1}{\dfrac{h}{t} + \dfrac{h_1}{l_2}} \tag{2a}$$

which is the same as Eq. (2).

This may be put in another form. Calling b the number of panels in l_2, a the number in AP, n the number in the span l, and $hr = h_1$, or $l = np$, $l_2 = bp$, $t = ap$, $n = a + b + 1$, Eq. (2a) becomes

$$l_1 = l_2\left(\frac{n}{b + ar} - 1\right) \tag{2b}$$

If both chords are curved, as in Fig. 93, the construction is the same. To find the neutral point in any panel PS, produce the top chord to C and D, draw DS and CP to meet in T, which will be in the vertical through the neutral point. The demonstration may be followed through as for Fig. 92. The bottom chord may be prolonged to meet the verticals through the reactions, instead of the top chord, as before.

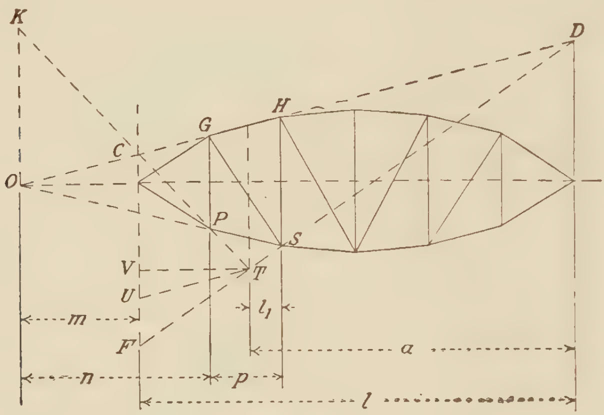


Fig. 93.

The influence line for the stress in GS is of the shape $A'P'S'B'$, shown in Fig. 92, D' being the neutral point for the stress in GS, or the point where a single load causes no stress.

4. Chord Stresses.—Taking a section through any panel as SY (Fig. 92), and calling the heights h_1 and h_2, and the moments at S and Y, M_1 and M_2,

Horizontal component in top chord $= \dfrac{M_2}{h_2}$

Horizontal component in bottom chord $= \dfrac{M_1}{h_1}$

Horizontal component of *tension* in diagonal $= \dfrac{M_2}{h_2} - \dfrac{M_1}{h_1}$

The stress in the diagonal will be tension if $M_2/h_2 > M_1/h_1$, and *vice versa*.

To avoid reversal of stress in a diagonal, counters may be used. If they are used, in order to find a chord stress it is necessary first to ascertain which diagonal is in action for the loading considered. If the

diagonals sloping upward toward the left are called the main diagonals (those shown in Fig. 92),

Main diagonal is in action if $\frac{M_2}{h_2} > \frac{M_1}{h_1}$

Counter is in action if $\frac{M_2}{h_2} < \frac{M_1}{h_1}$

The proper procedure, after finding the maximum moment at any joint, is to find, for the same loading, the corresponding moment at the adjacent joint on each side. Thus, we find the maximum moment (maximum M_1) at S, and the corresponding moment M_2 at Y and also the corresponding moment at P, and the maximum moment (maximum M_2) at Y and the corresponding moment M_1' at S and also the corresponding moment at Z (see Fig. 92a).

Suppose the loading to be that causing maximum M_2.

1. Then if

$$\frac{\text{maximum } M_2}{h_2} > \frac{M_1'}{h_1}$$

the horizontal component in top chord HR, for this loading, is $\frac{\text{maximum } M_2}{h_2}$, and the horizontal component in bottom chord SY, for this loading, is M_1'/h_1.

2. But if $\frac{\text{maximum } M_2}{h_2} < \frac{M_1'}{h_1}$

Horizontal component stress in top chord HR for this loading $= \frac{M_1'}{h_1}$

Horizontal component stress in bottom chord HR for this loading $=$

$$\frac{\text{maximum } M_2}{h_2}$$

Suppose now that the loading is that causing maximum M_1.

3. Then if $\frac{M_2'}{h_2} > \frac{\text{maximum } M_1}{h_1}$

Horizontal component of stress in top chord for this loading $= \frac{M_2'}{h_2}$

Horizontal component of stress in bottom chord for this loading $=$

$$\frac{\text{maximum } M_1}{h_1}$$

4. But if $\frac{M_2'}{h_2} < \frac{\text{maximum } M_1}{h_1}$

Horizontal component of stress in top chord for this loading $=$

$$\frac{\text{maximum } M_1}{h_1}$$

Horizontal component of stress in bottom chord for this loading $= \frac{M_2'}{h_2}$

There are apparently these four possibilities; but if case 2 exists, case 4 must *a fortiori* exist, and case 3 is impossible; hence there are really but two possibilities, namely, either case 1 or cases 2 and 4. It is now easy to perceive that if the following four quantities are calculated, *viz.*,

$$\text{Group } A \begin{cases} \text{maximum } \dfrac{M_1}{h_1} \\ \text{corresponding } \dfrac{M_2'}{h_2} \end{cases}$$

$$\text{Group } B \begin{cases} \text{maximum } \dfrac{M_2}{h_2} \\ \text{corresponding } \dfrac{M_1'}{h} \end{cases}$$

then

Horizontal component of *maximum* stress in top chord = *greatest of these four quantities;*

Horizontal component of *maximum* stress in bottom chord = greater of the smallest in each group.

For illustration, if

$$\text{Group } A\colon \frac{M_1}{h_1} = 20; \frac{M_2'}{h_2} = 19$$

$$\text{Group } B\colon \frac{M_2}{h_2} = 21; \frac{M_1'}{h_1} = 18$$

Then

Horizontal component of maximum in top chord = 21
Horizontal component of maximum in bottom chord = 19

This result may not be exactly correct for the bottom chord. The precise value would be the maximum M_2/h_2 consistent with the counter being in action, or the maximum M_1/h_1 consistent with the main diagonal being in action; but it is rarely necessary to try to find this exact value.

5. Diagonals.—We have seen that a neutral point may be found in each panel, being the point at which a single load must lie to produce no stress in the main diagonal in that panel. For a uniform load, the maximum stress in that diagonal will occur when the load extends from the right end to that neutral point, and the maximum stress in the counter when it extends from the left end to the neutral point, considering only the panels to the left of the center of the span. Or, if the neutral point in each panel of the entire bridge is found, then the maximum in the diagonal sloping upward to the left in any panel (main diagonals to the left of the center and counters to the right) will occur when the load extends from the right end up to the neutral point.

Position of a Set of Concentrated Loads for Maximum Stress in a Diagonal.—For a set of concentrated loads, this position must be found before the stress itself can be computed. This will not be necessarily the same

position as for the diagonal in a truss with parallel chords, but may be easily found by the same methods. Consider the diagonal a (Fig. 94). The origin of moments is O, and the stress in the diagonal will be the moment, about O, of all outer forces to the left of the section, divided by the perpendicular distance from O to the bar a. As a train moves up from the right, the only outer face to the left of the section will be the left reaction until the first load reaches S, and that reaction, and consequently the stress in a, will increase until that first load reaches S. If it passes beyond S, there will be a load on the joint P which will have a moment about O opposite in direction to the direction of the moment of the reaction, and, as the train moves to the left, both the reaction and the load at P will increase in proportion to the distance moved, though at different rates. Hence the question is, whether the second load P_2 shall be moved up to S.

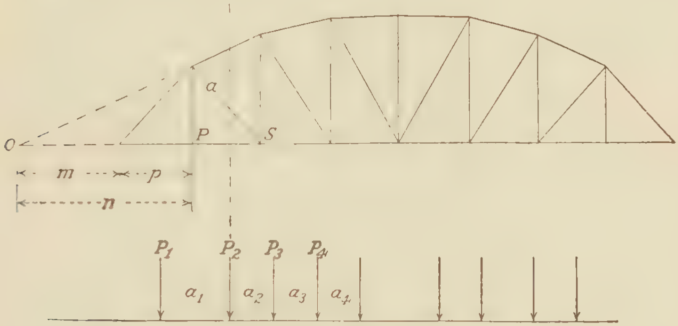


FIG. 94.

Let R = total load on the span when P_1 is at S. Then, if the loads be advanced a distance a_1, bringing P_2 to S, the reaction will be increased by $Ra_1/l + \Delta$, and its moment by $(Ra_1/l + \Delta)m$, if Δ is the reaction due to any loads coming on the span at the right. The load at P is P_1a_1/p if $a_1 < p$, and its moment is Pa_1n/p. Hence the stress in a is increased provided

$$\left(R\frac{a_1}{l} + \Delta\right)m > P_1\frac{a_1}{p}n, \text{ when } a_1 < p$$

or

$$\left(R\frac{a_1}{l} + \Delta\right)m > P_1(n + p - a_1), \text{ when } a_1 > p$$

If $\Delta = 0$, or is neglected, the stress is increased if

$$R\frac{a_1}{l} \cdot m > P_1\frac{a_1}{p}n; \text{ or } \frac{Rm}{l} > \frac{P_1n}{p}, \text{ when } a_1 < p$$

$$R\frac{a_1}{l} \cdot m > P_1(n + p - a_1), \text{ when } a_1 > p$$

Similar formulae may be written to ascertain whether P_3 shall be moved up. The maximum in a will clearly occur when some load is at S. If the

method is understood, the reader can easily solve any case by framing the conditions himself, without reference to formulae.

Example.—Let the truss be that shown in Fig. 95, with load system indicated.

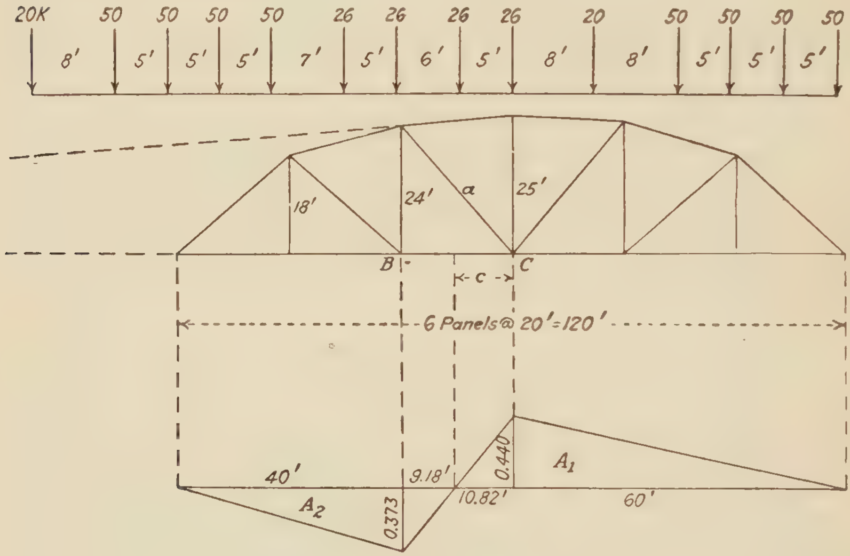
For maximum in bar a: With P_1 at center, total load on span is 244 kips. The origin is 24 panels to the left of B_1 or 22 panels from the left support = 440 feet.

$$\frac{244 \times 8}{120} \times 440 + \Delta \gtrless 20 \times \frac{8}{20} \times 480; > \therefore \text{ move up } P_2$$

with P_2 at center, total load = 344

$$\frac{344 \times 5}{120} \times 440 + \Delta \gtrless 70 \times \frac{5}{20} \times 480; < \therefore \text{ do not move up } P_3$$

Hence maximum is with P_2 at C.



Influence line for vertical component in bar a

FIG. 95.

Influence Line for Diagonals.—In Fig. 96 the influence line is shown for diagonal Db and in Fig. 97 that for diagonal Db'. The maximum stress for uniform load is found from these by finding the areas between the influence line and the axis. Thus, for Db, if A_1 and A_2 are the areas of the triangles indicated, and g and p respectively the dead and live loads per horizontal foot,

$$\text{maximum tension in } Db = (p + g)A_1 - gA_2$$
$$\text{minimum tension in } Db = gA_1 - (p + g)A_2$$

If the last value is negative, it shows compression or the necessity for a counter. *But the expression does not give the maximum stress in the counter*, which must be found from the influence line for that counter in Fig. 97.

$$\text{maximum tension in } Db' = (g + p)A_1' - gA_2'$$

For a set of concentrated loads it is generally simplest to find the maximum stress in all the diagonals sloping in the same direction, beginning at the left end. For the counters to the right of the center (Fig.

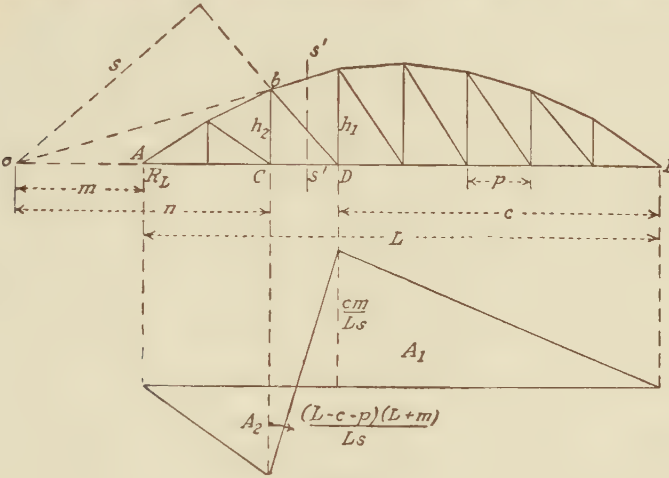


FIG. 96.

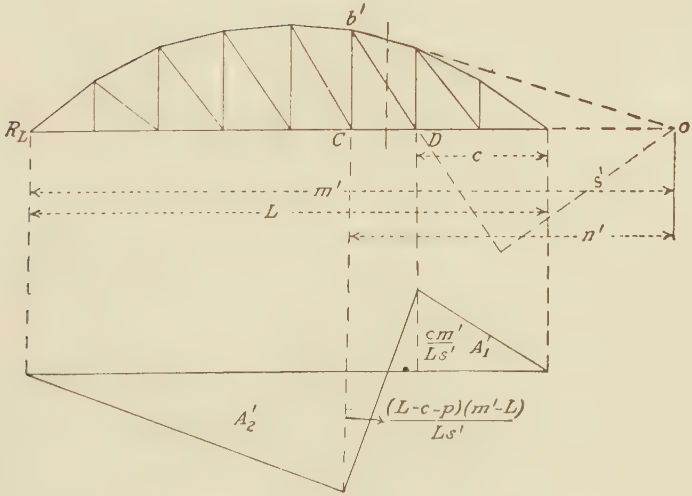


FIG. 97.

97) the origin of moments is at 0, to the right of the right support. The condition for increasing the stress, by moving P_2 up to D, is

$$\left(R\frac{a_1}{l} + \Delta\right)m' > P_1\frac{a_1}{p}n', \text{ if } a_1 < p$$

$$\left(R\frac{a_1}{l} + \Delta\right)m' > P_1(n' - p + a_1), \text{ if } a_1 > p$$

or similar to the expressions for the main diagonals.

6. Methods of Finding Stress in Diagonals.—After finding the position of the loads for maximum stress in a diagonal, the stress itself may be found by various methods.

1. For a uniform load, the influence line may be drawn, and the stress found from the areas.

2. Take the section $S'S'$ (Fig. 96) and find the stress by taking moments about O. It is generally simplest to divide this moment by OD, and so obtain the vertical component of the stress. This avoids the necessity of finding the lever arm s.

3. Find the shear in the panel CD, and the vertical component of the stress in the top chord, by finding the moment about D of all the forces to the left of the section and dividing it by OD. Subtract this vertical component (algebraically) from the shear, and the difference will be the vertical component in the diagonal. Great care must be taken to observe the character (direction) of the shear, and the character of the stress in the top chord, or quantities will be subtracted that should be added, or *vice versa*.

4. Find the *horizontal component* in the diagonal by the equation $\Sigma H = 0$ (Fig. 96). If the loads and reactions are vertical, there is no horizontal component of the outer forces to the left of the section. If

$$\left.\begin{aligned} M_1 &= \text{moment about } D \\ M_2 &= \text{moment about } C \end{aligned}\right\} \begin{aligned} &\text{of all the loads to the left} \\ &\text{of the section } S'S' \end{aligned}$$

then

$$\text{Horizontal component in top chord} = \frac{M_1}{h_1}$$

$$\text{Horizontal component in bottom chord} = \frac{M_2}{h_2}$$

Hence

$$\text{Horizontal component in diagonal} = \frac{M_1}{h_1} - \frac{M_2}{h_2}.$$

This is generally the simplest method.

Example.—Consider the truss used in the example of the last article (Fig. 95), but assume a uniform live load of 3,000 pounds per running foot. For the bar a, the neutral point in the panel, by Eq. (2), will be defined by the equation

$$c = \frac{20 \times {}^{24}\!/_{40} + 24 - 25}{{}^{24}\!/_{40} + {}^{25}\!/_{60}} = 10.82 \text{ feet}$$

Considering the live load alone, from the right to the neutral point,

$$\text{Left reaction} = \frac{3{,}000 \times \overline{70.82}^2}{2 \times 120} = 62{,}693 \text{ pounds}$$

$$\text{Load at } B = \frac{3{,}000 \times \overline{10.82}^2}{2 \times 20} = 8{,}780 \text{ pounds}$$

The influence line for bar a is shown as follows: When a load unity is at C, *left* reaction is ½, and vertical component in a is

$$\frac{1.440}{2.500} = 0.44$$

When unit load is at B, *right* reaction is $\frac{1}{3}$, and vertical component in a is

$$\frac{1.560}{3\ 500} = 0.373$$

Area $A_1 = \frac{1}{2} \times 70.82 \times 0.44 = 15.58$
Area $A_2 = \frac{1}{2} \times 49.18 \times 0.373 = 9.172$
For stress (vertical component) in a, for live load 3,000 pounds per foot:

1. By influence lines:

$$\text{Maximum V. C. in } a = 15.58 \times 3{,}000 = 46{,}741$$
$$\text{Minimum V. C. in } a = -9.172 \times 3{,}000 = -27{,}516$$

2. $\text{Maximum V. C. in } a = \dfrac{62{,}693 \times 440 - 8{,}780 \times 480}{500} = 46{,}741$

3. For maximum in a:

$$\text{V. C. upper chord} = \frac{62{,}693 \times 60 - 8{,}780 \times 20}{500} = 7{,}171$$
$$\text{Shear} = 62{,}693 - 8{,}780 = 53{,}913$$
$$\text{V. C. in } a = 53{,}913 - 7{,}171 = 46{,}742$$

4. $$\text{H. C. in } a = \frac{62{,}693 \times 60 - 8{,}780 \times 20}{25} - \frac{62{,}693 \times 40}{24}$$
$$= 38{,}951$$
$$\text{V. C. in } a = 38{,}951 \times \frac{24}{20} = 46{,}641$$

All these methods should be clearly understood.

7. Verticals.—The maximum stresses in the verticals are best found by the method of moments. They cannot, in general, be found from the

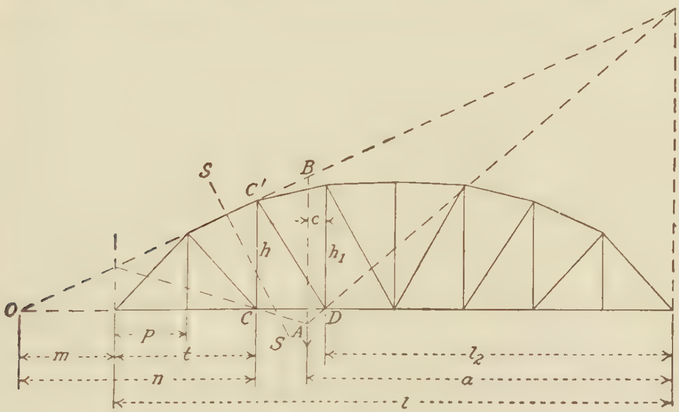


Fig. 98.

stresses in adjacent diagonals, as they can be in trusses with parallel chords, because here the loading giving maximum stress in a vertical may be different from the loading giving maximum stress in the diagonal which meets it at the straight chord (if one chord is straight).

Neutral Point for a Vertical.—It is easy to find the neutral point for a vertical; that is to say, the point where a single load must lie in order to cause no stress in that vertical. The neutral point for a vertical will not be the same as the neutral point for any diagonal. In Fig. 98 let it be

required to find the neutral point for the vertical CC'. The section is SS, the origin of moments O. The neutral point will be at A in the panel CD, such that

$$\frac{l_2 + c}{l} \cdot m = \frac{c}{p}(m + l - l_2 - p)$$

from which

$$c = \frac{p}{\frac{l - p}{l_2}\left(\frac{m + l}{m}\right) - \frac{l}{m}} \tag{3}$$

Produce the chord piece cut by the section to meet the verticals through the supports, and from these points draw lines through D and C; they will meet on the vertical through the neutral point, as may be proved by the reasoning in Art. 3.

For a set of concentrated loads, the first load would be placed at D, and then, by methods already explained, it would be determined whether to move the second load up or not. The stress itself would be found by taking moments about O.

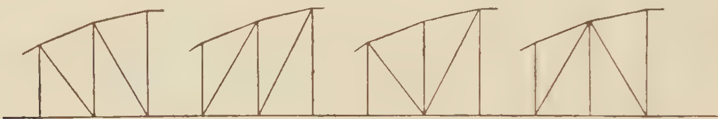


Fig. 99.

The solution for the verticals is thus very simple if there are no counters in either adjoining panel, because the proper section SS is obvious. This is not so if there is a counter in either adjoining panel. If there is a counter in each adjoining panel, there are four theoretical possibilities (Fig. 99), just as in trusses with parallel chords (see Art. 42, Chap. IV); but here all four cases may be really possible, and the maximum stress consistent with each condition must be found unless it can be seen that it would be less than one already found. Thus in case 4 the maximum stress will be the maximum lower chord panel load, provided this load may act and at the same time the diagonals shown may be in action, which is not likely for tension diagonals; but if in case 1 a stress in the vertical has been found greater than said maximum panel load, then case 4 drops out anyway. In the same way case 3 may drop out. Cases 1 and 2 should be settled first, in any event.

In drawing the influence line for any diagonal, it must be drawn as already explained, as though the counter did not exist; and for any vertical it must be first drawn assuming that certain diagonals are in action, and then drawn assuming that other diagonals are in action.

With this explanation, the reader should be able to solve all the verticals.

8. Curved-chord Bridges with Multiple Web Systems.—These are statically undetermined, as in parallel-chord bridges, but there is greater error here in assuming the systems to act separately, because the curved chord is not straight (originally) between the adjacent joints of one system, while in parallel chords both chords are straight. Hence in curved-chord bridges the loads on one system affect the other more than in parallel-chord bridges. Thus, in Fig. 100, it would be quite obviously erroneous to divide the truss into two systems, and assume the bar *ab* of the heavy system to be straight.

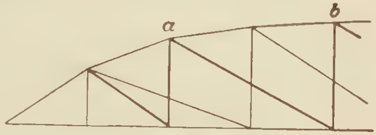


Fig. 100.

The writer does not believe there is any economic justification for building bridges of this type, though in Europe large spans have been built with even three web systems. Types with subdivided panels may be used much more advantageously. A swing bridge over the Harlem River in New York is of this type.

SPECIAL TYPES OF CURVED-CHORD BRIDGES

9. The Parabolic Truss.—In this truss, one chord only is curved, the joints of the curved chord lying on a parabola having a vertical axis through the center of the span. The separate pieces of the chord are of course assumed to be straight, between joints.

10. *The parabola is the curve which a flexible cord would assume if loaded with a uniform load per horizontal foot.*

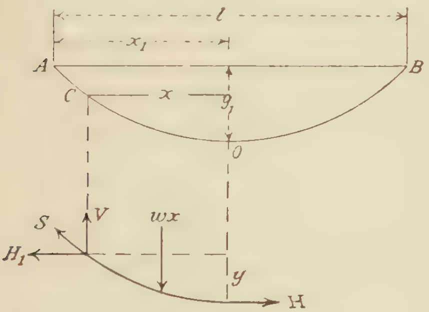


Fig. 101.

Let such a cord be hung from two points on the same level, A and B (Fig. 101). The lowest point O will be at the center, and the curve will be horizontal there. Let the uniform load be w per horizontal foot. The part OC is in equilibrium under the forces wx, S, and H, the last being horizontal, and S along the tangent to the curve at C. Resolve S into components H_1 and V. Then

$$\Sigma H = 0 \text{ gives } H_1 = H$$

Hence the horizontal component of the stress at any point is the same.

$$\Sigma V = 0 \text{ gives } V = wx$$

Hence the vertical component of the stress at any point is the total load between that point and the center.

$$\Sigma M = 0 \text{ about } C \text{ gives } Hy = \frac{wx^2}{2}$$

Hence

$$H = \frac{wx^2}{2y};\ y = \frac{wx^2}{2H} = (\text{constant})x^2$$

Hence the curve is a parabola with vertical axis and vertex at O. Since A and B are on the same level

$$H = \frac{wx_1^2}{2y_1} = \frac{wl^2}{8y_1} \tag{4}$$

The above supposes the load applied continuously to the cord. If, however, the load is supported on stringers and floor beams, which,

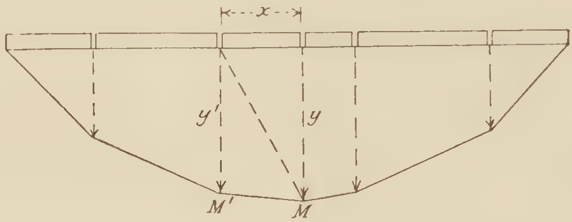


FIG. 102.

through vertical pieces, as in Fig. 102 carry concentrated loads to the cord, the latter will take the shape of a series of straight lines. In Fig. 102 the panels are unequal, but there is a joint at the center of the span M. The joints or angles will all lie on a parabola with vertex at M, and the horizontal component of stress in the cord will be constant, as before.

This can also be seen from the principles of Art. 6. Let Fig. 103 represent a truss with straight lower chord, and upper chord curved with

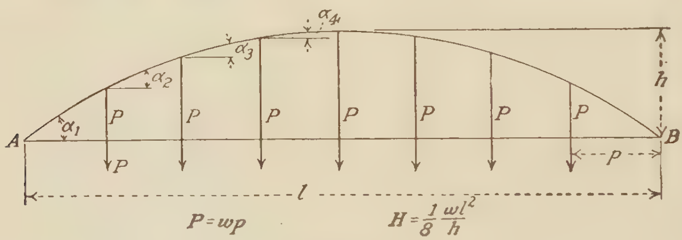


FIG. 103.

the joints all lying on a parabola with vertical axis. Then the ordinates at the joints are the ordinates of a parabola. The moment curve is also a parabola going through A and B, with a vertical axis, so that the ordinates to the moment curve are the ordinates of a parabola, for full loading uniform per horizontal foot. Then the horizontal component of the stress in any diagonal is (see Art. 6) $M/h - M'/h'$; but since both M and h are ordinates to a parabola,

$M/h - M'/h' = 0$, and there is no stress in any diagonal. This is true whether panels are equal or unequal.

In Fig. 104 let h_1 and h_2 be the heights at distances x_1 and x_2 from the center respectively.

M, M_1, M_2, the moments at the center and at h_1 and h_2 for full loading, M_1' and M_2' the moments for loading giving maximum in main diagonal a M_1'' and M_2'' the moments for loading giving maximum in counter Then

$$h_1 = h\left(1 - \frac{4x_1^2}{l^2}\right) : h_2 = h\left(1 - \frac{4x_2^2}{l^2}\right)$$

$$M_1 = M\left(1 - \frac{4x_1^2}{l^2}\right) : M_2 = M\left(1 - \frac{4x_2^2}{l^2}\right)$$

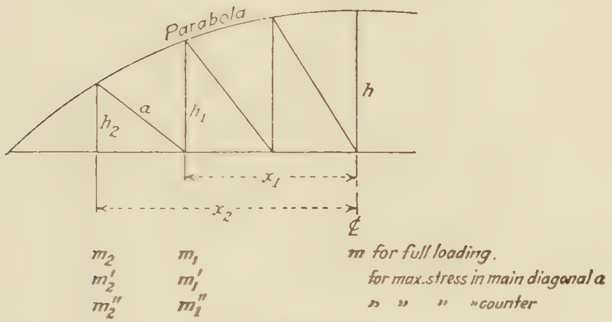


FIG. 104.

Horizontal component maximum stress in main diagonal $= \frac{M_1'}{h_1} - \frac{M_2'}{h_2}$

Horizontal component maximum stress in counter diagonal $= \frac{M_2''}{h_2} - \frac{M_1''}{h_1}$

But

$$M_1' + M_1'' = M_1;\ M_2' + M_2'' = M_2$$

$$\frac{M_2''}{h_2} - \frac{M_1''}{h_1} = \frac{M_2 - M_2'}{h_2} - \frac{M_1 - M_1'}{h_1} = \frac{M_2}{h_2} - \frac{M_1}{h_1} + \frac{M_1'}{h_1} - \frac{M_2'}{h_2}$$
$$= \frac{M_1'}{h_1} - \frac{M_2'}{h_2}$$

11. From the above, the following results are clear for a parabolic truss loaded with a uniform load per horizontal foot which may cover any part of the span:

1. None of the diagonals are in action under a full load.
2. A uniform dead load causes no stress in any diagonal.
3. The maximum tension in a diagonal (for a load from the right up to the neutral point) equals the maximum compression in the same diagonal (for a load from the left up to the neutral point), there being no counters.
4. If diagonals are tension bars only, a counter is needed in every panel.
5. If there are counters, the horizontal component of the maximum stress in a counter equals that of its main diagonal.

6. The above results are also true if the maximum stress is found by assuming all joints on one side fully loaded and all on the other side without load.

7. The stress in each vertical, for a full loading, is a panel load.

8. The stress in the straight chord, for a full loading, is constant and equals $H = wl^2/8h_c$, where h_c = ordinate to the parabola at the center. If the number of panels is even, h is the center height; if odd, and if h is the height at the two verticals nearest the center (Fig. 105),

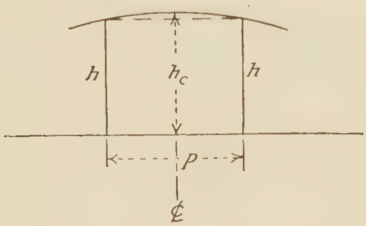


FIG. 105.

$$h_c = h\frac{l_2}{l^2 - p^2}$$

9. The stress in the curved chord, or full loading, has a constant horizontal component H. The actual stress in any piece is $H \sec \alpha$ (Fig. 103).

12. The Sickle-shaped Parabola Truss (Fig. 106).—Trusses of this general form may have the joints lie on parabolas or on any other curves. If not on parabolas, the computation is similar to that for curved-chord trusses in general. If, however, the joints of each chord lie on parabolas, the distances between the chords are proportional to the ordinates of a parabola. The ordinate y from the horizontal AB to a parabola at a distance x from the center, if the center ordinate is h, is

$$y = h\left(1 - \frac{4x^2}{l^2}\right)$$

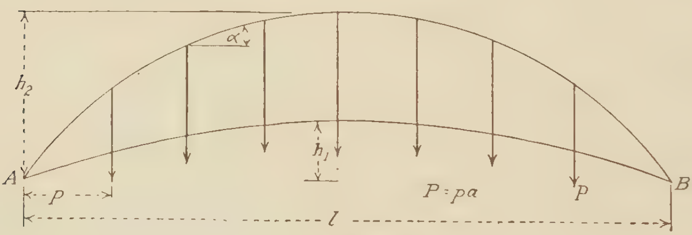


FIG. 106.

Hence if h_1 and h_2 are the center heights to the two parabolas, and y_1 and y_2 ordinates at x from the center

$$y_1 = h_1\left(1 - \frac{4x^2}{l^2}\right)$$

$$y_2 = h_2\left(1 - \frac{4x^2}{l^2}\right)$$

$$y_2 - y_1 = (h_2 - h_1)\left(1 - \frac{4x^2}{l^2}\right)$$

or the distances between chords are ordinates to a parabola with center ordinate $(h_2 - h_1)$.

It follows that the diagonals of this truss are not in action under a full load uniform per horizontal foot.

a. Let the load w per foot be applied to the bottom chord. Separating the two chords, as in Fig. 107, the stress in each vertical will be the same, which we call K, and the upper chord is loaded with a uniform load w_1 per foot, such that

$$K = w_1 p$$

There will be an inward reaction H_2, on this chord, such that

$$H_2 = \frac{w_1 l^2}{8h_2} = \frac{Kl^2}{8ph_2} \tag{5}$$

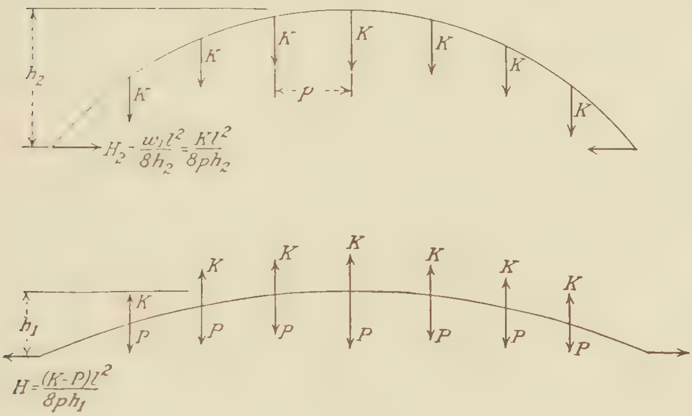


FIG. 107.

The bottom chord will be loaded with a downward load of w per foot, and an upward load of w_1 per foot. This chord must exert an inward pull at the supports in order to balance the outward thrust of the top chord; that is, it must have outward reactions $H = H_2$, so that w_1 must be greater than w. The full panel load is $P = wp$, and $P < K$, assuming the verticals to be in tension.

The reaction is

$$H = H_2 = \frac{(w_1 - w)l^2}{8h_1} = \frac{(K - P)l^2}{8ph_1} = \frac{Kl^2}{8ph_2} \tag{6}$$

Whence

$$K = \frac{Ph_2}{h_2 - h_1} \tag{7}$$

If $h_1 = 0$, $K = P$, as in the ordinary parabolic truss. K is evidently greater than P, so that the verticals are really in tension, as assumed.

Having found K from Eq. (7), $H = H_2$ is found from Eq. (6); and in any chord piece, of either chord, the stress for full loading is $H \sec \alpha$, where α is its angle with the horizontal.

b. Let the load w per foot be applied to the top chord. The top chord must be in compression, and the bottom chord in tension; hence the verticals must be in tension. The stress in the verticals K is found by the condition

$$\frac{Kl^2}{8ph_1} = \frac{(K+P)l^2}{8ph_2}; \text{ or } K = P\frac{h_1}{h_2 - h_1} \tag{8}$$

If $h_1 = 0$, $K = 0$; for in this case the entire load is carried on the parabolic top chord, and the verticals are not in action, since there is no load on the joints at their lower ends.

13. The Fish-belly or Double Parabolic (or Lenticular) Truss. *a. Load Applied to Bottom Chord w per Foot* (Fig. 108).—The verticals must

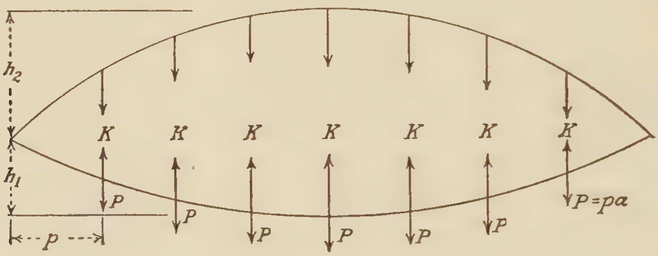


FIG. 108.

be in tension in order that there may be compression in the top chord.

$$\frac{Kl^2}{8ph_2} = \frac{(P-K)l^2}{8ph_1} \quad \therefore K = \frac{Ph_2}{h_2 + h_1} \tag{9}$$

Thus $K < P$; and when $h_1 = 0$, $K = P$.

b. Load Applied to Top Chord.—The verticals must be in compression, to have bottom chord in tension.

$$\frac{Kl^2}{8ph_1} = \frac{(P-K)l^2}{8ph_2}; \; K = \frac{Ph_1}{h_2 + h_1} \text{ (compression)} \tag{10}$$

Here $K < P$, and, as before, when $h_1 = 0$, $K = 0$.

14. Example of a Parabolic Truss (Fig. 109).—Let the live panel load be 15 tons, the dead 3 tons, and no locomotive excess. The load per foot is 2,400 pounds. Then

$$\text{Maximum in bottom chord} = \frac{2{,}400 \times 120 \times 120}{8 \times 32} = 135{,}000 \text{ pounds}$$

The maximum compression in each top chord is 135,000 sec α.

The maximum tension in a vertical is 36,000 pounds.

The maximum horizontal component in each diagonal is due to live loads alone, and is the same in all, and equals

$$\frac{2{,}000 \times 120 \times 120}{8 \times 32 \times 8} = 14{,}063$$

or the maximum live-load lower chord stress divided by the number of panels, if the maximum diagonal stress is found by assuming the panel point on the right of the panel fully loaded and that on the left entirely unloaded, not using the neutral point.

This last result will now be shown: First diagonal: load up to 2 on the right, using live loads only.

$$R_e = \frac{21}{8} \times 15 = 39.375$$

$$M_1 = 39.375 \times 15 = 590.625; \frac{M_1}{h_1} = \frac{590.625}{14} = 42.187$$

$$M_2 = 39.375 \times 30 = 1{,}181.25; \frac{M_2}{h_2} = \frac{1{,}181.25}{24} = 49.218$$

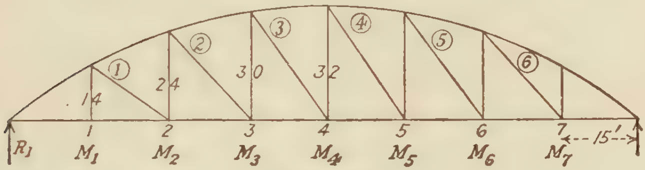


FIG. 109.

Maximum H. C. diagonal = 49.218 − 42.187 = 7.031 tons = 14,062 pounds. Second diagonal: load up to 3 on the right.

$$R_e = \frac{15}{8} \times 15 = 28\frac{1}{8}$$

$$M_2 = 28.125 \times 30 = 843.75; \frac{M_2}{h_2} = \frac{843.75}{24} = 35.156$$

$$M_3 = 28.125 + 45 = 1{,}265.625; \frac{M_3}{h_3} = \frac{1{,}265.625}{30} = 42.187$$

Maximum H. C. in diagonal = 42.187 − 35.156 = 7.031 tons = 14,062 pounds.

The next diagonal will give the same result.

This is a very curious property of the diagonals of this truss, when figured for a uniform load in this way. It shortens greatly the time of computation if the law is known. If the neutral points are used, the results will be different.

Each diagonal stress is the H. C. multiplied by the secant of the angle with the horizontal; and each diagonal has equal maximum tension and compression, since a full loading causes no stress in it, assuming that there are no counters.

If the diagonals are made as tension bars only, unable to take compression, the following principle is obvious:

In the parabolic truss, counters are required in every panel except the end panel; and, in any panel, the maximum horizontal components of main diagonal and counter are the same.

This can be checked by finding the stress in the counter shown in Fig. 109 in panel 4-5. Load up to 5 on the right; then $R_e = \frac{6}{8} \times 15 = 11.25$.

$$\frac{M_5}{h_5} = \frac{11.25 \times 75}{30} = 28.125$$

$$\frac{M_4}{h_4} = \frac{11.25 \times 60}{32} = 21.094$$

$M_5/h_5 - M_4/h_4 =$ H. C. in diagonal shown $= 28.125 - 21.094 =$ 7.031, as before.

Verticals.—The verticals must be computed for partial loading. For this purpose compute the lever arms AO, AO_1, AO_2, in Fig. 110.

$$AO + 15 : AO + 30 :: 14 : 24 \quad \therefore AO = 6$$
$$AO_1 + 30 : AO_1 + 45 :: 24 : 30 \quad \therefore AO_1 = 30$$
$$AO_2 + 45 : AO_2 + 60 :: 30 : 32 \quad \therefore AO_2 = 180$$

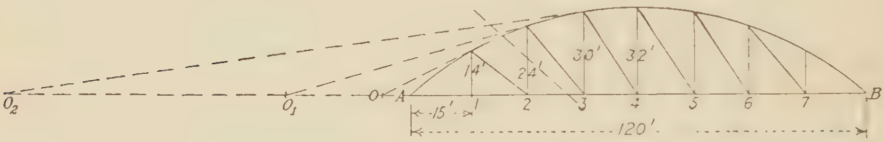


Fig. 110.

Or this may be done mentally: AO_2 drops 2 feet in a panel, hence it must go 15 panels to drop 30 feet, hence O_2 is 12 panels $= 180$; and so with the others.

First vertical: this has only a tension of 18 tons.

Second vertical: load up to 3 from the right.

$$R_l = \frac{15}{8} \times 15 + 3\frac{1}{2} \times 3 = 38\frac{5}{8};$$

$$-\ 38.625 \times 6 + 3(21+36) - 36\,S_2 = 0;\ \therefore S_2 = -1.69 \text{ (\textit{i.e.}, compression)}$$

or shear in section is $38\frac{5}{8} - 6 = 32\frac{5}{8}$, acting up.

$$\text{H. C. upper chord} = \frac{38.625 \times 30 - 3 \times 15}{24} = 46.41$$

$$\text{V. C. upper chord} = 46.41 \times \frac{10}{15} = 30.94, \text{ acting down.}$$

Hence stress in vertical $= S_2 = 32.625 - 30.94 = 1.69$ acting down, and therefore compression, as before.

Third vertical: load up to 4 from the right.

$$R_l = \frac{10}{8} \times 15 + 10.5 = 29.25:$$

$$-29.25 \times 30 + 3(45 + 60 + 75) - 75S_3 = 0; \therefore S_3 = -4.5 \text{ (i.e., compression).}$$

Fourth vertical: load up to 5.

$$R_l = \frac{6}{8} \times 15 + 10.5 = 21.75:$$

$$-21.75 \times 180 + 3(195 + 210 + 225 + 240) - 240S_4 = 0$$

$$S_4 = -5.44 \text{ (i.e., compression)}$$

Fifth vertical: load up to 6.

$$R_l = \frac{3}{8} \times 15 + 10.5 = 16.125:$$

The origin of moments is 180 feet to the *right* of B.

$$16.125 \times 300 - 3(285 + 270 + 225 + 240 + 225) + 225S_5 = 0$$

$$S_5 = -4.5 \text{ (i.e., compression), or the same as the third}$$

Sixth vertical: load up to 7.

$$R_l = \frac{1}{8} \times 15 + 10.5 = 12.375:$$

The origin of moments is 20 feet to the right of B.

$$12.375 \times 150 - 3(135 + 120 + 105 + 90 + 75 + 60) + 60S_6 = 0$$

$$S_6 = -1.69 \text{ (i.e., compression), or the same as the second}$$

Seventh vertical: Clearly there can be no compression in this bar. When the main diagonal on its left is in action, it has a tension equal to the load at 7; when the counter shown is in action, it has a tension also. as is seen at once from the joint at its upper end, but this tension will be less than that for full loading, because the chord stresses at the top will be less.

Hence the maximum tension in each vertical is 18 tons, and the maximum compression is the value just found.

15. Formulae for the Parabolic Truss.—Formulae may be deduced for the web stresses in the parabolic truss for a uniform load, using the (impossible) loadings as above and as in the trusses heretofore considered, but they will not be given. They merely encourage the practice of substituting numerical values, which is a bad practice to get into. It is much better to work out the stresses concretely, as done above. It is only desirable to illustrate the method by deducing the peculiarity, above shown, that the maximum horizontal component in each diagonal is the same and equal to the maximum horizontal component in the chords divided by the number of panels.

In Fig. 111 let m = number of panels
$P = wp$ = live panel load
p = length of a panel
mp = span

Since the dead load causes no stress in the diagonals, it need not be considered. The height at a distance x from the left abutment is

$$h = h_c\frac{4x}{l}\left(1 - \frac{x}{l}\right)$$

Consider any diagonal, sloping downward to the right, whose lower end is x from the left abutment, and the upper end $x - p$. Let x be n_1 panels, or $x = n_1p$, $x - p = (n_1 - 1)p$. Then, for the maximum stress in this diagonal, there are $(m - n_1)$ loaded joints. The left reaction is

$$R_l = wp(m - n_1)\frac{m - n_1 + 1}{2m}$$

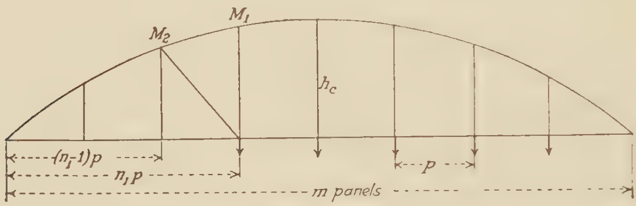


FIG. 111.

The moment M_1 at the lower end of the diagonal, and the moment M_2 at its upper end are

$$M_1 = R_ln_1p = wp^2n_1(m - n_1)\frac{m - n_1 + 1}{2m}$$

$$M_2 = R_l(n_1 - 1)p = wp^2(n_1 - 1)(m - n_1)\frac{m - n_1 + 1}{2m}$$

The heights are

$$h_1 = \frac{4h_cn_1}{m}\left(1 - \frac{n_1}{m}\right)$$

$$h_2 = \frac{4h_c(n_1 - 1)}{m}\left(1 - \frac{n_1 - 1}{m}\right)$$

Whence

$$\frac{M_1}{h_1} = \frac{wp^2m}{8h_c}(m - n_1 + 1)$$

$$\frac{M_2}{h_2} = \frac{wp^2m}{8h_c}(m - n_1)$$

$$\frac{M_1}{h_1} - \frac{M_2}{h_2} = \frac{wp^2m}{8h_c} = \text{maximum H. C. in diagonal}$$

This value is independent of n_1, and therefore is the same in all diagonals. It may be written

$$\text{Maximum H. C. in each diagonal} = \frac{Pl}{8h_c} = \frac{wl^2}{8mh_c} = \frac{\text{maximum H. C. in chord}}{m} \qquad (11)$$

In a similar way it may be shown that the maximum *compression* in a vertical at a distance n_1p from the left reaction is, if g is the dead load per foot,

$$\text{Maximum comp. in vertical} = \frac{wp}{2m}(m - n_1 - 1)(n_1 - 1) - gp \qquad (12)$$

and it may be proved that the maximum compression is the same on two verticals equally distant from the center of the span.

$$\text{Maximum tension in a vertical} = (w + g)p$$

The above formulae hold for the parabolic truss with horizontal bottom chord, and load on bottom chord. They will not all hold for the parabolic truss with top chord horizontal, the formulae for the verticals being

$$\text{Compression in vertical for full loading} = (w + g)p$$

$$\text{Maximum compression in vertical for partial loading} = \frac{wp}{2m}(m - n_1 - 1)(n_1 + 1) + gp$$

and it may be shown that the maximum compression for partial loading is greater than the compression for full loading.

Formulae may also be deduced for the fish-belly truss.

16. Parabolic trusses with one straight chord have been largely built. The roadway is on the straight chord. If the bottom chord is straight,

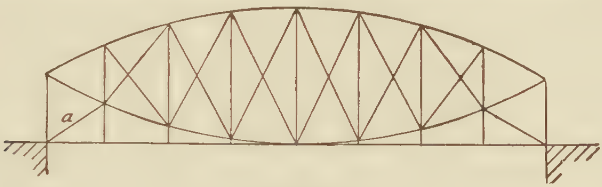


FIG. 112.

such trusses have the considerable disadvantage that no upper lateral bracing is possible for some distance from each end. Such bridges are not desirable except for short spans, which can be built as "pony" trusses, or with no upper lateral bracing whatever.

To overcome this defect, fish-belly or lenticular trusses have been used, supported on verticals at the ends. If these verticals are long enough, the top laterals can be carried to the ends, that is, to the top of the posts, but the lateral force can be carried only to the abutments by the rigidity

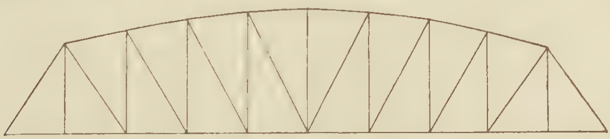


FIG. 113.—Camel-back truss.

of the posts, acting as beams. Figure 112 represents a type of this kind, which was largely used a few decades ago, for highway bridges, by the Berlin Iron Bridge Company. The bar a is to steady the end of the bridge.

Instead of using the complete parabolic truss, it is in all respects better, the bottom chord being straight, to use inclined end posts, and to curve the top chord between the tops of these posts, making the *camel-back* truss (Fig. 113). The joints of the top chord may lie on a parabola

or any other curve. Such a truss does not possess the properties of the true parabolic truss, which requires the parabola to continue to the ends. The diagonals will be in action under full load. But in such a truss bridge the upper laterals may be carried to the upper ends of the end posts, as in any through bridge, and there may be portal bracing if the end height is sufficient. The camel-back truss is computed like any other curved chord truss, as explained in Arts. 4 to 7.

17. The true parabolic truss is economical of material, for the straight chord stress is uniform from end to end, while the stress in the curved chord is nearly constant, increasing toward the end as the angle with the horizontal increases. The chords may therefore be made of nearly or quite uniform section from end to end, closely fitting the stress. The web is very light. Owing to this fact, the height may be economically made greater than in bridges with parallel chords, since the material in the chords diminishes as the height increases, while that in the web increases as the height increases.

18. The Pauli Truss.—It has been shown that, in the parabolic truss with one straight chord, the maximum stress in the straight chord is the same throughout its length, while the horizontal component of the maximum stress in the curved chord is the same in every panel and the actual stress increases from the center of the span toward the ends.

It is possible to design a truss in which the stress in the curved chord is constant throughout. This would involve, in comparison with the parabolic truss and with the same center height, increasing the other heights, since the stresses toward the end must be reduced and hence the lever arms increased. In this case the joints would not lie on a parabola, the diagonals would be in action under a full load, and the stress in the straight chord would not be constant. There is little or no advantage in making the stress in the curved chord constant rather than that in the straight chord.

With a double parabolic or lenticular truss it is possible to give dimensions that will make the maximum stress in each of the curved chords constant throughout. Such a truss is called the *Pauli* truss, from its inventor, and it involves a slight theoretic saving in material, too small, however, to be of consequence. This truss has been used considerably in Germany and perhaps Austria, but nowhere else, so far as the writer knows. The principal bridge of this type was that built in 1857 over the Rhine at Mainz, with four spans, each of 105.2 meters.

19. The Schwedler Truss.—In parallel-chord bridges the diagonals near the center have a minimum stress which is negative, thus requiring counters if they can carry only one kind of stress, while near the ends the minimum stress is positive. Thus the web members near the center must be supplemented by counters, or else they must be proportioned for reversal of stress, in either case requiring more material. In the para-

bolic truss, on the other hand, there must be a counter in every panel, or else every diagonal must be proportioned for reversal.

The principle of the Schwedler truss is to give the curved chord such a shape, the other chord being straight, that the minimum stress in all diagonals is just zero. This puts upon the curved chord as much of the duty of carrying shear as is consistent with having no reversal of stress in the diagonals.

The heights of the verticals are easily found, starting with a given height at the center, assuming a uniform load.

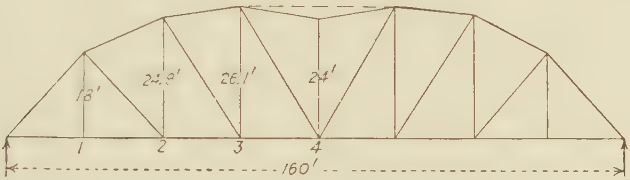


FIG. 114.

In Fig. 114, let the live panel load be 20 tons, the dead 8 tons, and let all diagonals be tension members. Then for the minimum stress in the diagonal of panel 3-4, the live load must be on joints 1, 2, 3, and

$$R_l = \frac{18}{8} \times 20 + 3\frac{1}{2} \times 8 = 73$$

$$M_4 = 73 \times 80 - 28(20 + 40 + 60) = 2{,}480$$
$$M_3 = 73 \times 60 - 28(20 + 40) = 2{,}700$$

The stress in the diagonal will be zero if

$$\frac{2{,}480}{24} = \frac{2{,}700}{h_3} \; \therefore h_3 = 26.1$$

For minimum stress in diagonal in panel 2-3, load joints 1 and 2; $R_l = 60.5$; $M_2 = 1{,}860$; $M_3 = 1{,}950$.

$$\therefore \frac{1{,}950}{26.1} = \frac{1{,}860}{h_2}; \; h_2 = 24.9$$

For panel 1-2: $R_l = 45.5$; $M_1 = 910$; $M_2 = 1{,}260$.

$$\therefore \frac{1{,}260}{24.9} = \frac{910}{h_1}; \; h_1 = 18$$

This truss has a depression in the top chord in the middle, but it has always been built with the chord horizontal over this depression, and in this portion, counters may be necessary, so that the full advantage of the principle is not obtained.

The theoretical advantages of this truss are considerable. As compared with the parallel-chord truss, curving the top chord down at the ends and making it take much of the shear relieves the web greatly without correspondingly increasing the chord. Many such bridges

have been built in Germany, but none outside, so far as the writer knows. They effect a considerable saving of material over parallel-chord trusses.

20. Trusses with Inclined Chords.—Instead of having the joints of a chord lie on a curve, they often lie on inclined straight lines, as in Fig. 115. Figures 116 and 117 show similar trusses much used in roofs. Such

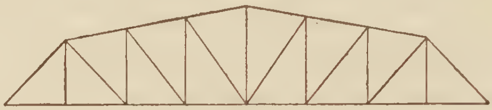


Fig. 115.

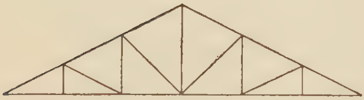


Fig. 116.

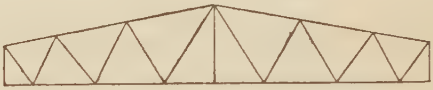


Fig. 117.

trusses have the same general advantages as curved-chord trusses, though in different degree, and they are generally easier to construct than curved-chord trusses.

The economy of a truss depends not merely upon the amount of material, but even more upon the labor cost. Hence forms which involve the least labor are apt to be preferred, even at the expense of material.

CHAPTER IX

ECONOMICS OF SIMPLE TRUSSES

1. In all design, economy should be sought. The mistake must not be made, however, of supposing that this means economy of material alone. Many other things must be considered, including labor, ease of erection, convenience of transportation, ease of maintenance and repairs, convenience of shop handling, etc. Years ago, when labor was cheap and material dear, material was perhaps the most important element, and a saving of material was sought by every practicable means. But today, when labor is very dear, it is much more important, and material much less important, than formerly. Rigidity must also be considered, and ease of inspection. Appearance is also an element for which, alone, it is often desirable to spend money.

All these things must be in the mind of the designer, who must not only be able to save material, but who must be familiar enough with methods of fabrication to know how to save labor costs and secure convenience in the shopwork.

The proper design will often reduce to a question of opinion, for the different elements cannot always be accurately compared. Yet there are some principles that may be stated; and economy of material should be aimed at so long as it does not involve disadvantages in other matters.

2. A Mathematical Minimum.—There is one mathematical principle that is often of service, which should be familiar to the engineer.

Let a quantity y, whose minimum is sought, involve a variable x in such a way that the expression for y consists of a constant, plus a term which varies directly with x, plus another term which varies inversely with x; or, if a, b, c, are constants,

$$y = a + bx + \frac{c}{x}$$

Then y will be a minimum when $dy/dx = 0$, or

$$\frac{dy}{dx} = 0 = b - \frac{c}{x^2}$$

or

$$bx = \frac{c}{x}$$

or *the two terms containing x are equal.*

3. A Long Bridge.—If a long bridge is to be built, the location of the piers will often be fixed by local conditions, such as the depth of water,

the character of the foundations, the presence of islands or reefs, or by governmental restrictions. Sometimes, however, the character of the foundations is uniform, and it is merely a question of the number of spans to be used. The following demonstration is often given for this case. There will be two abutments in any case, and the floor will cost practically the same, no matter how long the spans, so that these elements need not be considered. It is merely a question of the cost of the intermediate piers and the main supporting trusses or girders. If the spans are long and the foundations difficult, the cost of a pier will be nearly the same, no matter how long the spans, though of course the weight on the pier will increase with the length of span, and so the dimensions of the pier, and the number of piles under it, will increase with the length of span; but the labor costs of putting down the pier and procuring the equipment will be so great that the total cost of a pier will be nearly constant.

Let A = cost of one pier, assumed constant
L = total length to be bridged
l = length of one span, center to center of piers
n = number of spans = L/l
p = cost of main trusses per foot of length = al

Then the total cost Y will be

$$Y = (n - 1)A + nal^2 = \frac{LA}{l} - A + Lal \qquad (1)$$

Then by the principle of the last article for minimum Y,

$$\frac{LA}{l} = Lal$$

or

$$A = al^2 = \text{cost of one span} \qquad (2)$$

Hence for minimum cost, the cost of one pier must equal the cost of one truss span (not including floor).

This rule is only approximate, because the two assumptions on which it is based are inaccurate. The cost of a pier will increase somewhat with the span; yet for even considerable variations on either side of the minimum cost, the cost of the pier will not vary much, so that this assumption is probably quite close. The discussion in Art. 8 indicates that the cost of a truss alone (not including floor) varies as the square of the length if the ratio l/h is taken as constant; or the cost per foot varies as the length. This is also the basis of Dilworth's formulae (see Chap. III). On the other hand, Kunz assumes the cost of a truss per foot to be $b + al$, or the total cost $bl + al^2$, if a and b are constants. If this relation is used,

$$Y = (n - 1)A + nbl + nal^2 \qquad (3)$$
$$= \frac{L}{l}A - A + Lb + Lal$$

and, for minimum Y,

$$\frac{L}{l}A = Lal$$

or

$$A = al^2; \quad l = \sqrt{\frac{A}{a}} \tag{4}$$

Here the cost of one pier is less by bl than the cost of one truss span.

The value of l may be found if A and a are known, but these will vary, not only with the locality, but with the time.

These principles may serve as a useful guide in some cases, as in that of an elevated railroad, where the cost of one pair or bent of columns, with cross-girder, should nearly equal the cost of one truss span, or more accurately by Eq. (4).

The most accurate method of determining economical ratios is of course to make complete comparative estimates, but the above principles are useful in some cases.

4. Later in this work some comparisons will be made regarding the relative economy of different systems, as simply supported spans, continuous trusses, cantilevers, arches, and suspension bridges. It is proper at this point, however, that several inquiries regarding simple spans should present themselves, such as the following:

Is the triangular system (all web members inclined) more economical than the rectangular system with columns vertical?

In the rectangular system, should columns or ties be vertical?

In the triangular system, should all web members be equally inclined, or should columns have a different inclination from that of ties?

What is the economical inclination of web members?

Are curved-chord trusses (members straight but joints on a curve or inclined line) more economical than parallel-chord trusses?

In curved-chord trusses, should one chord only be curved, or should both be curved?

Are multiple-web systems economical?

Are inclined end posts economical?

It will be attempted in the following discussion to shed a little light on these points, and to show the principles involved.

5. Triangular System vs. Rectangular System.—In studying the economy of material in trusses, the point of view taken must be to regard a load as divided into two parts, namely, the parts supported at the two points of support, and to consider the truss as carrying, *i.e.*, transporting, these parts from the point of application of the load to the ends. Thus, in Fig. 118, a load at the center is divided into two equal parts, one going to the left abutment and the other to the right. These parts travel entirely through the web, if the chords are parallel and horizontal as assumed, because a vertical force can travel in direct stress only

through a member which has a vertical component. A vertical force can travel horizontally through a beam, in flexure, but in a truss there is supposed to be only direct stress, and with hinged and frictionless joints there can be only direct stress (with loads applied at the joints). Hence the load travels to the abutments entirely through the web; the chords are only there for the web to pull against, or to resist the web horizontal components. One-half the load P, then, in Fig. 118 goes up a, down b, up c, and so on. In Fig. 119, all of P goes up a, then one-half goes down b, up c, and so on.

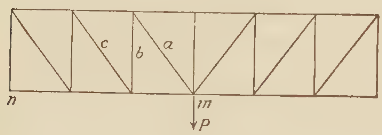


Fig. 118.

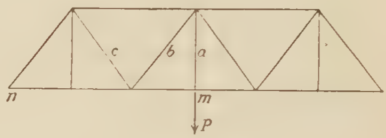


Fig. 119.

Now in the rectangular system the verticals do nothing toward carrying the load toward its destination; they merely carry it down faom the top chord to the bottom, so that the inclined ties can give it another step onward. It is as if a man, in walking, should step forward with only the right foot, each time merely bringing the left foot up to the same point. The presumption is, therefore, that in the triangular system, where every web member carries the load toward its destination, the economy is greatest.

In carrying a load P (Fig. 120) from m to n, it may be done in numberless ways. It might be done in one step only, as in the figure. But

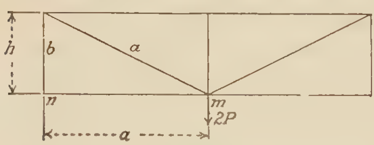


Fig. 120.

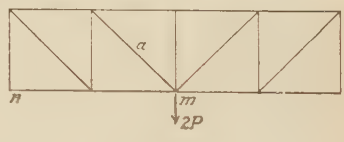


Fig. 121.

unless the span were very short, this would be very uneconomical. The bottom chord would have no stress, but the top chord would have the same stress throughout, and the same that it has in the middle; and there would be a great excess of material in the long stringers and the heavy floor beam. But even the web members alone might be uneconomical. The stress in a would be $P\frac{\sqrt{a^2+h^2}}{h}$, and the amount of material in it $P\frac{a^2+h^2}{hf_t}$; and the material in b would be Ph/f_c. Hence the total material would be

$$Q = \frac{P(a^2+h^2)}{hf_t} + \frac{Ph}{f_c}.$$

If the load were carried in two steps, as in Fig. 121, the total material in the web would be

$$Q_1 = \frac{2P\left(\frac{a^2}{4} + h^2\right)}{hf_t} + \frac{2Ph}{f_c}$$

It will be found that, if $f_c = f_t$, which is here allowable, since the gross area of a tension piece should be taken,

$$\text{if } a > 2h, \quad Q > Q_1$$

The use of long diagonals, taking the load directly from the point of application to the ends, was the principle of the Bollman truss (Fig. 135) which was deservedly abandoned years ago. It is not only wasteful of material, but the long, sagging diagonals were objectionable.

Generally $a > 2h$ and $Q > Q_1$; and, in considering economy, the floor must also be considered. There is an economical panel length, though this is dependent upon the truss as well as upon the floor.

It is obvious that the material in the chords is greater in Fig. 120 than in Fig. 121; and also that there is some inclination of the diagonals which is most economical.

It is also clear that, as regards material, the triangular truss is more economical than the rectangular. The vertical shear at each point must be carried by some diagonal, and it makes little difference, in general, whether it is a tie or a strut. In Fig. 122 the total length of diagonals is the same in the two figures, and the stresses are the same; but in Fig. 122*a* there are two more necessary web members than in Fig. 122*b* (in one-half the truss), not counting the secondary p; or the two trusses may be compared more in detail, as follows: bars a, c, d, e, h, g, and n are precisely the same in both; b is in compression while b' is in tension, and b may have slightly more material; f' has more stress than f, but k has the same excess over k'; if bars o and q are used to support the top chord, they have less material than o' and q'; while p in tension, carrying a panel load, has probably less than p'; clearly (b) is more economical of material.

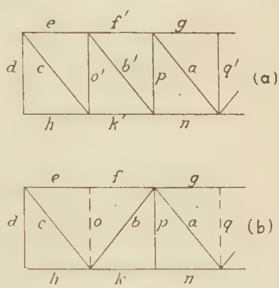


FIG. 122.

6. Vertical or Inclined Columns.—In the rectangular system, the columns should be vertical, and the ties inclined, for then the columns are shortest, and the stress in them is smallest. If the ties are inclined, their deflection due to their own weight is reduced by the tension, while if the columns were inclined, the deflection would be increased by the compression. The connections are simpler with ties inclined.

7. Inclination of Web Members in Triangular System.—As the allowable stress in a column decreases as the length increases, while in a tie it is independent of the length, it is clear that for theoretical economy the columns should be steeper than the ties. The Post trusss (Fig. 139) was based on this fact. Using the column formula for cast-iron columns, the proper economical angles of the ties and columns were determined.[1] It was found that the columns should make an angle of 39°49′ with the vertical.

This truss, however, on account of the uncertainties of the stresses, which were far more objectionable than the theoretical saving by using the economical angles, has gone entirely out of use.

It is possible, however, to make some reasonable theoretical study of the best inclination of the web members. In practice, both web ties and struts are inclined at the same angle.

8. Material in Simple Trusses.—The chords carry the moments at the joints. Let Fig. 123 *b* represent a rectangular bridge truss and let the curve above be the moment curve divided by *h*, which will be approximately a parabola. Then the material in the chord $f'g'$ will be propor-

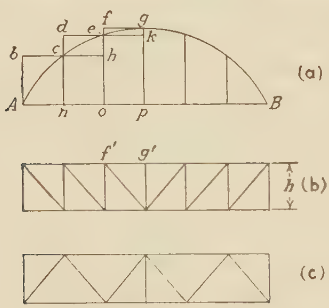


FIG. 123.

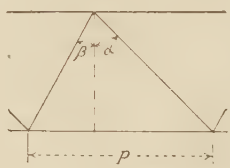


FIG. 124.

tional to the area *opgf*, and the total material in half the top chord will be represented by the area *Abcdefgp*, while that of the bottom chord will be represented by *nchekp*. There will be an end bottom chord piece, though not required for the vertical loads. For practical purposes the total material in both chords will be approximately represented by twice the area of the curve *AgB*. This will also be true for a triangular truss *c*; here the material in the top chord will be *ndkp*, and that in the bottom chord *Abhfgp*. Calling the material in the chords Q_c and f the unit stress, and calling k a constant greater than unity, which allows for excess above the theoretical value, such as weight of details, gussets, rivet heads, etc.,

$$Q_c = \frac{2k}{hf} \cdot \frac{2}{3} \cdot l \cdot \frac{wl^2}{8} = \frac{wl^3k}{6hf} \tag{5}$$

where w is the total load per foot. If the dead and live loads are w_d and w_l, and the unit stresses f_d and $_lf_l$

$$Q_c = \frac{l^3k}{6h}\left(\frac{w_d}{f_d} + \frac{w_l}{f_l}\right) \tag{5a}$$

[1] See MERRILL, W. E., "Iron Truss Bridges for Railroads." D. Van Nostrand Co., New York, 1870.

From this it follows that the material in the chords varies inversely with the height. Double the height with the same span, and the material in the chords is halved. The weight of chords *per foot* will vary with l^2/h; and if l/h is a fixed ratio, it will vary with l, while the total weight of chords will vary with l^2.

For the web, let p in Fig. 124 be a panel to the left of the center of the span, the web ties being inclined α and the struts β to the vertical. Let S be the maximum shear in the panel. Then the material in one strut and one tie will be

$$Q_w = \frac{Sh}{f}(\sec^2 \alpha + \sec^2 \beta) \tag{6}$$

The allowable stress in tension is greater than that in tension, but the net area must be taken in tension, so that it is here allowable to use the same unit stress for both, though this would be modified if the reduction by the column formula is considerable.

The length of the panel is $h(\tan \alpha + \tan \beta)$; hence the weight of the web per foot is

$$\frac{Q_w}{p} = \frac{S(\sec^2 \alpha + \sec^2 \beta)k_1}{f(\tan \alpha + \tan \beta)} \tag{7}$$

This will vary with S, and in the different panels of the bridge, but it is independent of h if α and β are fixed. Hence it would appear that with constant α and β, the higher the truss the less the material in inverse proportion, and that the greatest economy would occur with a very large height (depth). Actually, however, the increase of material in web struts, as the depth increases, and the material necessary to provide lateral rigidity and stability put a limit to the height, the value of h/l varying from about $\frac{1}{5}$ to $\frac{1}{10}$, perhaps averaging $\frac{1}{7}$ or $\frac{1}{8}$. It is thus seen that the column formula and the requirements of lateral rigidity are important, probably vital, elements determining the economical depth.

In Eq. (7), Q_w/p will be a minimum when

$$\alpha = \beta = 45°, \text{ and the minimum value is } \frac{2S}{f} \tag{7a}$$

This is for the triangular system.

In the rectangular system $\beta = 0$, and

$$\frac{Q_w}{p} = \frac{S(2 + \tan^2 \alpha)}{f \tan \alpha} \tag{7b}$$

and the minimum value will occur when $\tan \alpha = \sqrt{2}$. This minimum value is

$$2\sqrt{2}\,\frac{S}{f} = 2.828\frac{S}{f}, \text{ for } \alpha = \text{about } 55° \tag{7c}$$

Hence it appears that, for the most economical proportions in both cases, the amount of material in the web of the rectangular system is

1.414 times that in the triangular system. The angle of 55°, however, is too flat.

Equation (7) gives the weight of the web, per foot, in a panel in which the shear is S. We may find an expression for the total material in the web, approximately, by multiplying Eq. (7) by dx, integrating over half the span, and multiplying by two. The shear curves for a uniform load are shown in Fig. 125, the lower curve for dead load and the upper for maximum live shear. The area of the left half of these curves, which does not allow for counters, is $l^2/48$ $(6w_d + 7w_l)$. To allow for counters, let us replace $(6w_d + 7w_l)$ by $8(w_d + w_l) = 8w$. Then the total material in the web is

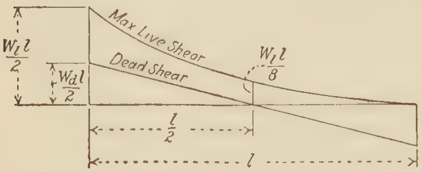


FIG. 125.

$$\text{Total } Q_w = \frac{wl^2(\sec^2 \alpha + \sec^2 \beta)k_1}{3f(\tan \alpha + \tan \beta)} \tag{8}$$

If $\alpha = \beta$,

$$\text{total } Q_w = \frac{2wl^2k_1}{3f \sin 2\alpha}$$

If $\beta = 0$,

$$\text{total } Q_w = \frac{wl^2k_1(2 + \tan^2 \alpha)}{3f \tan \alpha}$$

The total material in the truss is then:

If $\alpha = \beta$,

$$Q = \frac{wl^3k}{6hf} + \frac{2wl^2k_1}{3f \sin 2\alpha}$$

If $\beta = 0$,

$$Q = \frac{wl^3k}{6hf} + \frac{wl^2k_1(2 + \tan^2 \alpha)}{3f \tan \alpha}$$

or, if a_1 and b_1 are constants:

If $\alpha = \beta$,

$$Q = l^2\left(a\frac{l}{h} + b\right)$$

If $\beta = 0$,

$$Q = l^2\left(a_1\frac{l}{h} + b_1\right)$$

The constants depend upon w, k, f, k_1, and α.

Generally l/h may be considered a constant; in this case, if c is a constant, the total material in a truss may be expressed in the form

$$Q = cl^2 \tag{9}$$

and the material per foot is cl.

The total material in a truss span, including the floor, may then be expressed in the form

$$Q = fl + cl^2 \tag{10}$$

if f represents the weight of the floor per foot.

This is the principal result of the foregoing discussion. See also the chapter on Loading.

8a. Economic Depth.—It has been shown that the total material in the chords varies inversely as the depth and that, with the economic inclination of web members, the total material in the web is independent of the height, if no account is taken of the effect of the column formula. Practically, the material in the web will increase as the height increases.

The precise economic depth can be determined only by comparative designs. It is not important, however, to determine it precisely. Merriman and Jacoby find that "the depth may vary 10 per cent from the economic depth without increasing the material as much as 1 per cent."[1]

Since the stiffness increases with the depth, and since the economic depth is large, the tendency of practice has been to use greater depths than formerly. Often the depth is made so great that it is desirable to stiffen vertical columns by intermediate horizontal braces.

9. Economy of Curved-chord Trusses.—In the truss with parallel chords, the transverse shear is carried entirely by the web. This shear is small at the center and large at the ends. The chords carry the moment, and have a large stress at the center and a small stress at the ends: often there is an excess of material near the ends. By giving the chords a small inclination, their stress would not be much increased, and yet their vertical components may be considerable, thus relieving the web of a part of the shear. For long spans, one chord at least should be curved. It is decidedly economical, and generally done.

If the joints of the top chord lie on a parabola, we have seen that there is no stress in web diagonals under a full uniform load, that the stress in the straight chord is the same throughout, that the stress in the curved chord increases toward the ends, and that counters are required in every panel. This is not in general an economical form of truss, except for light bridges, and if the top chord of a through bridge is curved, the lateral bracing is interfered with. It is desirable to have complete lateral and portal bracing in a long span; but beginning at the top of the end posts the depth should gradually increase to the center, giving a form of truss sometimes termed the *camel-back*.

10. One or both chords may be curved. For convenience of floor connections, however, it is customary to make the chord carrying the floor straight, that is, the bottom chord of through bridges and the top

[1] A Text Book on Roof and Bridges, Part III, Bridge Design, p. 31, New York, John Wiley & Sons, 1909.

chord of deck bridges. This is true economy, and enables all the advantages of the curved chord to be gained.

For small highway bridges, a form of truss with both chords curved was largely built by the Berlin Bridge Company. This had the form shown in Fig. 126. It is, however, little used at present, though it was not uneconomical of material; but the top lateral bracing was interfered with, unless, as in the figure, the end vertical supports were made high enough to give clearance.

Fig. 126.

11. Multiple Systems of Web.—Such systems had for their object to keep the panel length small while maintaining a reasonable inclination of the diagonals. But this may be done just as easily by subdividing the panels of a primary single system and keeping the structure statically determinate. Because of the great uncertainty of the stresses in multiple systems, they have properly been discarded in good design.

It is singular that progress should consist so largely in adopting at first something complex, and then approaching simplicity. The earliest long-span trusses, both in Europe and in America, were generally of multiple systems. The Europeans were greater sinners in this respect than we were. In this country, the Whipple and Linville trusses were used by their inventors, with double systems. In Europe, more systems were used in the early French and German bridges, some being merely lattice systems in iron, like the American Town trusses in wood.

12. Economy of Inclined End Posts.—In a through rectangular system, is it economical to use inclined end posts? Compare the two ends in Fig. 127. Bars a are the same in either case. Tie e is replaced by strut e' having the same length and stress but probably slightly more

material. Bar d is unnecessary except to carry stress from wind and friction on bearings. Column c is replaced by tie d' of the same length and stress, and d' will surely have considerably less material than c

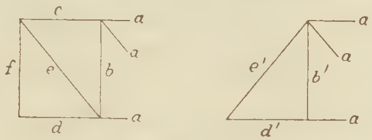


FIG. 127.

and d together. Column b is replaced by tie b', having less material. Column f is dispensed with.

Obviously there is a considerable saving in material by using the inclined end post, with no offsetting disadvantages.

CHAPTER X

A SKETCH OF THE DEVELOPMENT OF STRUCTURAL FORMS

1. The oldest bridges of importance were of stone. The Romans, and even more ancient peoples were acquainted with the principle of the arch. Assyrian arches of wedge-shaped brick have been disclosed by excavations. To the ancient Romans, however, is due the special development of the arch, and its use on a large scale in the construction of

FIG. 128.—The Pont du Gard.

viaducts, aqueducts, sewers, buildings, and bridges. The Cloaca Maxima, a sewer still in use, consists of three rings of stones. One of the Roman bridges over the Tiber had a span of 84 feet. The Pont du Gard (Fig. 128) was one of the earliest arch bridges built by the Romans outside of Italy. It carried an aqueduct supplying Nimes with water.[1]

2. Timber was also used for bridges in ancient times, but only in the form of beams. Julius Caesar built a trestle bridge across the Rhine,

[1] See paper by author on "The Historical Development of Stone Bridges," *Jour. Assoc. Eng. Soc.*, pp, 117–143, 1896; also HOWE, MALVERD A., "History of the Stone Arch; The Rose Technic," December, 1898; NEFF, F. H., "The Development of Masonry Bridges," *The Integral*, vol. V, No. 8, 1897.

the drawings for which have been reproduced from the description given in the "Commentaries."[1]

After the fall of the Western Empire there was little bridge building in Europe until the twelfth century.

Palladio, a famous Italian architect, 1560–1580, used the truss idea for the first time, in the forms shown in Fig. 129, but without a knowledge of the principles involved. From that time until the nineteenth century

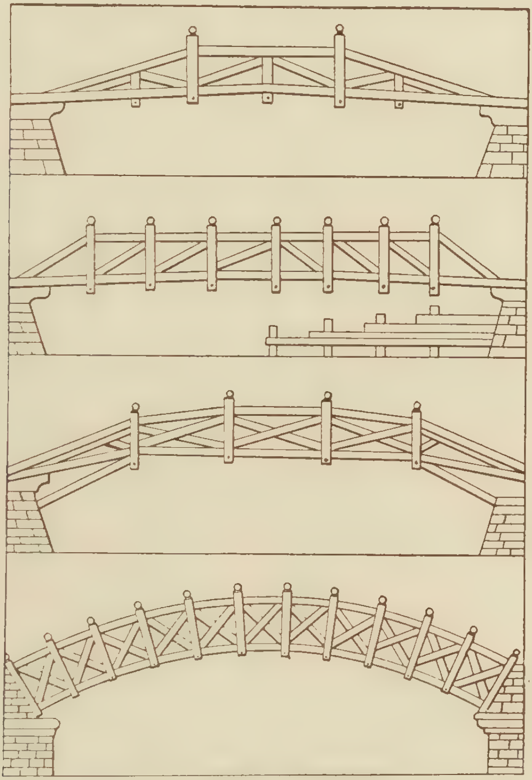


FIG. 129.

wooden bridges were built, in Europe and in America, generally with complicated forms (Figs. 130 to 133). The development in the United States is well described in Mr. Cooper's paper.[2] The Town lattice, shown in Fig. 133, had constant section throughout. The Colossus bridge (Fig. 132) had a span of over 340 feet, but the loads were light at

[1] See RONDELET, "L'Art de Batir," GAUTIER, "Traité des Ponto," Paris, 1728, plate XIV; and GAUTHEY, "Traité de la Contruction des Ponts," Paris, 1813.

[2] See JOHNSON, BRYAN, and TURNEAURE, "Modern Framed Structures," vol. 1, pp. 7–14; MERRIMAN and JACOBY, "Bridge Design," pp. 1–22; COOPER, "American Railroad Bridges," *Trans. Am. Soc. C. E.*, vol. XXI, pp. 1–58, 1889.

that time. Most of these bridges were combinations of a truss with an arch. The theory of neither was understood.

Following the Town truss, the Long truss, patented in 1830 and 1839 by Brevet-Lt. Col. Long, of the U. S. Engineers, was somewhat used, either by itself or combined with the arch. Like the Town truss, it could be made entirely of wood, by framing the parts together, and using wooden keys or treenails.

In 1840 the Howe truss was patented by William Howe, and soon became the standard form of wooden truss, used even to the present day.

In 1844 the Pratt truss was patented, having the diagonal tension members of iron, and the vertical members of wood. It was not so well

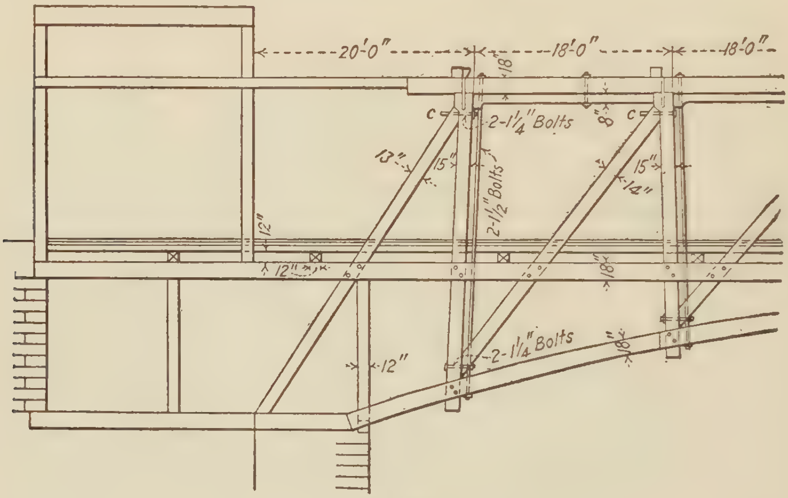


FIG. 130.

suited for combination bridges as the Howe truss, but became the favorite type for iron bridges.

3. Iron bridges are more recent than those of stone or wood. The first iron bridges were of cast iron, and it was not until the invention of the puddling process by Henry Cort, about 1780, that wrought iron could be produced in sufficient quantity and at low enough cost to be extensively used; nor was it until after the invention of the Bessemer process by Sir Henry Bessemer, about 1856, that steel could be similarly made available.

The stone arch suggested the cast-iron arch, the wooden beam suggested the cast-iron beam, and the suspension bridge, which was known even in primitive times, suggested the wire-cable suspension bridge.

The first iron bridge was the cast-iron arch at Coolbrookdale, over the Severn, in England, built by John Wilkinson and Abraham Darley, in 1773–1779, with a span of 100 feet and a rise of 42 feet. This was followed

by many cast-iron arches in England, France, and Germany, in some of which the arch was made of cast-iron voussoirs of an I-section, flanged and bolted together, like the bridge over the Thames at Southwark, with a

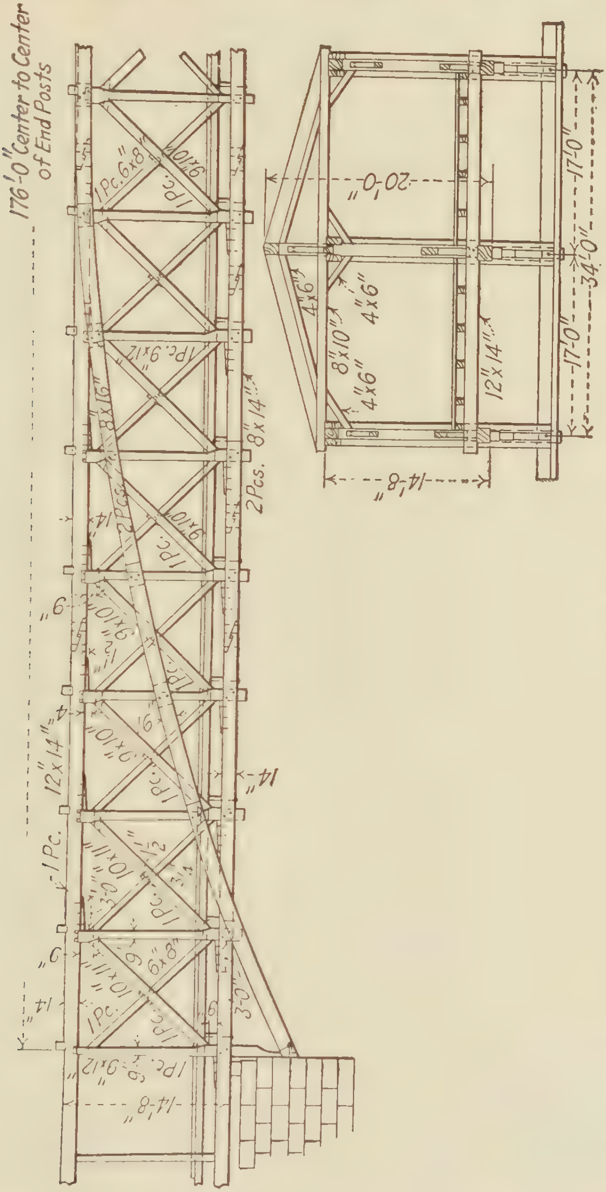


Fig. 131.

span of 210 feet, built by Rennie in 1814–1819; while in others the section was a hollow tube. The Pont du Carrousel over the Seine at Paris, built by Polonceau in 1834–36, consisted of three spans of 47.7 meters

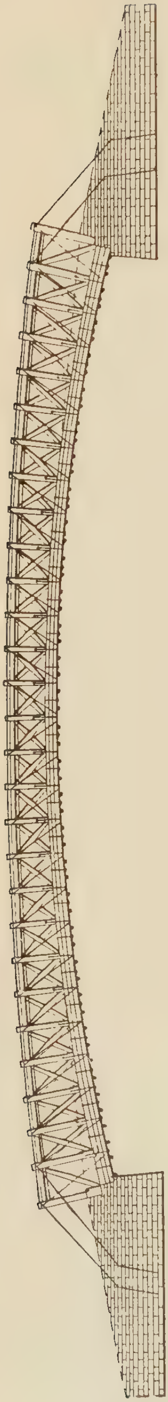


Fig. 132.

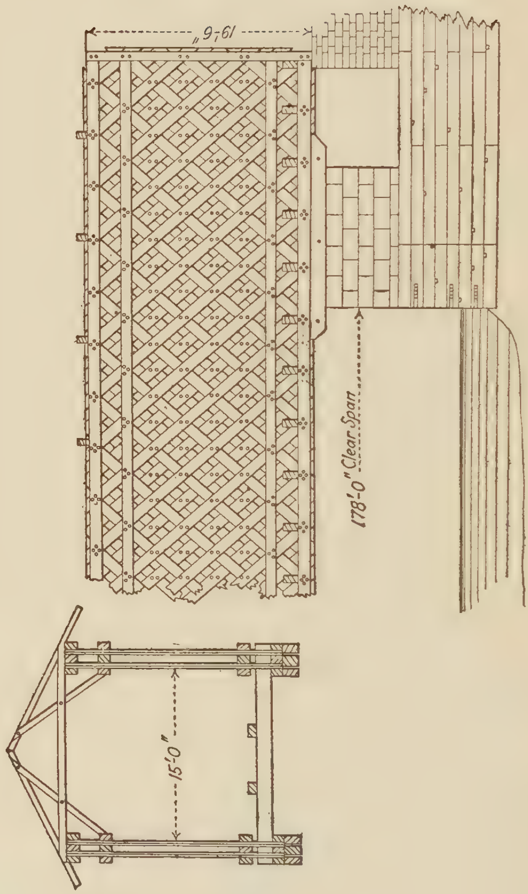


Fig. 133.

each; the section was a hollow elliptical tube enclosing a wooden arch, the two halves being separate, and bolted together along thetop and bottom.

Cast-iron beam bridges were built on the early railroads of England, in some cases with one straight flange and one curved flange, the tension flange larger than the compression flange.

4. When wrought iron came into use, it was employed in combination with cast iron, using cast iron in compression and wrought iron in tension. Plate girders were built up to 90 feet in span, having the top flange of cast iron, and with double webs of wrought iron, bolted to the top flange and riveted to the wrought-iron bottom flange. Cast-iron beams were trussed with wrought-iron rods. One bridge of this kind, built by Robert Stephenson, collapsed, owing to a faulty connection at the end between the rods and the beam.

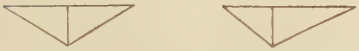


Fig. 134.

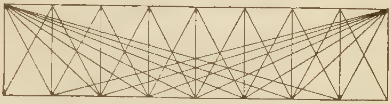


Fig. 135.—Bollman truss.

5. The first iron-truss bridges were combination bridges, the compression members of cast iron and the tension members of wrought iron. In this country the theory of the truss was not understood at first, although trusses had been built. Perhaps the only truss that was even approximately understood was the simple King post (Fig. 134). At all events, the earliest type of iron truss was the Bollman truss, invented by Wendell Bollman, and largely used between 1840 and 1850 on the Baltimore and Ohio Railroad. It is merely a combination of King-post trusses (Fig. 135). Similar was the Fink truss (Fig. 136), introduced about 1851 by Albert Fink. In both these trusses the compression members were at first of cast iron.

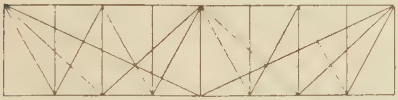


Fig. 136.—Fink truss.

Fig. 137.—Whipple truss.

6. The first near approach to the modern truss in this country was made by Squire Whipple, who built, in 1852, a bridge of 152-foot span, near Troy, N. Y., on the Rensselaer and Saratoga Railway. This was a double system Pratt truss, that is, a Whipple truss (Fig. 137), with compression members of cast iron, and pin connections. Whipple, too, was the first in this country, to give a correct analysis of the stresses in a truss. Johnson, Bryan, and Turneaure, in their excellent work on "Modern Framed Structures,"[1] make the following statement:

[1] See vol. I, p. 12.

Squire Whipple, C. E., a philosophical instrument maker of Utica, N. Y., seems to have the distinguished honor of being the first man who ever correctly and adequately analyzed the stresses in a truss, that is, in a framework by which loads are carried horizontally from joint to joint by means of chord and web systems and finally delivered vertically upon the abutments. This is the true function of a truss; and although Palladio, Long, Howe, and Pratt had already designed and built such structures, they had never known what stresses the members were subjected to, and did as all engineers and builders did in those days—dimensioned their parts in accordance with such experience and judgment as they could bring to bear upon the problem. For instance, the vertical tie rods in the Howe truss were at first made of the same size from end to end. Mr. Whipple's work is preserved in a small book of 120 pages entitled "A Work on Bridge Building, consisting of Two Essays, the one elementary and general, and the other giving Original Plans and Practical Details for Iron and Wooden Bridges." By S. Whipple, C. E., Utica, N. Y., 1847.

This book had been published three or four years when Hermann Haupt wrote his work on bridges. Apparently Mr. Haupt had never seen a copy of it, since he claims his work as original also, and there is no internal evidence that he had seen Whipple's book. His methods of analysis are much cruder than Mr. Whipple's and far less complete.

The theory of the stone arch, and of arch and suspension bridges under fixed or uniform loads, was early developed, but the true theory of truss action seems to have originated with Mr. Whipple. His manuscript was written in 1846. See Development of the Iron Bridge, by S. Whipple in *Railroad Gazette*, April 19, 1889; and Discussion by A. P. Boller in *Transactions of A.S.C.E.*, vol. XXV, p. 362. Also American Railroad Bridges, by Theodore Cooper, *Transactions of A.S.C.E.*, vol. XXI, p. 1.

7. In Europe, combination trusses were built as far back as 1846. A triangular system was used by the Belgian engineer Neville, in that year, with curious connections and with the diagonals all alike. The English Captain Warren patented the triangular Warren truss, and a combination bridge on this system was built across the Trent at Newark, in 1851, with a span of 240.5 feet, having pin connections.

In this country, besides iron trusses with parallel chords and cast-iron compression members, a number of bridges were built with curved cast-iron top chord and wrought-iron bottom chord. These may still be met with in country districts.

8. When the use of cast iron for main members was abandoned on account of its brittleness, it was still used for minor parts and for blocks at the joints. The Phoenix truss, large numbers of which were built up to 1880, had all members of wrought iron and pin connections. The compression members were of Phoenix columns, and at each top chord joint and at each end of the inclined end posts there was a cast-iron joint block against which the chord pieces and the vertical posts abutted, and through which the pin passed. The web diagonals thus had to be entirely outside the chord, and the pins were in many cases too small; yet few have given any trouble. John W. Murphy was perhaps the first to use wrought iron for all bars, with cast-iron joint blocks, in 1863.

9. In Europe, when cast iron was abandoned, the types used may be called extreme modifications of the plate girder. The first extreme was represented by the tubular bridge, in which the flanges were widened and made of cellular form extending clear across the roadway, the train going through the tube. Such were the famous Britannia and Conway tubular bridges, and the Victoria bridge across the St. Lawrence River at Montreal, all built by Robert Stephenson. The first two are still in existence, but the Victoria bridge was replaced in 1890 by a modern truss bridge. The Britannia bridge consisted of two tubes or tunnels each carrying one track; it has two spans of 460 feet and two of 230 feet. The Victoria bridge had 24 spans of 242 feet and one of 330 feet. The Britannia bridge was at first intended to be a chain suspension bridge with the tubes as stiffening girders, but tests showed the tubes to be so strong that the chains were not used. The spans of this bridge were floated out to between the piers, and raised to their final positions.

Following these bridges, perhaps the two most notable bridges in England before the modern era were that over the Wye, a tributary of the Severn, at Chepstow, and that over the Tamar at Saltash, near Plymouth. The Chepstow bridge had a span of 305 feet and was a through bridge of three panels only, the curved top chord being a circular tube 9 feet in diameter, and the tension members eyebars. It was built in 1850-1852 by Brunel, who also built, before 1850, the Saltash bridge. This was lenticular in shape, with curved upper and lower chords, the top chord a single elliptical tube reaching over the entire single track, 17 feet in horizontal diameter and 12 feet in vertical diameter. It had two main spans of 455 feet each. The liking of English engineers for tubular compression members is shown by their use of this section in the Forth bridge.

The other extreme in the development of the long-span wrought-iron bridge from the plate girder consisted in making the web, not a plate, but a close lattice with small bars very close together, that is, a multiple system with many systems, somewhat like the Town wooden lattice. This form was used in England, but mainly on the continent, as in the bridge over the Weichsel at Dirschau with 385 feet clear span, that over the Nogat at Marienburg, both built in the early fifties, and the bridge over the Rhine near Kehl. The Pauli patent for a lenticular truss with constant chord stress (see Chap. VIII) was taken out in 1856, and the theory of the truss began to be developed, as it had been a little earlier, but less thoroughly, in America.

10. The oldest bridges entirely of wrought iron, however, were suspension bridges. Wrought iron was known to the ancients, but could not be made in large pieces. The first iron suspension bridges were therefore made of chains. A publication of 1667 mentions one such in China.

The first suspension bridge in America was built by Finlay in 1796, with a span of 70 feet. He patented this in 1801, and it is said that up to 1811 forty similar bridges were built, the largest with a span of 306 feet over the Schuylkill. In 1809, John Tempelmann built one of these chain bridges, with 240 feet span, over the Merrimack River near Newburyport, Mass.[1]

A number of other suspension bridges were built in this country in years following, that over the Ohio at Wheeling, built by Charles Ellet, one of the greatest engineers of his day, having a span of 1,010 feet and a central ordinate of 61 feet. Other long suspension bridges were built, culminating in the Brooklyn bridge, completed by Roebling in 1877, with a span of 1,595 feet, which for a time was the longest suspension span in the world, but has since been surpassed by the Williamsburg bridge over the East River, with a span of 1,600 feet, and by the bridge over the Delaware River at Philadelphia, with a span of 1,750 feet.

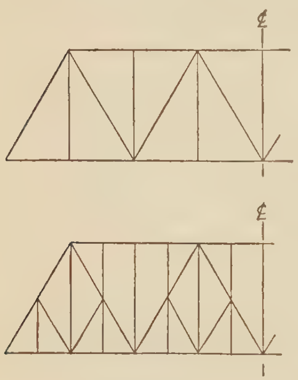


Fig. 138.

The first iron suspension bridge in England was a short one of 60 feet span, built in 1741; the roadway, for foot passengers, lay directly upon the chains.

The suspension bridge is thus the oldest type of bridge of iron or steel, and it long antedated the iron truss bridge in this country.

11. The first truss bridges of considerable span in this country were for the most part of the Whipple type, with cast-iron compression members. Many of these were built by J. H. Linville, engineer of the Keystone Bridge Company of Pittsburgh. In 1861, Linville first used eyebars and wrought-iron posts in the web system, with cast-iron upper chords. Later, wrought iron was used throughout.

The Whipple double-system type offered the advantage of moderate panel length, economical height of truss, and economical inclination of the diagonals. Being statically undetermined, however, they have given way to the subdivided triangular or rectangular systems, (Fig. 138) which have the above advantages and are statically determined. Such a type was used by Albert Fink, in 1868–1870, in his bridge over the Ohio River at Louisville, and it has become the prevailing type in cases where the simple Pratt type will not apply economically.

[1] The chains of this bridge gave way in 1927 under the load of a heavily loaded oxcart, but it was rebuilt in 1909, when it was superseded by a wire-cable suspension bridge built by the county commissioners of Essex County, under the direction of R. R. Evans, County Engineer, the new structure having been designed by the writer.

The first wire-cable suspension bridge was one built in 1815 across the Schuylkill at Philadelphia, with a span of 408 feet.

There were other early types, such as the Post truss (Fig. 139), which had for its object to incline the web posts so that they would transfer the loads toward the ends and not merely carry them down from top to bottom chord. The economical angle for the posts was investigated, using the formula for cast-iron columns. The stresses, however, are very uncertain, and this form of truss has long been abandoned.

The prevailing form of steel truss in this country is now the Pratt truss or the subdivided Pratt truss (sometimes called the Baltimore or Pettit truss). The triangular truss is often used, subdivided if necessary, but the connection of floor beams is not so convenient as in the Pratt truss. For long spans one chord is practically always inclined.

It is interesting to note that in the development of bridge forms, as in other things, complexity has preceded simplicity. The early wrought-iron trusses were the undetermined Whipple or lattice systems. The simple triangular or rectangular systems, subdivided if necessary, come later, although the Warren truss, as above shown, was an early form.

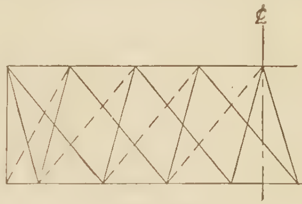


Fig. 139.

12. Cantilever bridges were early used in America. The first was the Kentucky River bridge, with three spans of 375 feet, built in 1876–1877 by C. Shaler Smith. This was followed by the bridge over the Mississippi near St. Paul, built by the same engineer in 1879–1880, with two spans of 270 feet and one of 324 feet; and by the Niagara bridge, built in 1883, the Fraser River bridge in 1885, the St. John bridge in 1885, and many others, culminating with the Quebec bridge in 1918, with two side spans of 515 feet and a central span of 1,800 feet, the longest single span in existence.

Steel-arch bridges have also been widely used. The first important one was that over the Mississippi at St. Louis, with spans of 502 and 520 feet, built in 1868–1874 by James B. Eads.

Suspension bridges have already been mentioned.

Continuous girders have found little favor in America, though largely used in Europe. With the increasing tendency to adopt statically undetermined forms, however, some continuous bridges of long span have recently been built, where the foundations were solid enough to prevent undue settlement of the piers, as in the bridge over the Ohio at Sciotoville, on a branch of the Chesapeake and Ohio Railway, built by the McClintic-Marshall Company of Pittsburgh, from designs of Gustav Lindenthal. This has two continuous spans of 775 feet each, and was the first large continuous bridge in America and the largest and heaviest fully riveted truss. It was completed in 1917.

Another recent continuous bridge is that at Kettle Rapids in Canada, on the Hudson Bay Railway, with a channel span of 400 feet and two side spans of 300 feet each, built in 1918.

An older continuous bridge was that of the Canadian Pacific Railway over the St. Lawrence at Lachine, completed in 1887 from designs by C. Shaler Smith. This bridge had four continuous spans on five supports, the end spans deck, the two center spans through. The foundation was rock, but in order to be able to maintain the assumed conditions, there were adjusting screws under the pier supports. The adjustment was never needed. This bridge was replaced in 1913 by four simple spans, two through center spans of 408 feet, and two deck side spans of 269 feet.[1]

With the extensive use of concrete and reinforced concrete for bridges, many concrete arches and girder bridges have been built, within the past 30 years, the longest arch having a span of 280 feet, though much longer spans have been proposed.

13. Pin vs. Riveted Trusses.—For many years the typical American bridge had pin connections, while European bridges had riveted connections. We have seen, however, that the Warren truss in England, as in the Trent bridge at Newark, had pin connections as far back as 1851. Still, the pin bridge never has gained much foothold in Europe. In this country it was almost universally used after eyebars came into use, largely because of the ease and speed of erection. A pin bridge of 160 feet span, which in Europe as a riveted bridge would have required 10 or 12 days for its erection, has been erected in 8½ hours. In the early days of railroads, and over rivers subject to sudden floods, this is a great advantage.

Of late years there has been a change of opinion, and riveted connections are now generally preferred for spans up to from 150 to 300 feet (see A.R.E.A. specifications). Indeed, some railroad companies have for many years preferred riveted bridges.

> In 1859, Howard Carroll commenced building riveted lattice bridges for the New York Central Railroad and its connections . . . This early work of excellent character gave the preference to this class of bridges upon this road and other neighboring systems.[2]

This change of feeling has probably been due mainly to the recognition of two facts: first, that in a pin bridge the stresses are not strictly statically determined, but that the friction on the pin may produce appreciable bending moments in the bars at the joints; and, second, that riveted bridges are more rigid than pin-connected bridges.

The history of iron bridges is well treated, more completely than in any other work, in Heinzerling's "Die Brücken in Eisen."[3]

[1] See paper by MOTLEY, P. B., *Trans. Can. Soc. C. E.*, vol. XXVIII, part 1, 1914.

[2] COOPER, "American Railroad Bridges."

[3] Another valuable work is MEHRTENS, GEORG, "Der deutsche Brückenbau im XIX Jahrhundert," Julius Springer, Berlin, 1900, beautifully illustrated.

The development of bridge building in America is best treated in Mr. Cooper's paper, already referred to; and more in detail in four works or reports by foreign engineers.[1]

[1] Steiner, F., "Ueber Brückenbauten in den Vereinigten Staaten von Nord Amerika," a report on the Philadelphia Exposition of 1876, Faesy & Frick, Vienna, 229 pp., 1878; Ritter, W., "Der Brückenbau in den Vereinigten Staaten Americas," a report on the Chicago World's Fair of 1893, Fritz Haller & Co., Bern, 66 pp., 1894; Comolli, L. Ant., "Les Ponts de l'Amerique du Nord," Ambroise Lefèvre, Paris, 1879, one volume of 247 pp. and atlas; Lavoinne, E., and E. Pontzen, "Les Chemins de Fer en Amerique," Dunod, Paris, 1880, with a good chapter on bridges, and atlas.

The peculiarities of American bridges are well brought out and discussed in these works of foreign engineers.

There is a particularly good chapter on the evolution of bridges in Part III of Merriman and Jacoby's work on "Roofs and Bridges," and also a good chapter in Johnson, Bryan, and Turneaure, Part I.

CHAPTER XI

GRAPHICAL STATICS. GENERAL PRINCIPLES

1. Graphical statics is a branch of mechanics which has grown up principally within the past 50 years, and treats of the graphical solution of the problems of statics. There is probably no statical problem that cannot be solved completely with the aid simply of the drawing board, triangles, and scale. A similar branch, Graphical Dynamics, deals with the graphical solution of the problems of kinetics.

The entire sciences of graphical statics and graphical dynamics rest on the principle that both *time* and *force* may be represented by straight lines. Mechanics deals with time and with force, while mathematics deals with numbers and dimensions; but taking the second and the pound (or ton) for units, any length of time may be represented by a line proportional to the number of seconds, and any force by a line whose length is proportional to the force and whose direction shows the direction of the line of action of the force. The graphical representation of a force is therefore much simpler than its analytical representation.

In statics we have nothing to do with time, but simply with forces. In kinetics, however, we have to do with time, and therefore in solving graphically the problems of kinetics it is necessary to represent time in a graphical way.

2. Notation.—We shall designate a force by the letter P, and its point of application by A, distinguishing forces by subscript figures, thus P_1 acting at A_1, P_2 at A_2. Where a force is represented by a line, an arrow shows the direction of the force. Where, however, forces are represented together in a closed or open force polygon we number the ends of the lines, beginning with o; thus o1 will be the force P_1 and it will act in the direction *from* o *toward* 1; 12 will be P_2, acting *toward* 2. Where a force is designated by the numbers at the ends of the line representing it, always show in this way, by the order of the numbers, the direction of the force. Thus the force 12 is equal and opposite to the force 21. The resultant of a number of forces will be designated by R, with the proper subscript numbers; thus R_{1-5} means the resultant of P_1, P_2, P_3, P_4, P_5.

3. Composition of Forces in the Same Straight Line.—If a number of forces act in the same straight line, their resultant is found by laying off the forces, in order, from any point o, in the proper direction; the line on will represent the resultant in magnitude and direction. In order that there may be equilibrium, on must equal o; *i.e.*, n and o must coincide; or $\Sigma P = $ o.

4. Forces in a Plane and at a Point.—The resultant of any two forces P_1 and P_2 (Fig. 140) if found by completing the parallelogram of forces, oa representing P_1 and ob P_2. The diagonal oc is the resultant. This is also obtained by laying off P_1 and P_2 successively as oa, ac, and the line from o to c represents the resultant. In general, then (Fig. 141) to find the resultant of n forces, lay them off in order, forming the force polygon

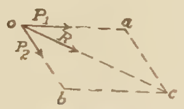


FIG. 140.

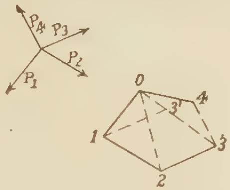


FIG. 141.

o123 . . . n, in which o1 is P_1, 12 is P_2, and so on, and on is the resultant R_{1-n} or no is the force necessary to hold R_{1-n} in equilibrium. The resultant acts, of course, through the point at which all the forces are applied.

5. Forces in Space and at a Point.—When the forces are not in a plane, but are applied at a point, the above also holds true, but the polygon o . . . n will of course not be in a plane.

In both cases it follows that the projection of the resultant on any line or plane is equal to the algebraic sum of the projections of the separate forces on the same line or plane.

6. The polygon o12 . . . n is called the *force polygon* of the forces P_1, P_2, P_3 . . . P_n. When the forces are in a plane, the force polygon can be drawn at once; when they are in space, it must be determined by the projections of the vertices on the two planes of projection, by the use of the principle that equal and parallel lines have equal and parallel projections. Figure 142 shows this construction for four forces. o′ and o″ are the projections of the point of application, the projections of P_1 are o′P_1' and o″P_1''; of P_2, o′P_2' and o″P_2'' and so on; the projections of the force polygon are o′1′2′3′4′ and o″1″2″3″4″; of the resultant, the projections are o′4′ and o″4″; that is, it acts *from* o′ to 4′ and *from* o″ to 4″.

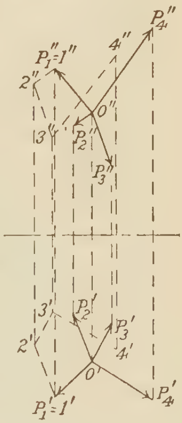


FIG. 142.

The order in which the forces are laid off is immaterial; for if in Fig. 141 the forces P_1, P_3 and P_2 are laid off in the order named, the force polygon will be o13′3, and the resultant will be o3, as before. Hence any two successive forces may be interchanged; and by such interchange of forces two by two any final order of the n forces may be obtained.

The force polygon gives not only the resultant R_{1-n}; it gives also the resultant of any number of *successive* forces. Thus in Fig. 141, o2 is

R_{1-2}; o3 is R_{1-3}; 13 is R_{2-4}; 14 is R_{2-4}, and so on. The force polygon, however, does not give the resultant of P_1, P_3, and P_4, or of any forces *not consecutive.*

7. Equilibrium of Forces at a Point.—Since the resultant of all the forces is represented by on, it follows that in order that the forces may be in equilibrium, that is, for the resultant to be zero, on must be zero, or n and o must coincide; in other words, *the force polygon must close.* This, therefore, is the graphical condition of equilibrium of forces at a point, either in a plane or in space. By virtue of the theorem that the projection of the resultant on any line equals the algebraic sum of the projections of components, this single graphical condition of equilibrium replaces the three analytical conditions $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$.

8. Resolution of a Given Force into Components Acting at a Point, Which Must of Course Be on the Line of Action of the Given Force.—Any force may be resolved into any number of components at a point, the only condition being that the components, if plotted in order, shall give a resultant equal to the given force. Thus, in Fig. 143, the force P may be resolved into o1 and 12, or into o1′, 1′2′ and 2′3′, or in any other like manner. If a force is to be resolved into two components whose directions are given, it is only necessary to draw, from the ends of the line representing the given force, two lines having the directions of the components; the intersection of these two lines will form the triangle of forces, and the sides parallel to the components will represent those components in magnitude and direction: their point of application may be any point in the line of action of the given force. Their direction is found thus; the given force being on, then o1 and 1n are the components. Further n1 and 10 represent two forces, having the given directions, which will *hold the given force P (= on) in equilibrium.*

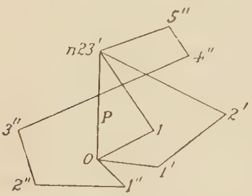


FIG. 143.

If a given force is to be resolved into two components, both components and the given force must lie in a plane. If it is to be resolved into more than two components, they need not all lie in a plane.

If a given force P (= on) is to be resolved into three forces which act at a point but not in a plane, and whose directions are given, imagine a line drawn from o parallel to one of the components, and a line from n parallel to the second; then only a single line can be drawn between these last two lines and parallel to the third component; hence the problem is perfectly determinate, and may be solved by the aid of descriptive geometry.

The following is a good problem on which to exercise the imagination. Suppose a given force P_{1-3} = on, is to be resolved into three components at a given point in the line of action of P_{1-3} but not in a plane,

P_1, P_2, and P_3. Suppose the magnitude and direction of P_1 known, while P_2 must pass through a given point A, and P_3 must cut a given line M. Imagine that you lay off, from the given point o, the force o1 $= P_1$; drawn on' parallel to and equal to 1n, and then on' represents the magnitude and direction of the resultant of P_2 and P_3; hence the plane o$n'A$ must contain these forces. Let M' be the point where the line M cuts this plane o$n'A$; then draw the lines oA and oM'; then draw through n' a line parallel to oM', meeting oA at 2; then the two remaining components are o2 (passing through A), and 2n' (cutting M), all acting at o. This problem may therefore be solved completely.

9. Composition of Forces in a Plane, but Not at a Point (Fig. 144).—Suppose that it is desired to find the resultant of the four forces P_1, P_2, P_3, P_4. Clearly, we may proceed thus; produce P_1 and P_2 until they intersect, and from their intersection lay off the forces and draw the parallelogram of forces, thus finding the resultant R_{1-2}; produce this resultant till it meets P_3, and find the resultant R_{1-3}; produce this till it meets P_4 and find in the same way R_{1-4}. Or we may begin by laying off the forces in order, forming the polygon o1234; then o2 represents R_{1-2} in magnitude and direction, and it is only necessary to find one point in its line of action, and this point is where P_1 and P_2 meet. Draw through this point a line parallel to o2, and where it meets P_3 will be a point through which R_{1-3} acts, and R_{1-3} is given in magnitude and direction by o3; and so on in the same way. In other words, the parallelograms of forces may be drawn all at once, at o1234, and thus the resultants in magnitude and direction determined from this figure, which is again called the force polygon. This avoids the trouble of having to construct parallelograms of forces at various points, as by the first method.

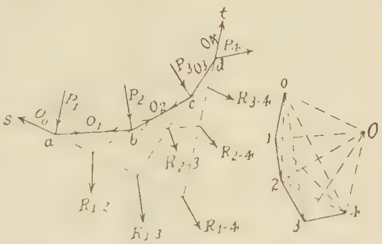


FIG. 144.

But it is clear that these methods would not be applicable when the given forces are parallel, or when they do not intersect on the paper. The force polygon may always be drawn at once, and the resultants determined in magnitude and direction, but it is essential to have some generally applicable method of finding a point in the line of action of any resultant sought, instead of having to determine such a point by prolonging some of the forces or resultants already known. Such a method is the following: Choose any point O, and draw the radiating lines Oo, $O1$, $O2$, $O3$, $O4$.[1] At *any point a*, let P_1 be resolved into two components oO, acting from s toward a, and $O1$ acting from b towards a; at the point b, which is therefore fixed as soon as O and a are fixed, let P_2 be resolved

[1] Be careful to distinguish between O (the letter) and o (the number zero).

into two components, $1O$ acting from a toward b, and $O2$ acting from c toward b. Then the resultant of P_1 and P_2 is the same as the resultant of oO, $O1$, $1O$, and $O2$, acting as described. But $O1$ and $1O$ are equal and opposite; hence they are balanced, and the resultant of P_1 and P_2 is identical with that of oO and $O2$, or it acts through the point of intersection of *sa* and *bc*, and is given in magnitude and direction by o2. This is shown by the force polygon, where o2 is the resultant of o1 and 12, and also of oO and $O2$. In the same manner, P_3 is resolved at c (which is fixed as soon as O and a are fixed) into $2O$ and $O3$; hence R_{2-3} is identical with the resultant of $1O$, $O2$, $2O$, and $O3$, or with that of $1O$ and $O3$; hence it acts through the point where *ab* and *dc* intersect. Further, R_{1-3} is identical with the resultant of oO, $O1$, $1O$, $O2$, $2O$, $O3$, or with that of oO and $O3$; hence it acts through the point of intersection of *sa* and *dc*, as shown. Continuing in the same way, P_4 is resolved at d, into $3O$ (acting along *cd*), and $O4$ (acting along *td*); hence R_{3-4} acts as shown; and since R_{1-4} is identical with the resultant of oO and $O4$, it acts through the point of intersection of *sa* and *td*. Thus the resultants are all given in magnitude and direction by the force polygon, and they act through points which are found by prolonging lines in the other figure, or the sides of the polygon *sabcdt*.

Since the resultant R_{1-4} is identical with that of the forces oO (acting along *sa*), and $O4$ (acting along *td*), it follows that R_{1-4} would be *held in equilibrium* by forces Oo (acting along *as*), and $4O$ (acting along *dt*); also that P_1 would be held in equilibrium by Oo and $1O$, and so on. The force polygon also goes to show that R_{1-4} or the system of forces $P_1P_2P_3P_4$ would be held in equilibrium by forces represented in magnitude and direction by Oo and $4O$, since these forces form with the given ones a closed polygon $o1234Oo$. It is these forces which hold the given forces in equilibrium which are shown by the arrows on the lines *sa*, *ab*, *bc*, *cd*, and *dt*.

It is important to observe that the point a may be anywhere on the force P_1, and that the point O may be chosen anywhere. This point O is called the *pole* of the force polygon o1234; the polygon *sabcdt* is called the *equilibrium* polygon for the given forces, or sometimes the *funicular* or *string* polygon; it represents a frame which would be in equilibrium if the given forces were applied at the joints, and the forces Oo and $4O$ in the strings *as* and *dt*. The lines of this equilibrium polygon are called the *strings;* the stresses in the strings are given in magnitude by the radiating lines of the force polygon. The radiating lines Oo, $O1$, etc., are called the *rays* of the force polygon. These rays are parallel to the corresponding strings of the equilibrium polygon.

We therefore have the following general method for finding graphically the resultant of any number of forces in a plane: Lay off the forces consecutively, to scale, forming the force polygon o1234 . . . n; choose

a pole O in such a position that the equilibrium polygon will come on the paper and of convenient shape; construct the equilibrium polygon by drawing, anywhere, a line parallel to Oo till it meets P_1; from that point a line parallel to $O1$ till it meets P_2; then a line parallel to $O2$ till it meets P_3; and so on, ending with a line parallel to On indefinitely. Then the resultant of any number of *consecutive* forces is given in magnitude and in direction by the force polygon, and its point of application is found by prolonging, in the equilibrium polygon, the strings which enclose the

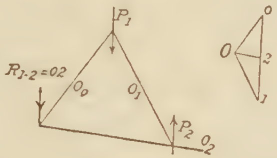


FIG. 145.

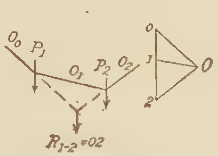


FIG. 146.

given forces, or in other words, the *extreme strings;* that is, R_{1-4} is represented by o4, and its point of application is given by prolonging the strings parallel to Oo and $O4$.

This general method should be thoroughly mastered by the student. It can be applied to all cases, though sometimes the pole O must be very carefully chosen in order that the equilibrium polygon of the intersections

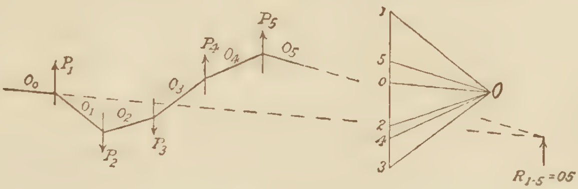


FIG. 147.

of its lines may come within the limits of the paper. For instance, Figs. 145, 146, and 147 show its application to parallel forces, in Fig. 145 the two parallel forces acting in opposite directions, and in Fig. 146 in the same direction.

10. Equilibrium of Forces in a Plane.—Taking the general case of n forces, it is clear from what has preceded that these forces are *held in equilibrium* by the forces nO and Oo in the force polygon, acting along the strings bearing the same letters in the equilibrium polygon (in Fig. 144 along Oo from a to s, and along $O4$ from d to t). In order that the n forces may be in equilibrium, these two forces which hold them in equilibrium must be themselves in equilibrium, or must be equal and opposite. Hence follow the following two graphical conditions of equilibrium:

1. *The force polygon must close, for then nO and Oo are equal and opposite, since n and* o *coincide.*

2. *The extreme strings Oo and On must lie in the same straight line; they will be parallel, of course, since n and O coincide.*

The first of these graphical conditions takes the place of the two analytical conditions $\Sigma H = 0$ and $\Sigma V = 0$.

The second graphical condition, as we shall see later, corresponds to the analytical condition $\Sigma M = 0$.

Further consideration shows, however, that if the extreme strings of the equilibrium polygon are in the same straight line, the force polygon *must* close, unless the pole is taken in the line on, in which case it may not close. If this position of the pole is avoided, the graphical conditions reduce to one:

1. *The equilibrium polygon must close; that is, the extreme strings must lie in the same straight line. This must be true of any and every equilibrium polygon which can be drawn for the given forces.*

It may therefore be said that, for equilibrium, both force and equilibrium polygons must close; but if the latter does, the former will also, if the pole is not on on.

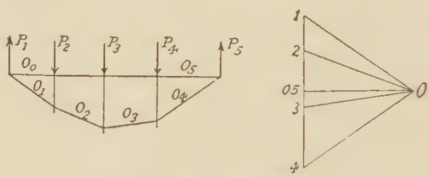


FIG. 148.

If the force polygon closes, however, it does not follow that the equilibrium polygon will; since the extreme strings, though parallel, may not be in a straight line. In this case the stresses which hold the given forces in equilibrium are two equal and parallel forces nO and Oo; and the *resultant* is therefore a couple equal and opposite to that constituted by these two forces. Hence:

If the force polygon closes, but the equilibrium polygon does not, the resultant is a couple whose moment is oO times the perpendicular distance between the end strings.

It will now be evident that the fact that the equilibrium polygon closes is equivalent to the analytical fact that $\Sigma M = 0$; for if it does not close, the resultant is a couple.

Figure 148 shows the force and equilibrium polygons for five forces which are in equilibrium, both polygons closing. The student should accustom himself to the conception of the equilibrium polygon as a *frame*, and should clearly see the character of the stresses in the different bars or strings. At any force, as P_n, the stresses in the strings act in the directions $O(n - 1)$ and nO; thus at P_3 in Fig. 147 they act in the directions O_2 and $3O$; hence the strings parallel to the radiating lines or

rays O_2 and O_3 are in *tension*. At P_1 the stresses are in the directions Oo and $1O$; hence the dotted string parallel to Oo is in *compression* and that parallel to O_1 is in *tension.*

When the resultant is a couple, we have seen that its moment $= Oo$ times the perpendicular distance between the end strings. Figure 149 shows the force and equilibrium polygons for two equal and opposite forces P_1 and P_2. The force polygon closes, and the moment of the couple is Oo times ab. It is easy to show that this equals the product of P and ac. For, producing the string O_2 and the force P_1 till they meet at e, the triangle afe is similar to the triangle Oo_1, ae representing P_1 and ef representing $Oo = O2$. But the area of this triangle $= \frac{ae \times ac}{2} = \frac{ef \times ab}{2}$. Hence any couple in a plane may be altered at will, as regards forces and lever arm, so long as the product of force and lever arm is constant, a result already familiar.

Figure 150 illustrates the case of a number of forces whose resultant is a couple whose moment equals $Oo \times l$. In this case the *resultant* couple

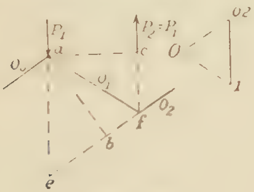


FIG. 149

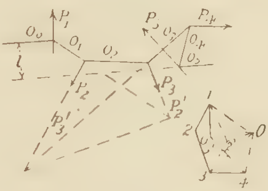


FIG. 150.

has a *right-handed* moment, since the forces in the strings Oo and $O5$, which *balance* the given forces, act in the directions Oo and $5O$, and have a *left-handed moment.*

The order in which the forces are laid off in the force polygon does not at all affect the resultant. In Fig. 144, for instance, the stress $3O$ in the string O_3 balances the forces $P_1P_2P_3$ and the stress Oo along the string Oo; and it can make no difference in the position of O_3 whether P_2 is laid off before P_3, or whether it is laid off after P_3.

11. Relation between the Force and Equilibrium Polygons and Effect of Position of Pole.—With any given set of forces P_1 to P_4 (Fig. 144), the resultant of these forces acts along a fixed line; hence, no matter where the pole O is taken, and no matter where the starting point a of the equilibrium polygon is taken, the end strings as and dt must intersect on this fixed line. The same is true of the extreme strings for any number of consecutive forces, which must always meet on the line of the resultant of those forces. *Hence, as the pole moves in any manner whatever, the intersection of any two strings moves along a straight line.*

Another theorem of great practical importance should be thoroughly grasped by the reader. In Fig. 151, let o1234 be the force polygon for the four forces P_1, P_2, P_3, P_4, and O the pole. Draw the equilibrium polygon a I II III IV e. Choose a second pole O' and draw its corresponding equilibrium polygon a I′ II′ III′ IV′ e, and investigate the relations between these two equilibrium polygons. At the point I, P_1 is held in equilibrium by Oo and 1O, acting along Ia and I II. At the point I′, P_1 is held in equilibrium by Oo′ and 1O', acting along I′a and I′ II′. It follows that the forces Oo (along Ia), 1O (along I II), oO' (along I′a), and O'1 (along I′ II′) are in equilibrium. (Note direction of forces enumerated.) From this it follows that the resultant of two of these four forces which are in equilibrium, Oo and oO', is equal and opposite to the resultant of the other two, 1O and O'2. But the resultant of Oo and oO', from the force polygon is OO', and it acts through a; and the resultant of 1O and O'1 is $O'O$, and it acts through b, where the strings 1O and O'1 meet; hence the line ab in the equilibrium polygon is parallel to the line OO' joining

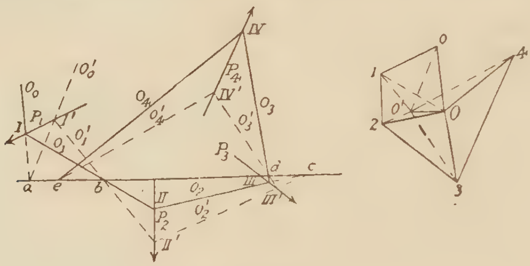


FIG. 151.

the two poles in the force polygon. In the same way, if c is the point of intersection of the strings 2O and O'2, it may be shown that bc is parallel to OO' in the force polygon. Hence the theorem:

If two equilibrium polygons are drawn with different poles, the intersections of corresponding strings of the two polygons all lie on one straight line parallel to OO′ in the force polygon.

This theorem is of great value, and is frequently used.

12. Pole on 0n. Resultant Polygon.—If the pole is chosen on the line 0n of the force polygon, the first and last strings of the equilibrium polygon will necessarily be parallel, but the stresses in them will not be equal and opposite unless the force polygon closes. If the force polygon does not close, the resultant of all the forces is the resultant of the two parallel forces in the end strings of the equilibrium polygon, and these strings will not be in the same straight line unless by chance the first string happens to have been chosen exactly in the line of action of the resultant of the forces. The position of the pole on 0n is therefore to be avoided if the resultant is sought.

If the pole, however, is at o, then the stresses in the *strings* of the equilibrium polygon, represented in magnitude by the *rays* of the force polygon, are themselves the resultants of the forces. Such an equilibrium polygon is called a *resultant polygon*, and if used, it is precisely the same as the method referred to in Art. 9, by which P_1 is prolonged to meet P_2, and their resultant found. Thus, Fig. 152 shows a resultant polygon for five forces, and R_{1-5} is o5, acting along the string $O5$. The polygon is II III IV V.

Suppose, in Fig. 152, that instead of the five forces o1, 12, 23, etc., there were a different P_1, represented by o$'$1, and suppose a second resultant polygon is drawn, with pole at O'; this polygon is II$'$ III$'$ IV$'$ V$'$. Now it is clear that so far as the forces P_2 P_3 P_4 and P_5 are concerned, these two equilibrium polygons bear the same relation to each other that they would if these four forces were the only ones in question, and O and O' were two

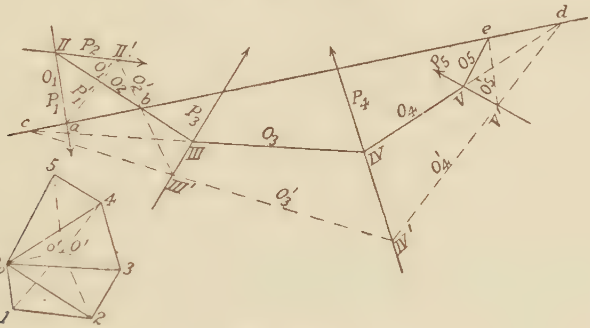


FIG. 152.

poles, with O_1 and O_1' the first rays: that is, corresponding strings, such as II III and II$'$ III$'$, III IV and III$'$ IV, IV V and IV$'$ V$'$, Ve and V$'e$, must all meet on a line *bcde* parallel to OO'; but this line also goes through the intersection of the first strings O_1 and O_1' or the forces P_1 and P_1', that is, through the point a; and the direction of the line, parallel to OO', is parallel to the resultant of P_1 and $(-P_1')$. Hence follows the theorem:

If two equilibrium and two corresponding force polygons have a certain number of consecutive forces in common, beginning at the point m in the force polygon (1 *in the above case*), *and if the equilibrium polygons are resultant polygons, then the intersections of corresponding strings corresponding to rays beyond m all meet on a straight line parallel to* oo$'$ (*or* OO'), *and this line goes through the point where the resultant of the partial polygon* 0*m meets the resultant of the partial polygon* o$'$*m, and is in fact the line of action of the resultant of* om *and m*o$'$.

13. Example: Line of Pressure in an Arch.—The following example will illustrate the preceding theorem (Fig. 153). Let there be a horizontal force P_1 and six vertical forces P_{2-7}, and let the resultant polygon II III IV, etc., be drawn. The force poly-

gon is $(Oo)1234567$. Now let it be required to draw a second resultant polygon, assuming a different horizontal force P_1', acting below the first one, and represented in magnitude by $O'1$. From the preceding paragraph we know that corresponding strings of the new and old polygons will meet on a line which is the line of action of the resultant of P_1 and $(-P_1')$, or of $O1$ and $1O'$, and hence upon a horizontal line. We may find that resultant graphically by taking any pole C, drawing a line parallel to Co till it meets P_1, thence a line parallel to C_1 till it meets P_1', thence a line parallel to CO' till it meets the first string which was drawn parallel to CO at k. On a horizontal line through k, all corresponding lines of the two resultant polygons must intersect.

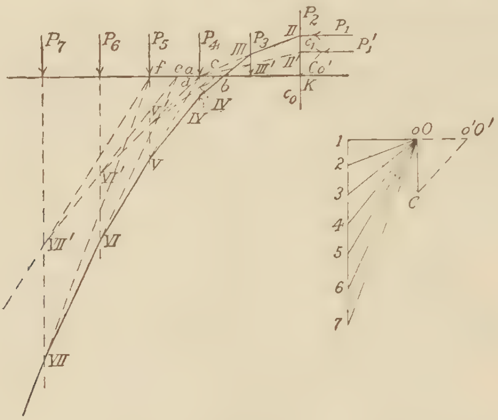


FIG. 153.

Hence producing II III to a, and drawing aII′ fixes the string, II′ III′; bIII′ gives the string III′ IV′; cIV′ gives IV′ V′, dV′ gives V′ VI′; eVI′ gives VI′ VII′, and fVII′ gives the last string.

14. Equilibrium Polygon through Given Points.—It is frequently necessary in the applications of graphical statics, to pass an equilibrium polygon through one or more fixed points.

a. Through One Point.—Any string may be made to pass through a given point, and an infinite number of polygons may be drawn satisfying this condition. Thus, if the string $O2$ is to pass through a given point a, this string may be drawn first, in any direction, passing through a. Then drawing the ray $2O$ in the force polygon parallel to this string, the pole O may be assumed anywhere on this ray, and the equilibrium polygon drawn each way from the string $O2$ first assumed. Similarly, any other one string may be made to pass through the point a, instead of the string $O2$.

b. Through Two Points.—Let the string $O2$ be required to pass through the point a, and the string $O4$ be required to pass through the point b (Fig. 154). Draw any string $O2$ through a, and a parallel thereto in the force polygon $2O$. Then the pole is on this line somewhere; *assume* it at O, and complete the equilibrium polygon as shown. The string $O4$ will not generally pass through the given point b; the problem is to alter

the polygon so that it will. As long as the string $O2$ is unchanged, the pole must be on $2O$, and as it moves along $2O$, the strings of the equilibrium polygon will all rotate about points in a line A parallel to $2O$ (see Art. 11); and if string $O2$ is not changed, this line A will be the string $O2$ itself. Now the string $O4$ cuts the line of the string $O2$ at n. Hence nb will be the new string $O'4$. passing through b, and if we draw, in the force polygon, the ray $4O'$ from 4 parallel to the string $O'4$, the new and correct pole O' will be located. The new strings $O'3$, $O'4$ and $O'5$ are thus drawn, from the intersections of the strings $O3$, $O4$, and $O5$ with the string $O2$; the new strings on the left of $O2$ must be drawn by first drawing $O'1$, and O'o parallel to the rays $O'1$, and O'o in the force polygon. Thus, starting with *any* assumed string $O2$ passing through a, it is possible to draw an equilibrium polygon in which $O4$ passes through b. Hence an infinite number of equilibrium polygons may be drawn fulfilling the given conditions, that is, as many as there are lines passing through a.

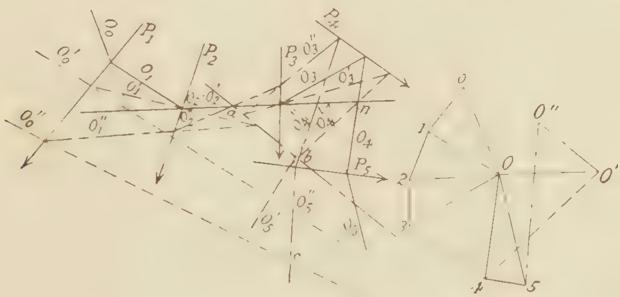


FIG. 151.

c. Through Three Points.—Let it be required to draw the equilibrium polygon so that one string (say $O2$) shall pass through a given point a, a second string (say $O4$) through a given point b, and a third string (say $O5$) through a given point c. First, as in Fig. 151, construct the polygon, as just explained, so that the strings $O2$ and $O4$ shall pass through the given points a and b, respectively; the pole is then O'. The problem now is, to change the pole and the polygons so that $O5$ shall pass through the point c, while $O2$ and $O4$ shall continue to pass through a and b. Draw the line ab. If the pole moves from O' along a line parallel to ab, corresponding strings of the polygons will revolve about points in some line parallel to ab; but since the strings $O2$ and $O4$ must continue to pass through a and b, that line must be ab itself. Hence proceed as follows: From the point where $O'5$ cuts ab, draw a line to c; this is the new string $O''5$ passing through c as required; draw, in the force polygon, the line $O'O''$ parallel to ab, and the line $5O''$ parallel to the string just constructed, and this fixes the final pole O''. The equilibrium polygon may be drawn

by passing $O''4$ through b and the point where $O''5$ meets P_5, $O''3$ through the intersection of ab with $O'3$, $O''2$ through a, $O''1$ through the intersection of ab and $O'1$, and O''o through the intersection of ab and O'o. Hence the given conditions are fulfilled. Only *one* polygon is possible which fulfils these conditions; for supposing the pole O'' found as shown, and the polygon drawn, then any other polygon with the same strings passing through a and b must have its pole on a line through O'' parallel to ab, and any other polygon with the same strings passing through b and c must have its pole on a line through O'' parallel to bc; hence it must be at O'' itself.

The above method is simple in principle, and only requires remembering the relation between two equilibrium polygons. The following method, which will be illustrated for forces nearly vertical, though it applies to any set of forces, is in some respects better and shorter. It

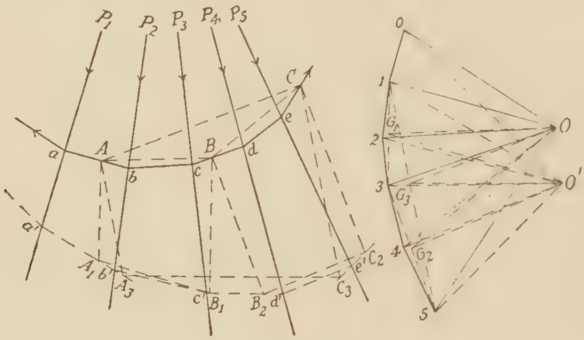


FIG. 155.

depends upon the principle of finding the locus of the pole for all equilibrium polygons in which the desired strings pass through a and b, and then its locus for all polygons in which the desired strings pass through b and c. The intersection of these loci is the pole for the polygon passing through a, b, and c. In order to discover how this may be done, consider the problem solved, and see what relations may be perceived. Let o5 (Fig. 155) be the force polygon for the five forces shown, and O be the pole, and let it be desired that the string $O1$ (say) shall pass through the point A, the string $O3$ through B, and the string $O5$ through C. The polygon as drawn fulfils this condition; the problem is to discover how the pole could be chosen so as to bring this about. The resultant force between A and B, 13 is R_{2-3}, and it is held in equilibrium by the stresses $3O$ and $O1$ in the strings $O3$ and $O1$. Draw lines through A and B parallel to 13, and draw through O a line parallel to AB, meeting 13 in G_1. Now if a force G_{11} be supposed acting at A, it will be held in equilibrium by forces $1O$ (in the string O_1) and OG_1 (in AB), and similarly $3G_1$ at B

will be held in equilibrium by forces G_1O (in AB) and $O3$ (in the string $O3$), and the polygon $AbcB$ will be a closed equilibrium polygon for the forces G_11, P_2, P_3, and $3G_1$, these forces being in equilibrium, since both force and equilibrium polygons close. In other words, G_11 and $3G_1$ are two forces acting at A and B, respectively, both parallel to R_{2-3}, which hold R_{2-3} in equilibrium. These forces are fixed and definite, and must be the same for every equilibrium polygon drawn for the given forces. Consequently, if $a'b'c'd'e'$ is an equilibrium polygon drawn with the pole O', if AA_1 and BB_1 are drawn through A and B to meet the extreme strings $a'b'$ and $c'd'$ for the forces between A and B, if A_1B_1 is drawn and also a line through O' parallel to A_1B_1, it will meet 13 in the force polygon at the same point G_1. Similarly if BB_2 and CC_2 are drawn through B and C parallel to 35 in the force polygon (the resultant of the forces between B and C), and if B_2C_2 is drawn, and a line through O' parallel thereto, this last line, no matter where O' is, will always meet 35 in the same point G_2. But it is evident from the first polygon that the true pole O is in the line G_1O parallel to AB, and similarly it is on the line G_2O parallel to BC; hence, after finding G_1 and G_2, the true pole O is found by drawing through these points lines parallel to AB and BC. This pole may be checked by considering the points A and C instead of A and B or B and C; thus draw AA_3 and CC_3 parallel to $R_{2-5} = 15$, join A_3 and C_3, draw $O'G_3$ parallel to A_3C_3, and through G_3 a line parallel to AC, which should also go through O.

This is probably the simplest method of solving the problem, though there are others.[1] It may be formulated thus:

Draw an equilibrium polygon for the given forces, with any assumed pole O'; through A and B draw lines parallel to the resultant of the forces between these points, to meet the extreme strings (produced if necessary) for these forces at A_1 and B_1; through B and C draw lines parallel to the resultant of the forces between these points to meet the extreme strings for these forces at B_2 and C_2; through A and C draw lines parallel to the resultant of the forces between these points to meet the extreme strings for these forces at A_3 and C_3; through O' draw a line parallel to A_1B_1 to meet the resultant in the force polygon of the forces between A and B at G_1, another line parallel to B_2C_2 to meet the resultant in the force polygon of the forces between B and C at G_2, and a third line parallel to A_3C_3 to meet the resultant in the force polygon of the forces between A and C at G_3; draw lines through G_1 parallel to AB, through G_2 parallel to BC, and through G_3 parallel to AC, and all of these lines will meet in the true pole O. With O as a pole draw one of the required strings passing through the point through which it is required to pass, and complete the equilibrium polygon.

There is evidently but one polygon which will fulfil the required condition.

[1] See *Der Civilingenieur*, p. 535, 1886.

The method just explained may be used to construct an equilibrium polygon through two given points, as A and B, by finding G_1 and drawing G_1O. If the pole is anywhere on this line, and the string O_1 goes through A, the string O_3 will go through B.

15. Resolution of Forces in a Plane into Components. *a. A Single Force Resolved into Two Components.*—The only condition here is that the components shall meet on the line of action of the force. If the components are given in direction, the triangle of forces gives the magnitudes; if one is given in direction and magnitude, the other is given by the line closing the triangle. If the components are parallel, it is necessary to draw the equilibrium polygon, as well as the force polygon. Thus in Fig. 156, if the force R_{12} = o2 is to be resolved into two parallel components along the lines shown, draw the string O_o parallel to the ray O_o to I, and produce to R_{12}, thence a parallel to O_2 through II, join I II and

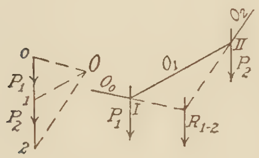


FIG. 156.

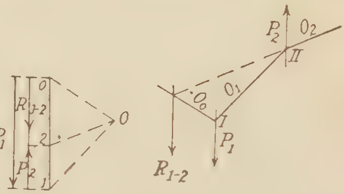


FIG. 157.

draw O_1 parallel to it; then o1 is P, and 12 is P_2, and both act in the same direction as P_{12}. In Fig. 157, both P_1 and P_2 are to act in lines on the same side of R_{12}, but the construction is the same; draw O_o till it meets P_1 and R_{12}; from its intersection with R_{12} draw O_2 till it meets P_2; then a line in the force polygon parallel to I II gives the point 1, and o1 = P_1 and 12 = P_2. If the magnitude of the parallel forces is given, but not their position, that is, if the points o, 1, 2 are fixed, draw the string O_o till it meets R_{12} and O_2 from this intersection; then *any* line as I II, drawn parallel to o1, will give two points I and II, in which the given forces may act. If either point I or II is given, the other may be found.

b. One Force Resolved into Three Components, Given by Their Lines of Action.—If the three components have such positions that two of them intersect on the line of action of the given force (or are parallel to it), while the third does not pass through this point, then the problem is impossible. The same is true if all three components are to meet in a point *not* on the line of the given force. If all three components meet on a point in the line of the given force, or are all parallel to it, then the resolution may be effected in an infinite number of ways; for in the former case the polygon of forces becomes a quadrilateral in which one side is given and only the directions of the other three; while in the second case the first and last strings of the equilibrium polygon are given in

direction, but not the two intermediate strings; hence the forces cannot be definitely determined.

If the lines of action of the three components meet in three different points, none of which is on the resultant; or, what is a case of the same thing, if two of the components are parallel, but not parallel to the resultant, and are cut by the third component, which may or may not be parallel to the resultant, then the resolution is possible in only a single way: for let the three components be P_1, P_2 and P_3 (Fig. 158), and connect the

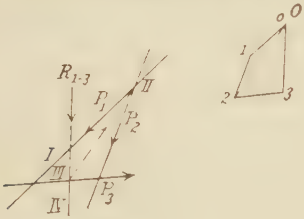


FIG. 158.

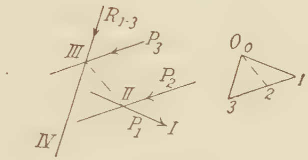


FIG. 159.

point of intersection of R_{1-3} and P_3 with the point of intersection of P_2 and P_1, and resolve R_{1-3} along P_3 and along the line just drawn (II III in Fig. 158), giving $P_3 = 23$ and $R_{1-2} = o2$; then resolve R_{1-2} along the given lines of action of P_1 and P_2 into $P_1 = o1$ and $P_2 = 12$. Figures 159 and 160 show further modifications of Fig. 158, Fig. 160 showing R_{1-3} resolved into an equal and parallel force and a couple. This method is really simply drawing first the *resultant* polygon I II III IV, and finding the force polygon from it, the pole O being at o.

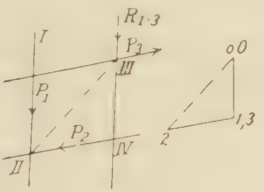


FIG. 160.

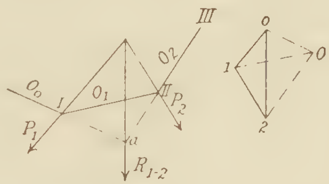


FIG. 161

The problem of resolving a force into four components in the same plane with it, if not impossible, is indeterminate. In resolving into three components, as above, it is also to be carefully observed that only the line of action of each force, and not its *sense*, is to be given.

In resolving a force into two components, even if they are not parallel to the resultant, the force polygon and *equilibrium* polygon are generally used. Thus, in Fig. 161, if the given force R_{1-2} is to be resolved into two components *whose lines of action are given*, choose a pole O, and draw the equilibrium polygon I II III. The first and last strings are given in direction, and must simply fulfil the condition of meeting at any point a

on R_{1-2}. Then R_{1-2} is the resultant of oO acting along the string Oo and $O2$ acting along II III. At I, where the string Oo meets P_1, resolve oO into o1 acting along P_1 and 1O acting along I II; and at II, where II III meets P_3, resolve $O2$ into $O1$ acting along I III and 12 acting along P_2. Then R_{1-2} is the resultant of o1, 1O, $O1$ and 12, or since 1O and $O1$ balance, it is the resultant of o1 and 12; hence o1 $= P_1$ and 12 $= P_2$. It is thus only necessary to draw the string Oo to a, and aIII parallel to $O2$, and join I and II where they meet the lines of action of the components P_1 and P_2; then draw the ray $O1$ in the force polygon parallel to II III, and from o and 2 draw o1 and 21 parallel to P_1 and P_2, and the two components are found.

All that has been said about resolving a force into components may be applied to finding the forces which will *balance* a given force, these forces being equal and opposite to the components hitherto referred to. Thus, in Fig. 161, R_{1-2} would be *held in equilibrium* by a force P_1 = 1o acting through I and a force P_2 = 21 acting through II. The student should go over the foregoing demonstrations with reference to finding forces to balance a given force, instead of finding its components.

To resolve a number of forces into two or three forces in the plane of the given forces, and having given lines of action, the resultant of the given forces may be found, and then resolved as already explained. The resultant being found by the equilibrium polygon, the extreme strings are the ones which hold the resultant in equilibrium (corresponding to Ia and II III in Fig. 161), and the components may be found precisely as in that figure.

The preceding theorems regarding resolution of forces find important applications in finding the reactions, shears, and stresses in bars, all of which involve merely resolving a given force into components, or finding balancing forces, or finding resultants. These will be explained in the following chapters.

CHAPTER XII

THE GRAPHICAL DETERMINATION OF REACTIONS

1. The problem of finding reactions is merely that of finding forces which will balance a given set of forces, or the resultant of those forces. It is thus merely resolving a given force (the resultant of the given forces) into components, for the reactions are equal and opposite to those components.

2. A body supported at one point will be in equilibrium only if the resultant applied force acts through that point, and the reaction must be equal and opposite to that resultant.

3. If a body is supported at two fixed points, at either of which the reaction may have *any direction*, the two reactions cannot be found by statics; for the only conditions they must fulfil are that their lines of action must meet on the resultant of the applied forces, and the three forces must form a closed triangle in the force polygon; these conditions may be fulfilled in an infinite number of ways. If the direction of each reaction is given, and if these meet on the line of action of the resultant, the problem is merely to construct the triangle of forces, except when the reactions are parallel to the resultant, in which case there is no triangle. If the direction of only one reaction is known, that reaction may be produced to meet the resultant of the outer forces, and the other reaction must act through this point, so that its direction will be known, and the triangle of forces may be constructed. This will not hold when the forces are parallel, and even if they are not, the intersection with the resultant will often not fall on the paper. In these cases other methods must be used, and these are furnished by the equilibrium polygon, by which, instead of dealing with the resultant of the outer forces, we may deal with the two components of that resultant, which act along the extreme strings of the equilibrium polygon.

The problem is merely this: The equilibrium polygon for the given applied forces being drawn, two forces (the two reactions) must be added to this set, such that a set of balanced forces will result, in which the force and equilibrium polygons will both close. The force polygon will be made to close by drawing from the ends of the force polygon for the applied forces, *i.e.*, from 0 and n, lines parallel to the directions of the reactions. *To close the equilibrium polygon, all that is necessary is to prolong the direction of one reaction till it meets one end string, and the direction of the other reaction till it meets the other end string, and to join these two points by*

what is called the closing line (because it closes the equilibrium polygon). *Then, in the force polygon, drawing a ray through the pole O, parallel to the closing line, the intersection K of the lines representing the two reactions must lie on this ray, and since the direction of one reaction is known* (*or of both, if they are parallel*), *the direction of the other is found.*

Either reaction may be produced to meet either end string, provided the other is produced to meet the other end string; and when the force polygon is closed by drawing a line from 0 or n parallel to that reaction whose direction is known, it must be drawn from the end of the ray corresponding to the string which that reaction was produced to meet.

But if the direction of only one reaction is known, how can the other be produced to meet one end string? Clearly it cannot be produced at all; and yet it is essential to find its intersection with one end string in order to solve the problem. This difficulty is overcome if *the equilibrium*

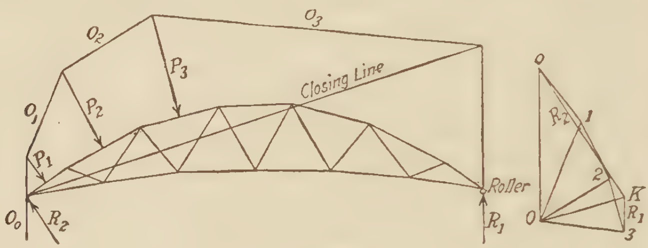


FIG. 162.

polygon is drawn with one end string passing through the point of application of the reaction whose direction is unknown, so that this point is the intersection desired, and is one end of the closing line.

Some examples will make this method clear.

4. Examples.—*a.* A horizontal beam loaded with three vertical loads, the reactions being vertical (Fig. 163). Draw the force polygon 0123, choose a pole O, and draw the equilibrium polygon. Then the resultant R_{1-3} is held in equilibrium by the forces O_0 and $3O$ in the end strings. From the point where the string O_0 meets the line of action of R_1 draw a line to the point where the string O_3 meets the line of action of R_2. This line is the *closing line.* Draw Ok parallel to this closing line; then $R_1 = k$o, and $R_2 = 3k$, for, at the left end of the closing line there is equilibrium between the reaction R_1, the stress oO in the string oO and the stress Ok in the closing string; hence the triangle of forces in the force polygon is Oko, and ko $= R1$. Similarly at the right end of the closing line $3k = R_2$. The closing line or string is in compression, and the other strings are in tension. The reader should note the similarity between Figs. 163 and 166, the latter showing a set of forces in equilibrium. The closing line is so named because it closes the equilibrium polygon; before drawing it, that polygon did not close, and the forces were not in equilibrium, the end strings being O_0 and O_3.

Another way of meeting the difficulty referred to is to find one component of the reaction whose direction is unknown, and consider this a known outer force. The equilibrium polygon may then be drawn anywhere, and the end string prolonged to

meet the direction of the undetermined component. Thus, in Fig. 162, though the direction of R_2 is unknown, it is evident that its horizontal component is the total horizontal component of the loads, or the perpendicular from 3 on the vertical through 0 in the force diagram. This may be considered as P_4, acting at the left abutment, and the end strings of the equilibrium polygon drawn for the four forces P_1, P_2, P_3, P_4 may be extended to meet the verticals through the supporting points, without drawing the equilibrium polygon through the point of support. In Fig. 163 if the end string O_0 is prolonged to meet R_2 (instead of R_1) and O_3 prolonged to meet R_1, the closing line

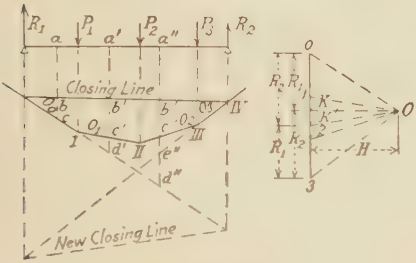


FIG. 163.

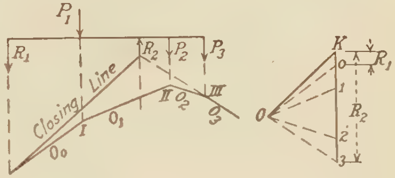


FIG. 164.

will be that marked "new closing line," and a line parallel to it through O in the force diagram will give the point k' (instead of k); but the reaction R_1 will now be that which forms a triangle with O_3 and the new closing line, or $3k'$, and will be found to have the same value found before.

b. Overhanging horizontal beam, with vertical forces and reactions (Fig. 164). The equilibrium polygon and closing line are drawn as before, and the point k found in the force polygon. The strings O_0, O_1, O_2, are in compression. At R_1, and the stress in O_0 acts toward the joint; hence the triangle of forces, in the proper direction, is

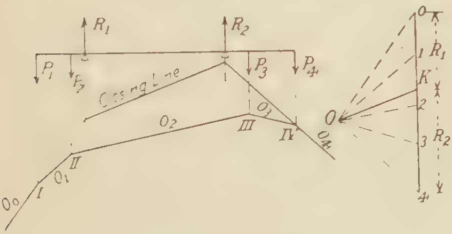


FIG. 165.

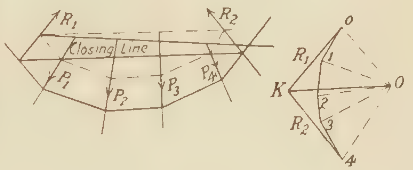


FIG. 166.

$_0Ok$, and $R_1 = k_0$, acting downward, while the closing line is in tension. Similarly $R_2 = 3k$, acting upward. The force polygon, taking the forces in order, is $0123k0$.

c. A horizontal beam overhanging both supports, with vertical loads and reactions (Fig. 165). This figure should be clear without explanation.

d. Loads and reactions inclined, the latter known in direction, and meeting on the resultant load (Fig. 166). If only the direction of R_2 were known, the equilibrium polygon would be drawn through the point of application of R_1.

e. A sickle-shaped roof truss, exposed to wind pressure on the left, and with the right end resting on rollers on a horizontal surface, or vertical reaction (Fig. 162). Here $R_1 = 3k$, $R_2 = k_0$.

THE GRAPHICAL DETERMINATION OF MOMENTS

1. Culmann's Theorem.—In Fig. 167, let *st* in the force polygon represent a force P, held in equilibrium by the forces tO and Os acting along the strings shown in the equilibrium polygon. If now we wish to find the moment of P about any point as a, draw through a a line parallel to P, till it meets tO and Os, either or both produced if necessary. Then the triangles dbc and Ost are similar, and, if H is the perpendicular from O on st,

$$H:st::l:bc \quad \therefore H:P::l:bc$$

$$\therefore P \cdot l = \text{moment of } P \text{ about } a = H \cdot bc$$

H being measured to the scale of force, and bc to that of distance. This is known as *Culmann's theorem*, and is thus expressed in words:

To find the moment of a force about any given point, the equilibrium polygon being drawn, draw through the given point a line parallel to the force, and let its intercept between the two strings holding the force in equilibrium be bc; then the required moment equals bc multiplied by the perpendicular distance H from the pole of the force polygon on the force in question.

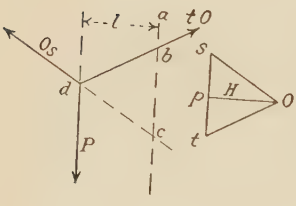


FIG. 167.

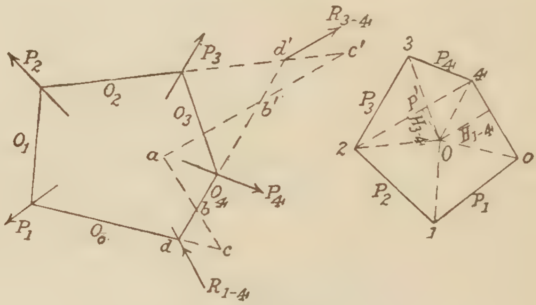


FIG. 168.

Thus, in Fig. 168, let P_1 . . . P_4 be four forces in a plane, and the equilibrium polygon drawn as shown. To find the moment of the resultant about any point as a, this is simply $H_{1-4} \times bc$. The moment of P_3 and P_4 about a is $H_{3-4} \times b'c'$. In this manner the moments of *any consecutive forces* about any point may be found from the equilibrium polygon.

2. Application to Beams.—In Fig. 163 the moment at any point on the beam is the moment of all the outer forces on either side of the point, that is, of a set of consecutive forces in the equilibrium polygon, and the preceding theorem is applicable. For vertical forces, the perpendicular from O on any force or resultant is the same, and indicated by H. The moment at the point a is the moment of the reaction R_1, and equals $H \times bc$. At a' the moment is that of the reaction minus that of P_1 or $H(b'd' - c'd') = H \cdot b'c'$. At a'' the moment is clearly $H \cdot b''c''$. Hence:

The moment at any point on a beam loaded vertically is equal to the vertical ordinate between the closing line and the equilibrium polygon, multiplied by H (the pole distance).

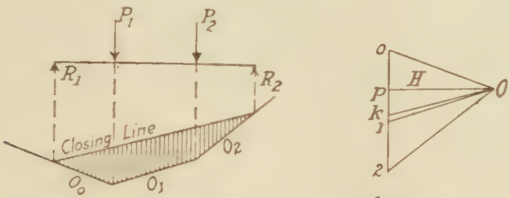


FIG. 169.

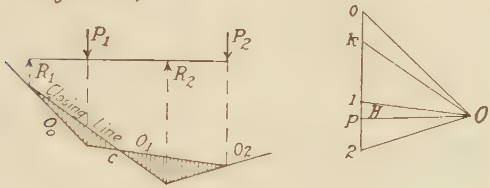


FIG. 170.

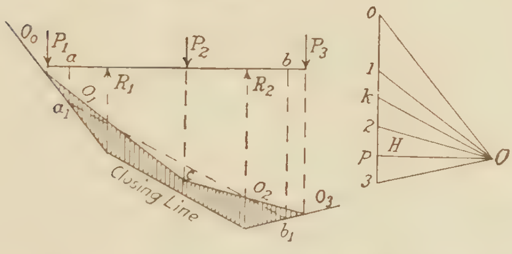


FIG. 171.

Figures 169, 170, and 171 show by the shaded areas the areas whose vertical ordinates represent moments, and they should be carefully studied. In Fig. 170, the moment causes tension in the upper fiber between the right-hand end of the beam and c, where it is zero. In general, when the equilibrium polygon is a tension polygon (pole to right of force line, and 0 at top, as usually drawn, with P_1 on left of paper), the moment is positive (compression in top fiber) if the closing line is above the equilibrium polygon and negative in the reverse case.

In Fig. 171, the moment is negative throughout, but numerically smallest at c. But if the supports were moved farther apart, until the closing line were represented by the dotted line a_1b_1, the moment would be positive in some portions of the beam.

3. Maximum Moment for Concentrated Loads.—The preceding principles render it easy to determine graphically the maximum moment at any point of a beam (Fig. 172). Construct first an equilibrium polygon for the given system of loads. Now instead of placing the loads at some given position on the span, and drawing a new polygon for each different position, draw one equilibrium polygon and shift the span. Thus, in Fig. 172, suppose it is desired to find the maximum moment at a point distant $\frac{1}{4}l$ from the left-hand end of a span of length l. First suppose the second load P_2 to be at the section, and to draw the closing line lay off $\frac{1}{4}l$ and $\frac{3}{4}l$ to the left and right of P_2 respectively, thus fixing the ends of the span when P_2 is at the point, and giving the closing line l_2l_2; then

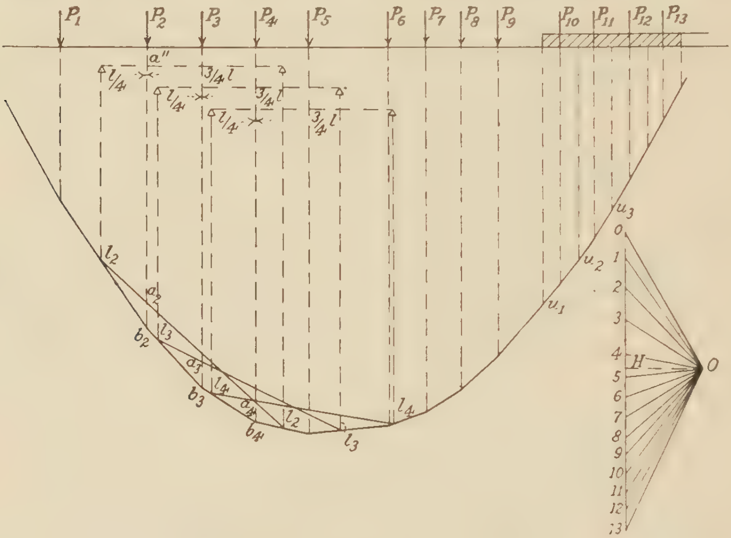


FIG. 172.

the ordinate at P_2, or a_2b_2, multiplied by H, represents the moment at the point in question; or, since H is constant, a_2b_2 represents the moment. Next try P_3 at the section, lay off $\frac{1}{4}l$ and $\frac{3}{4}l$, and draw the closing line l_3l_3; then a_3b_3 represents the moment with P_3 at the section. Finally a_4b_4 represents the moment with P_4 at the section. The maximum of these ordinates a_2b_2, a_3b_3, a_4b_4, multiplied by H, is the maximum moment at the given point, the ordinate measured to the scale of distance and H to the scale of force. Other loads may be tried if desired.

By this method, one equilibrium polygon may be made to answer for any number of spans, the load system remaining the same. In drawing the figure, always choose O so that H equals some even number of tons, say 10, 50, or 100, in order that the multiplication may be easy.

4. Treatment of Uniform or Distributed Loads.—In finding the resultant of a series of loads, a distributed load must be divided into small

parts, the resultant of each found, and its point of application, and each such resultant treated as a single load. In the case of any distributed load, the equilibrium polygon becomes a *curve* (Fig. 173). The load is divided into portions, as shown, and the area of the load curve enclosed between any two ordinates such as aa' and bb' represents the total load between those ordinates, acting at the center of gravity of the area referred to $aa'b'b$. These loads are treated as concentrated loads and the equilibrium polygon drawn as before. The equilibrium *curve* will be *tangent* to this polygon at the points a_1, b_1, c_1, etc., directly below the points assumed in dividing the load. Suppose that the equilibrium curve be correctly drawn, and that it be required to find the resultant of the loads between a' and b': this would be found in magnitude by taking the total load between a' and b', and its line of action would be determined by prolonging the *end strings* which enclose the loads in question; but

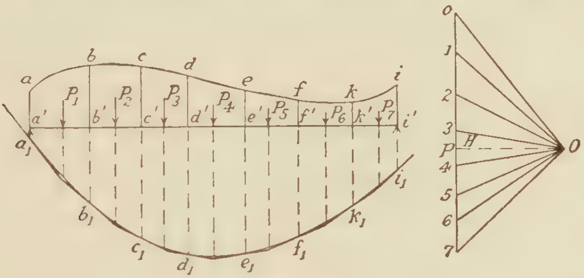


FIG. 173.

the strings of the *curve* are elements of the curve and are in the direction of tangents to the curve, and the end strings for the loads between a' and b' are the tangents to the curve at a_1 and b_1; hence these tangents meet on the load P_1, or the equilibrium curve is tangent to the polygon at these points of division. In Fig. 172 the uniform load at the right is treated in this way. The curve is tangent to the polygon at the points u_1, u_2, u_3, etc.

In treating uniform or distributed loads, the divisions may always be made so small that no essential error is introduced in using the polygon instead of the curve.

If the load is uniformly distributed, the resultant between any two points a' and b' will act at the center of $a'b'$. The equilibrium curve for such a load, therefore, has the property that the tangents to the curve at any two points meet on the vertical line which bisects the line joining the points of tangency. This is a peculiarity of the parabola; hence *the equilibrium polygon for uniformly distributed vertical loads is a parabola with vertical axis.*

CHAPTER XIV

THE GRAPHICAL DETERMINATION OF SHEARS

1. Since the vertical shear at any section of a horizontal beam or truss is the resultant vertical force on one side of that section, it may be found graphically, in magnitude and point of application, by the principles which have been explained.

In Fig. 174 a beam is again shown loaded transversely. Then the shear S_1 between a and b is equal to the left-hand reaction R_1, and represented by ko, acting at a. The shear S_2 between b and c is the difference of R_1 and P_1, and is represented by $k1$, and it acts at the intersection of

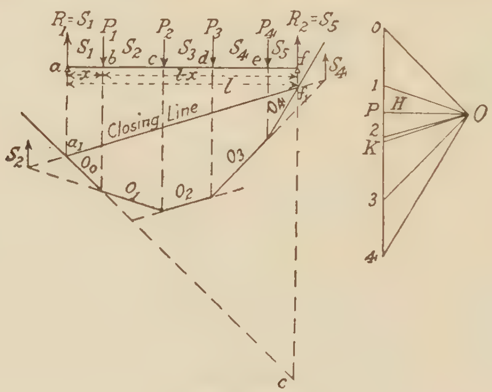


Fig. 174.

$O1$ and the closing line, as shown. In the same way the shear S_3 between c and d is represented by $k2$, acting at the point where $O2$ meets the closing line; and so on. From this it is obvious that *the moment is a maximum at the load where the shear changes from positive to negative, or, for a distributed load, where the shear is zero;* that the shear is a vertical force given by the force polygon, and that (with downward loads) its point of application when positive is outside the span on the left, and when negative outside the span on the right.

2. The following method of finding shear is very fruitful (Fig. 174). Let it be required to find the left-hand reaction, which is also the shear in ab, or the shear at b when the first load lies *just* to the right of b. This reaction is given in the force polygon by $Ko = R_1$; but it may also be found as follows: produce the first string Oo till it meets the vertical

through the right support. Then the triangles oOk and a_1f_1c are similar, and we have

$$R_1 : H :: f_1c : l \qquad \therefore \text{ if } b \text{ is at a distance } x \text{ from } R_1$$

$$R_1 = \text{shear at point } b = \frac{H}{l} \cdot f_1c$$

In this equation, H is measured to the scale of force, l and f_1c to that of distance; the result will be the same, however, if we measure H and l to the scale of distance, and f_1c to that of force. If, now, we make $H = l$ to the scale of distance, the above equation reduces to

$$R_1 = \text{shear at point } b = f_1c \text{ (to the scale of force)}$$

That is to say, H being made equal to l by the scale of distance, then *the shear at any point when the first load is at that point equals the ordinate between the equilibrium polygon and the first string produced, at a distance* $(l - x)$ *from the first load.* This theorem is very important. The closing line need not be drawn.

3. This theorem may be easily applied to find the maximum shear due to a system of moving loads which come upon the span from the right.

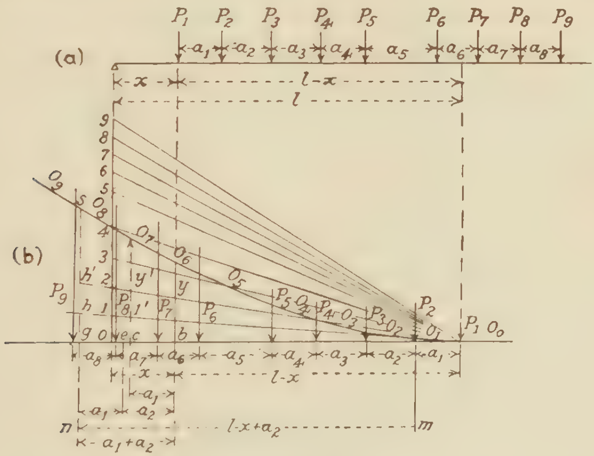


FIG. 175.

As shown in Fig. 175, l being the span, draw the equilibrium polygon with the loads reversed in position, that is, with P_1 at the right, and place P_1 over the right support. In the force polygon, lay off P_1 at the bottom of the line, at 01, P_2 above it at 12, and so on. Choose the pole at P_1 in the horizontal through o, and with $H = l$, to the scale of distance. In other words, oO being the span, lay off the loads in the vertical through O, take the pole at O, and reverse the loads, with P_1 at O. The first string will then be horizontal. The equilibrium polygon is precisely the same

in magnitude as the one we should obtain if we should lay off the forces as in Fig. 174. In that figure the pole might be chosen in the horizontal through o. Our new polygon is merely in a different position.

Now from the theorem just demonstrated, the shear at a distance x from the *left-hand* end of a beam, when P_1 *is just to the right of the point, and the loads approaching from the right* (a condition of things represented in Fig. 175*a*), will be the vertical ordinate y between the equilibrium polygon and the first string Oo, produced, at a distance $(l - x)$ from the first load P_1. But in Fig. 175*b* this point is directly beneath the point in the upper figure at a distance x from the left abutment. Hence the lower figure gives, by the ordinates between Oo and the equilibrium polygon, directly the shear at every point when the first load is just to the right of the point and the loads approaching from the right. In other words, *the equilibrium polygon in the lower figure is the curve of left-hand reactions for the given system of loads when* P_1 *is at any point and the loads on the right.* The reversal of the direction of the loads, placing P_1 at the bottom of the force line, is simply a device for bringing the proper ordinate at the point desired, for in Fig. 174 the ordinate for the shear at b, when the first load is there, is not at b, but at f.

From this figure we may easily find the maximum shear for any point at a distance x from the left support, for a given load system. Thus, y represents that shear if P_1 is at just to the right of the point. If P_2 is moved up to be just to the right of the point, P_1 moves ahead a distance a_1; hence lay off $bc = a_1$, and y^1 the ordinate at c represents the left-hand reaction when P_1 is at c, or in other words, when P_2 is at the point b. But the shear at x is $R_1 - P_1$; hence drawing a horizontal through 1, to meet y', gives $y' - P_1$, or the shear at b when P_2 is at b; supposing no loads to pass off the bridge. The same procedure may be continued till the maximum is found. In every case it is first necessary to find where P_1 lies; the ordinate at P_1 is the left-hand reaction, and to find the shear it is only necessary to subtract the loads between the reaction and the section.

If a load passes off the span at the left, the process must be somewhat modified, but is no less simple if the foregoing principles are kept in mind and applied with intelligence. Thus, suppose that $y' - 01$, which is the shear at b when P_2 is at that point, is greater than y. Then P_3 must be moved up on trial, and suppose that, in doing this, P_1 passes off the span, while P_2 remains on the span, its position being at $e(be = a_2)$. Then P_3 is at b, P_2 at e, and P_1 at g. Remembering the original theorem, that the shear at any point is the ordinate between the equilibrium polygon and the *first string produced*, at a distance from the first load equal to its distance from the right end of span, when the *first load* lies at the point, the first load (which is on the span) is now P_2, and the first string is O_1. Hence lay off, from P_2, the distance $mn = Oe =$ distance of first

load P_2 from the right support $= l - x + a_2$, which will bring n directly under g. The distance hs between O_1, produced, and the equilibrium polygon will now be the left reaction when P_3 is at b; hence the shear at b will be $hs - 12$, and if this is greater than $y' - 01$, the shear is increased by moving up P_3. Note that hs is the ordinate between $O1$ and the equilibrium polygon at the point where P_1 lies when P_3 is at b.

This method of finding shears may be used for any span, with the same load system, with the use of but one figure (see Fig. 176). The loads are laid off on a force line $0n$, including loads sufficient for the largest span

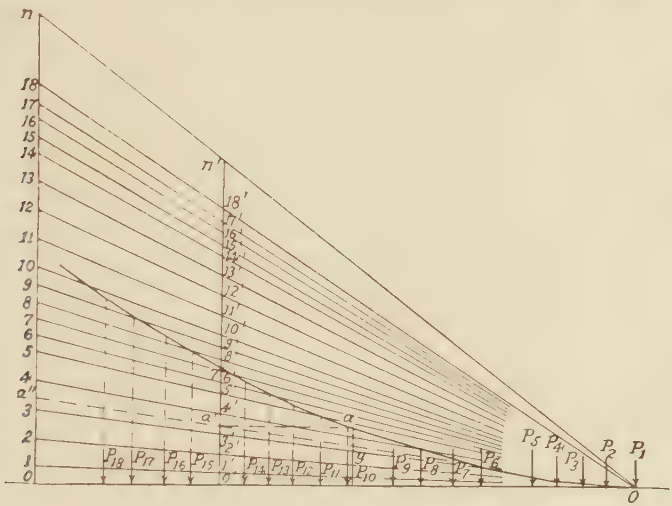


FIG. 176.

to be used, and these loads are placed, as before, with P_1 at O, O_0 being taken as large as the longest span to be used. Now if any given span $Oo' = l_1$ is to be used, the force line should be drawn at $0'$. Draw the vertical through this point; then $0'1'$, $1'2'$, etc. represent the forces P_1, P_2, etc., only the scale of force is different from the scale of force on the line $0n$. Hence this polygon may be used precisely as has been explained, measuring all forces to the scale shown on $0'n'$. This scale may easily be referred to the scale on $0n$ thus: If y is an ordinate, draw aa' horizontal, and $o'a' = y$; draw Oa' and produce it to a'' on the line $0n$. Then $0a''$ equals the value of y, measured to the scale of force used on $0n$. This is a convenient means of reduction, since the scale of $o'n'$ will be an odd scale, and y cannot easily be measured by it.

CHAPTER XV

THE GRAPHICAL DETERMINATION OF STRESSES BY THE METHOD OF JOINTS

1. This method is sometimes known as Maxwell's, sometimes as Cremona's, from two scientists who early discussed it. It consists in drawing a closed polygon of forces for each joint in a frame, beginning, as in the analytical method, with a joint where but two bars meet. These force polygons are arranged in such a way as to form one figure, known as the stress diagram. The notation used is known as Bow's, and consists in lettering or numbering the *spaces* between bars or forces in the figure of the frame, and the force representing the stress in any bar by the letters in the spaces bounding that bar. Thus, the bar between the spaces a and b is the bar ab, and the stress in it, in the force polygon or stress diagram, is the distance ab, the letter a being at one end of the line and b at the other. The application of the method is best illustrated by actual examples.

2. Application to Roofs (Figs. 177 to 181).—The graphical method of joints finds its principal applications in the case of roof trusses, for the reason that the different loadings to be provided for are few in number. In a structure in which the loading is different for the maximum stress in each bar, as in bridge trusses, the method is of little or no value, since it involves finding many stresses besides the particular one sought, these others, however, being of no value, since they are not maximum stresses. For roof trusses, however, there are only *four* loadings for which the stresses in all the bars must be computed, in order to determine the maximum stresses, *viz.*, (1) dead load, (2) snow load on one side, (3) wind on the right-hand side, and (4) wind on the left-hand side. The diagrams for these four loadings are therefore shown on the sheets. Having the stresses sealed from these diagrams, the maximum stresses are computed by making out a table as shown below.

Bar	Dead	Snow			Wind		Maximum	Area	Dimensions
		On right	On left	Total	On right	On left			
ab *bc* etc.									

The stresses for snow load for a symmetrical truss may all be found from one diagram drawn for snow on one side, for, since the reactions are vertical, the position of the roller makes no difference, and the stress in any bar, for snow load on the left, is the same as the stress in the corresponding bar on the other side of the roof (provided it is symmetrical) for snow on the right. Thus, in Fig. 177, the stress in DE for snow on the left is the same as that in JK for snow on the right. The stresses for total snow may be found by adding the stresses for snow on the right and for snow on the left. In the case of the wind, the diagram must be drawn for wind on each side, since the position of the roller affects these stresses. These columns being all filled out, the maximum stress is found. It is not necessary to consider the maximum wind pressure to act simultaneously with the maximum snow load, since such a wind would generally blow off the snow. Generally it is sufficient to consider the maximum wind pressure on one side only, and snow load on the other; but sometimes in the case of flat roofs, a total snow load will give greater stresses. In any case, the *maximum* stresses may easily be decided upon, and from them the area and dimensions of each piece.

For unsymmetrical trusses, more diagrams would be necessary, corresponding to the other loadings shown in the table.

3. The process of drawing the diagram should be reduced by the student to be a matter of routine similar to that of locating a point from two known points, having given the bearings (directions) of the lines connecting them. Thus to locate c from a and b, draw from a parallel to the bearing of ac, and from b parallel to the bearing of bc, and c will be located by the intersection of the lines so drawn.

Referring to Fig. 177, first find the reactions, either analytically or graphically, and letter (or number) the spaces between forces, whether loads, reactions, or stresses, as shown. Thus, the bar between E and F is the bar EF, and the stress in it will be designated in the force diagram as the distance ef. The force OX is the right reaction, shown by the distance xo; the force QR is the load at the joint between Q and R, and so on. If forces are resolved into components, a letter must be placed in the space between them. For wind and snow loads there are no loads on the bottom chord; hence one letter O is sufficient for the entire span. For the dead-load stresses, however, there is a load at each lower chord joint, as well as at every top chord joint, and numbers 1–8 have been used instead of letters.

Referring now to the diagram for wind on the left side, lay off the loads and reactions, forming the closed force polygon $zpqrstoz$. Then, at the joint at the left support there is equilibrium between the reaction, oz, the load at the joint zp, and the stresses in the bars PA and OA; the resultant of oz and zp, is op; therefore draw from p a line parallel to the bar PA and from o a line parallel to the bar OA, locating the point a.

The stress in PA is pa, and that in OA is oa, and since the reaction acts in the direction oz, the polygon of forces $ozpao$ must be traversed in the order of the letters as just named, in order to find all the forces in the direc-

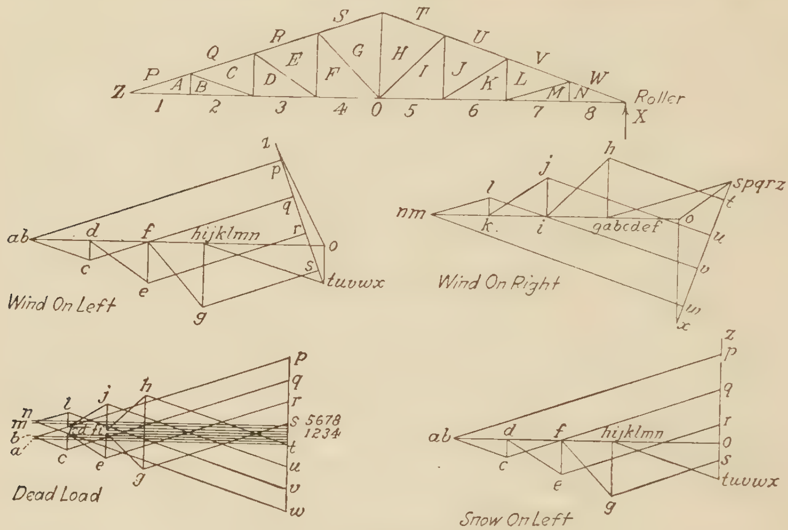


FIG. 177.

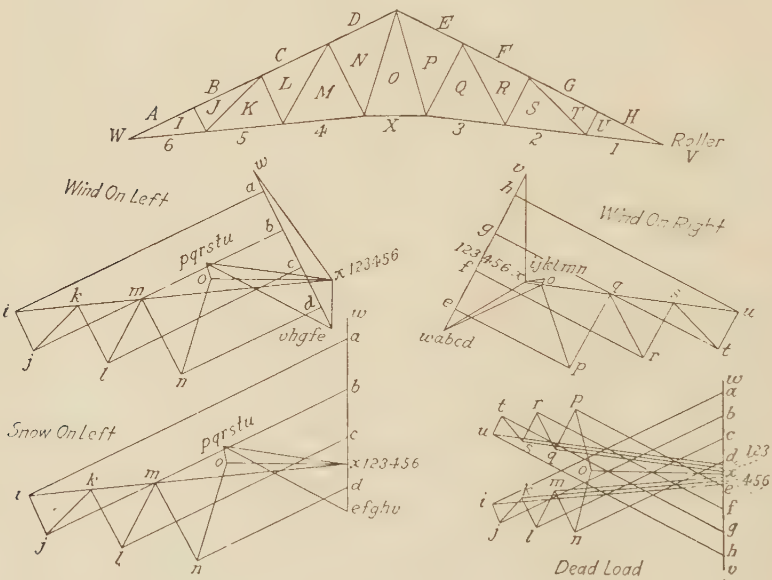


FIG. 178.

tions in which they act. Hence at the point considered the stress in PA acts from p to a, or is compression, and that in OA acts from a to o, or is tension. Having given a closed polygon of forces, in order to find the

direction of the forces, it is only necessary to know the direction of one force, which shows the direction to follow around the polygon.

Consider next the joint OAB. There is no load there; hence the stress in AB is zero, and that in OB equals that in OA; or, to follow the routine, draw from a parallel to AB and from o parallel to OB, which locates b at a. Consider next the joint $PQCBA$. The unknown stresses are those in QC and CB. Draw from b parallel to BC and from q parallel to QC, locating c. The force polygon is $pqcbap$, and since pq acts from p to q, this is the right way to go around the polygon. The stress in PA acts from a to p, or it is in compression, as at the abutment joint. The

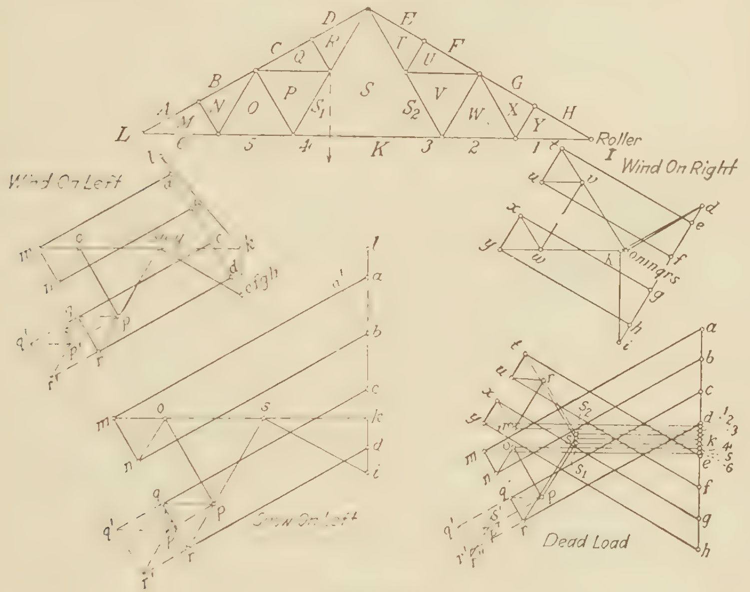


Fig. 179.

correct direction to go around any force polygon at a joint will always be indicated by the direction of the load, if a load acts at the joint; if not, it will be shown by the character of the stress in one of the bars previously determined, as PA in the case just treated.

With this explanation the diagrams in Figs. 177 and 178 can easily be followed through, and the reader should make himself thoroughly familiar with them, and with the routine method indicated.

3a. Ambiguous Cases.—In some forms of roof, the drawing of the diagram is likely to cause difficulty, in cases where analytically the stress in a certain bar must be found before the stress in another bar can be found; or where no section can be passed through a certain bar which does not cut more than three bars. Figures 177 and 178 should offer no difficulty, but Fig. 179 is one of the ambiguous cases. All ambiguous

cases, in which the frame is really statically determined, may be solved by the method which will now be explained. In the diagram for wind on the left (Fig. 179), begin by drawing *am* parallel to *AM*, and *km* parallel to *KM*, giving *m*; then from *m* parallel to *MN*, and from *b* parallel to *BN*, giving *n*; then parallel to *NO* and *KO*, giving *o*; but we cannot locate *p*, since we have neither *s* nor *q*: draw, however, from *o* a line parallel to *OP*; then *p* must be *somewhere* on this line; also draw from *c* and *d* parallel to *CQ* and *DR*; then *q* and *r* are *somewhere* on these lines. Imagine *q* and *r* to be at q' and r', $q'r'$ being parallel to *QR*. Then from q' draw parallel to *QP* and from r' parallel to *RS*; p' must be somewhere

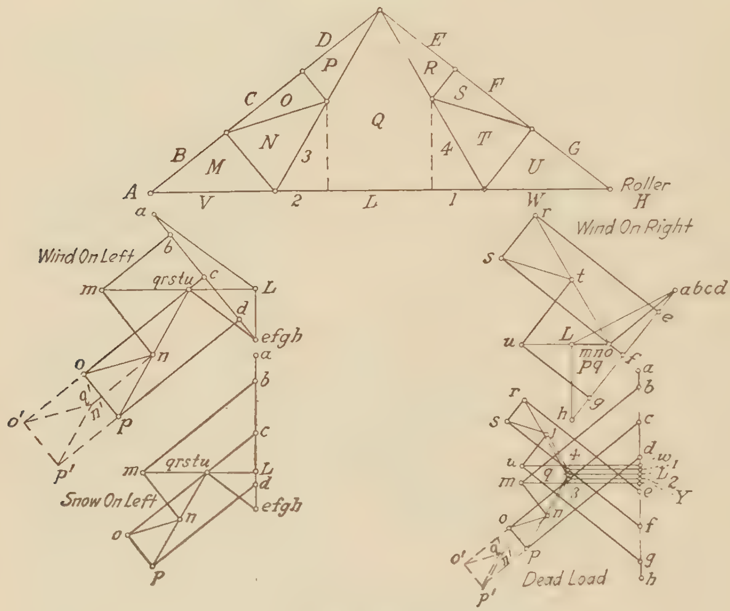


FIG. 180.

on the first line and both s' and p' on the second, since *SR* and *SP* are parallel, and hence *sr* and *sp* must coincide. The triangle $q'r'p'$, therefore, shows the relative location of *q*, *r*, and *p*; and since *p* must be on the line previously drawn from *o*, while *q* and *p* must be on the lines drawn from *c* and *d*, it is evident that by drawing through p' a parallel to cq' till it meets the line through *o* the triangle will be properly located at *pqr*. In other words, the triangle $q'r'p'$ is moved up parallel to cq' until p' is on *op* at *p*.

This method of treating ambiguous cases is general, and applies equally well to the dead stresses, even assuming, as suggested in the figure for dead load, a dead load at the joint *PQRS*. In this case a letter S_1 is placed to one side of this load. The stress in *KS* and in $4S_1$ will

be the same. In the force polygon, drawing a line from r' parallel to RS, the point s' (or s if r' were in the right place) would be on this line; ss_1 is the load at the joint; hence p' is vertically below the line $r's'$ by this load. Therefore lay off $r'r''$ equal to this load, and draw $r''p'$ parallel to $r's'$ and p' must lie on this line. But it also lies on the line through o parallel to op; hence the construction shown.

Figure 180 is another ambiguous case, solved in a similar manner. Figure 181 is still more ambiguous, but easily solved.

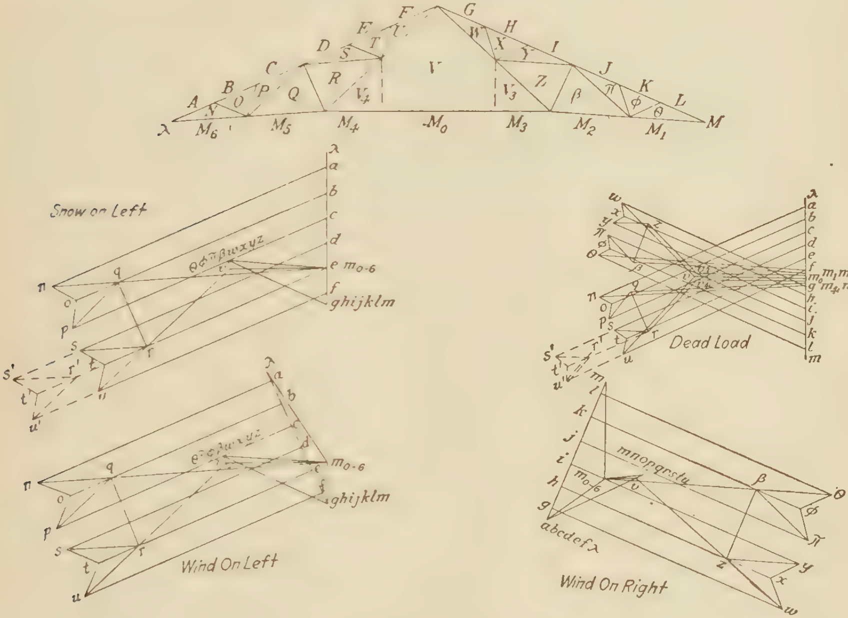


FIG. 181.

4. Treatment of Counters.—If counters occur, that is, two bars such that when one is in action the other is necessarily not in action, the method of procedure is to assume one of the two in action, find the stress in it, and observe whether it is of the kind that the bar will carry. If not, the other bar is the one in action, and the diagram must be altered.

5. The graphical method of joints is not often applied to bridge trusses subject to moving loads, on account of the fact that there are so many cases of loading to be considered, in finding the maximum stresses. It has been suggested to use the method by drawing the diagram for a single load at certain joints, and from the results tabulating the effect, in each bar, produced by these separate loads, and by the proper combinations finding maximum stresses. But such a method would be clumsy and wasteful of time.

6. In drawing the figures for this method, the points should be marked by needle pricks, small circles drawn around these points, and the lines drawn to the circles, not to the points themselves.

7. Comparison of Methods.—The graphical method of joints has been largely used, but is probably less used now than formerly. Its advantage is in the view that it gives of the relative stresses, and in the fact that at the end there is a check on the work in the closure of the last polygon. On the other hand, errors are cumulative by the graphical method, and may be difficult to trace. The preference for analytical or graphical methods is largely a matter of temperament. Some people like to draw; others prefer to compute. The writer much prefers analytical methods, and with him they prove more rapid and accurate than graphical methods. But that does not prove that others should prefer them, as men are not all cast in the same mold. The student of structures should master all fundamental methods, and be able to select in each case the one best suited.

CHAPTER XVI

THE GRAPHICAL DETERMINATION OF STRESSES BY THE METHOD OF MOMENTS

1. This method is frequently applicable, especially in the case of structures for which a diagram of the structure would generally be drawn to scale, and some form of graphical procedure adopted, such as arches. It is such a beautiful method of applying fundamental principles that its study is advisable for every student of structures, and if kept in mind, many opportunities will present themselves for its use, especially for those who incline to graphical treatment. It is very simple, involving only the principle of moments and the triangle of forces, and by it any bar can be selected, and the stress in it found, independent of any other bar, just as in the analytical method of moments; except that where a section through the bar cuts more than two other bars, the stress in one of them must be taken into account.

There are two ways of applying the graphical method of moments:

1. By taking the section and finding the resultant outer force R on one side. Then the moment of this resultant R, about the origin of moments O, equals and is opposite to the moment of the stress S in the bar considered, taken about the same point. In other words, the resultant of R and S goes through O. Having R thus found, and the direction of the resultant of R and S, we may easily find S by the triangle of forces.

2. By finding the moment, about the origin of moments, of the outer forces on one side of the section. This will be Hy if y is some distance in the equilibrium polygon, and H is a pole distance. Then $S = (H)y/l$ if l is the lever arm of the bar considered. This proposition may be plotted graphically.

By either of these methods it is possible to select any bar in a frame and determine its stress graphically, independent of the stress in any other bar, except in cases where no section through the bar can be taken which will cut only three bars.

An example will best show show how simple these methods are.

Figure 182 shows a projecting or cantilever truss. The direction of the reaction R_1 is supposed to be known, while that of R_2 is unknown. The equilibrium polygon is therefore drawn with the end string Oo going through the point of application of R_2, and the other end string $O5$ is prolonged to meet R_1, thus giving the closing line and the two reactions.

Suppose, now, that the stress in the lower chord bar x is to be found; taking a vertical section through x, the resultant outer force on the right of this section is R_{2-5}, acting at the point of intersection of the strings O_1 and $O5$. The resultant of R_{2-5} and of the stress in x must act at s, and must pass through t. Hence, draw in the force polygon the line $1a_1$ parallel to st till it meets $5k$ at a_1 and the stress in x is $5a_1$. Since R_{2-5} acts from 1 to 5, or downward, the stress in x which will produce a resultant (with R_{2-5}) along st must act from 5 toward a, on the portion of the truss to the right of the section, or it is compression. In the same way, by the simple triangle of forces, the stress in any bar may be found.

To use the second method, the moment of R_{2-5} about t is found by drawing through t a line parallel to R_{2-5} and finding the intercept ab

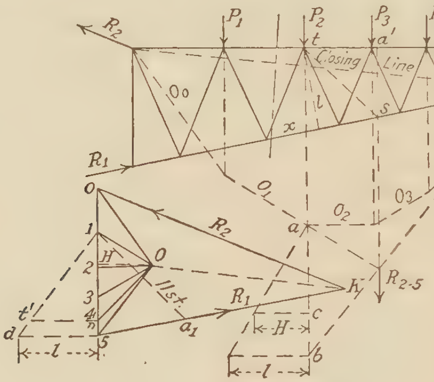


FIG. 182.

between the strings O_1 and O_5, which hold R_{2-5} in equilibrium. Then the moment of R_{2-5} about t is $H \times ab$. If l is the lever arm of x about t, the stress in x is $\frac{H \times ab}{l}$. Lay off l perpendicular to ab at b, and H as shown; then the stress in x is ac.

In this case, since the forces to the right of the section are parallel, the moment of the resultant about t is $R_{2-5} \times a't$, and this may be used instead of $H \times ab$. Thus, lay off, in the force polygon, the distance l at $5d$, draw $1d$, lay off $a't$ at nt', and the stress in x is $1n$.

In general, when the forces are not parallel, it is better to find the moment of the resultant as H times the intercept between the strings which hold the resultant in equilibrium.

If the reader desires more practice in this method, let him take any roof truss with assumed loads, and find the stress in each bar analytically and by this method.

CHAPTER XVII

GRAPHICAL DETERMINATION OF CENTER OF GRAVITY AND MOMENT OF INERTIA. LINEAR ARCHES

1. The principle involved in the relation of force and equilibrium polygons may be extended to cases having nothing to do with forces. That principle makes it possible to find graphically any quantity expressed by ΣAb, in which the values represented by b are distances, and the values represented by A either forces, areas, moments, or anything else. It is only necessary to consider or imagine the values A to be forces, and to represent them as such. In this way centers of gravity and moments of inertia may be found graphically.

2. Center of Gravity.—The distance of the center of gravity of an area from any given line is found by dividing the area into smaller areas whose centers of gravity are known, and multiplying each area by the

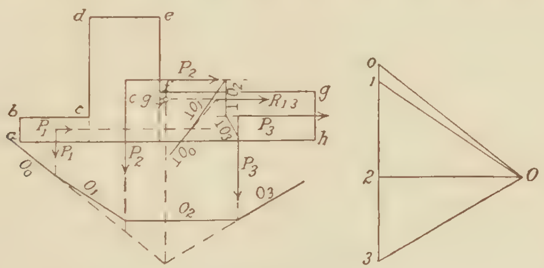


FIG. 183.

distance of its c.g. from the given line. The sum of these quantities is the total area multiplied by the distance of its c.g. from the line. In other words, if the smaller areas are considered as forces each acting at its c.g., the c.g. of the whole area will be in the line of action of the resultant of the smaller forces. Thus (Fig. 183) to find the c.g. of the area *abcdefgh*, divide it into three rectangles, find the area of each, lay off these areas as forces in the force polygon, supposing them first to act vertically, choose a pole, draw the equilibrium polygon, prolong the end strings, and the c.g. of the entire area lies in the vertical through the intersection. Now suppose the forces to act horizontally. The same force polygon will do, if it is supposed rotated 90°, and the new equilibrium polygon may be drawn by making its strings *perpendicular* to the strings of the first equi-

librium polygon. In this manner the horizontal line may be found in which the c.g. of the entire figure lies, and therefore its position is found.

3. Moment of Inertia.—The moment of inertia of an area about a given line is the sum of the areas of its parts each multiplied by the square of its distance from the given line, or ΣAx^2. But this is the same as $\Sigma(Ax)x$. If the areas A are considered as forces, each acting at its c.g. the values Ax will be the intercepts on the line, parallel to the forces, about which the moment of inertia is sought, and by considering these intercepts as new forces, each acting where its A acts, and drawing another equilibrium polygon, the moment of inertia will be an intercept.

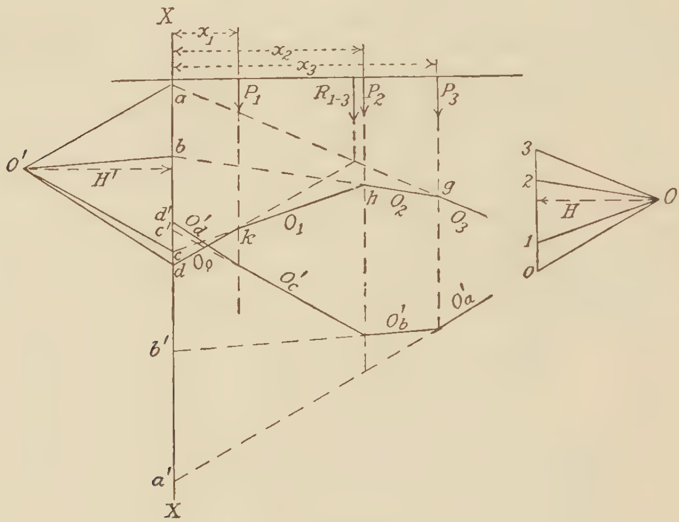


FIG. 184.

Example.—Let three forces act as shown in Fig. 184 and let it be required to find ΣPx^2 about the line xx. Draw the force and equilibrium polygons. Then $P_1x_1 = H \cdot cd$. If $H = 1$ (to the scale of force), $P_1x_1 = cd$; $P_2x_2 = bc$; $P_3x_3 = ab$, all to the scale of distance. Now let ad be a new force polygon, ab representing a force at P_3, etc.; take a pole O' and draw the new equilibrium polygon. Frequently it is not convenient to take $H = 1$; in general, designating by subscripts f and d the scales of force and distance, respectively,

$$P_3x_3 = H_f \cdot ab_d;\ ab_d = \frac{P_3x_3}{H_f}$$

or

$$ab_f = \frac{P_3x_3}{H_d}.$$

Also

$$ab_f \cdot x_3 = H'_f \cdot a'b_d = \frac{P_3x_3^2}{H_d}$$

or

$$P_3x_3^2 = H'_f \cdot a'b'_d \cdot H_d = H'_d \cdot H_d \cdot a'b'_f.$$

This method of finding moment of inertia, by using two equilibrium polygons, was first given by Culmann.

Again, since $P_3x_3 = H_f \cdot ab_d$, it follows that

$$P_3x_3^2 = 2H_f \text{ (area } abg)$$
$$\Sigma Px^2 = 2H_f \text{ (area } aghkd)$$

This method was given by Mohr.

To find the moment of inertia of an area, divide it into strips parallel to the axis assumed, find the area and c.g. of each strip, consider the area as a force acting at the c.g. parallel to the axis, draw the two equilibrium polygons, and by Mohr's method $I = 2H_f$ (area), the area being that corresponding to that shown in Fig. 184, or that included between the last equilibrium polygon, the axis, and one extreme string. If I is found about an axis through the c.g. of the area, the area will be that included between the equilibrium polygon and the two extreme strings, as the reader will see if he will work out a case. In using this method, take $2H$ equal to some even number. By Culmann's method, $I = H \cdot H'$ (an intercept).

It has thus been shown, for completeness, how centres of gravity and moments of inertia may be found graphically. There seems little reason for using these methods, however, since the results may be obtained so much more easily and quickly by computation. The writer, in a long experience, has never had occasion to use them; but he is glad that he knows them.

LINEAR ARCH

4. It has been shown that a closed equilibrium polygon represents a frame that would be in equilibrium under the action of the loads at the joints, and that if it does not close, it is in equilibrium under the loads at the joints and the stresses in the two end strings. The rays of the force polygon represent the stresses in the bars of the frame. Suppose now that this frame or equilibrium polygon is given, and that it is required to find the loads at the joints which will hold it in equilibrium. Clearly such a frame is unstable except for forces at the joints which have certain relations to each other, but for suitable loads it will be in equilibrium, and it is called the *linear arch* or *linear polygon* for those loads, whether it is really an arch (a compression polygon), or a tension polygon. A linear arch, then, is a flexible cord, or a frame composed of bars hinged at their ends, that is, either a curve or a polygon, capable of bearing no bending moment at any section of the cord or frame, and held in equilibrium by certain loads. The resultant force on any section of the cord is normal to the section at its center of gravity, but may be either tension or compression.

Let such an equilibrium polygon be given. Choose a pole and from it draw rays parallel to the strings of the polygon. The polygon will be held in equilibrium by any set of forces represented by a continuous straight or broken line cutting these rays, with the angles, if it is a broken line, on the rays. Let the polygon I, II, III (Fig. 185) be the linear arch, O the pole, and Oo, $O1$, $O2$, $O3$ the rays and strings. If all the forces at the joints are to be vertical, draw *any vertical* line 03; then

forces $P_1 = 01$ at I, $P_2 = 12$ at II, $P_3 = 23$ at III, etc. will hold the frame in equilibrium, and the stresses in the bars of the frame will be the rays Oo, O_1, etc. But clearly, this will hold good for any other vertical line $0'3'$, except that the stresses will be different. So long, therefore, as the loads at the joints are *in proportion to* 01, 12, 23, etc., they will hold the frame *in equilibrium*, though the stresses in the frame will depend upon the *magnitude of the loads*. *Equilibrium* is independent of *stress*. If the directions of the forces were not vertical but were given, the relative magnitudes could be found by drawing $0''1''$ parallel to P_1, $1''2''$ parallel to P_2, and so on.

These considerations are useful in enabling us to see the kind of load that a given linear arch is suited for, and they apply also to arches which are not linear arches, but are capable of carrying a bending moment; for clearly *an arch will be most suitable when the shape of its axis is such that it corresponds to the linear arch for the given loads*, if those loads are unvarying. If the loads vary, of course, no arch can carry the varying loads as a linear arch, but must carry bending for all loads except those for which it is the linear arch.

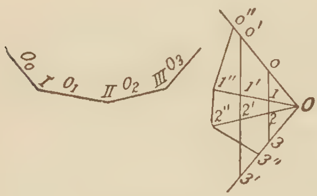


FIG. 185.

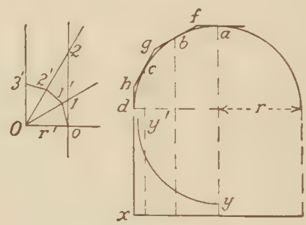


FIG. 186.

5. Semicircular Linear Arch.—Let a semicircular linear arch be loaded vertically (Fig. 186). Divide the curve into any number of equal parts, at a, b, c, d, and draw tangents at those points; then the resultant load between a and b acts through f; that between b and c acts through g, etc. Draw, from a pole O, rays parallel to these tangents, and draw any vertical line 012. Then 01 represents the total load on ba, 12 that on bc, and 23 (which is infinite) that on cd. Clearly, the load per horizontal foot increases from crown to springing, where it is infinite. The curve showing the necessary distribution of load on such a linear arch is therefore some curve similar to yy', which has the vertical at x for an asymptote. A circular arch loaded vertically therefore tends to spread at the haunches, since an infinite load per foot is impossible. This may be prevented if the load is made horizontal or inclined at and near the springing.

Let the above arch be loaded normally; then the loads are $01'$, $1'2'$, and $2'3'$; and as the points $abcd$ are taken closer to each other, the line $01'2'3'$ approaches an arc of a circle, and at the limit becomes an arc of

a circle; and the normal load per unit of length of the curve of the arch is constant, while the stress throughout the arch is also constant. Hence, the circle is the curve of equilibrium for uniform normal pressure, and under such pressure the stress is constant throughout the arch. The relation between normal load and stress may be easily found. Let $p \cdot ds$ be the normal load on ds of the arch (say on cb); this is represented by $1'2'$, and $1'2'$ bears the same proportion to the quadrant $01'2'3'$ that ds bears to the quadrant $abcd$; hence if $Oo = r'$, we have

$$p \cdot ds : r' :: ds : r, \text{ or } pr = r'$$

But $r' = Oo, = O1', O2'$, etc. and represents the constant thrust in the arch; hence, calling this T, we have, as the student already knows from Mechanics (Navier's principle)

$$T = pr$$

These principles, if kept in mind, will prove useful. For instance, since there can be no angle in the equilibrium polygon without a concentrated load at that point, it follows that a Gothic or pointed arch is, structurally, only correct and adapted to a concentrated load at the crown and a distributed load between the crown and the springing. The arch structurally adapted to a concentrated load at the crown alone would be two inclined straight lines. Hence, since architecture should be based upon structural truth, a Gothic arch should always have a concentrated load at the crown; otherwise, it will not *look right* to the eye which perceives things as they are. The dead weight alone will, if there is no other load, constitute the distributed load over the rest of the span.

CHAPTER XVIII

DIMENSIONING

1. Having found, by calculation from assumed loads, the maximum stress of each kind which can exist in a piece, and having found by experiment the breaking strength and other properties of the material, the engineer must decide what stress intensity is safe.

Practice in this matter has been somewhat rough and unscientific. It was at once recognized that the safe or *working load* should be less than the ultimate strength, but should be the latter divided by a so-called factor of safety; but as to what that factor should be there was, and is still, much uncertainty.

2. Factor of Safety.—It is obvious that this factor should vary according to circumstances, such as the character of the load. A quiescent load is different from a moving load, and moving loads differ according to the velocity and the shock with which they may be accompanied. Accordingly, different factors of safety were used for buildings, highway bridges, and railroad bridges.

There are two ways of allowing for these differences: (1) by using a variable factor of safety and unit working stress, the total stress being the sum of that due to dead (quiescent) and live (moving) loads; and (2) by reducing the total stress to an equivalent quiescent or dead stress. This is done by multiplying the live loads, or the live stress, by a factor dependent on conditions, which will reduce it to an equivalent dead load or stress, and adding this to the dead stress, to obtain the total stress, using in all cases the same factor of safety and unit stress, which is that suitable for a dead load only.

The first method was practiced for a long time, but the second is more scientific, and is now generally used. It is necessary, in using it, to find (1) the proper factor by which to multiply the live load or stress, and (2) the proper factor of safety or unit stress for a dead load.

3. The recognition of the greater destructive effect of a live load than of a dead load led to several suggestions of reducing the live stress to an equivalent dead stress. Gerber, a leading German engineer, proposed in 1863 to add the dead stress to three times the live stress, and to allow the sum to produce a unit stress equal to the elastic limit; in other words, the unit stress for dead load was to be the elastic limit and for the live load one-third as much. Many bridges were dimensioned in this

manner. It is objectionable, however, in that there should be, even for the dead load, some margin below the elastic limit, to allow for deterioration, defects, etc. Later, Unwin suggested that the live stress be multiplied by two, added to the dead stress, and the sum treated as a dead stress, using whatever unit stress is considered proper for a dead stress. This suggestion was endorsed by John Griffen and T. C. Clarke,[1] in 1872, in a paper before the A.S.C.E. The objection to this is that it is not necessary, in long spans, to multiply the live load by as large a factor as two.

4. Factor of Safety.—The factor of safety is necessary in order to allow for the following uncertainties:

1. Inaccuracies in the computations of stress, effect of changes of temperature, and initial stresses.
2. Possible increase of loads in the future.
3. Effect of repetitions of stress.
4. Variations in the material and inaccuracies in workmanship.
5. Defects in the material.
6. Deterioration of the material, as by rust or decay.
7. The effect of sudden application, dynamic action, shocks, vibration, etc., which cannot be computed; that is, the effect of live load as compared with dead.

Each of these elements is necessary because of our ignorance of its precise effect; and the factor of safety has therefore often been justly termed a factor of ignorance. The efforts of engineers have been directed toward attempting to allow for each of these elements separately, so far as possible, by some method based on experiment, experience, and judgment. If this is done, the allowance for each is definite, and can be changed as further experience may dictate.

Considering each element, the reasons for it will be stated, and the methods of allowing for it will be explained.

5(1) Inaccuracies of Computation. Computations of stress involve many assumptions which are more or less in error. Beam and column formulae are not exact; trusses are computed on the assumption that there are frictionless hinges at each joint, and therefore no bending moment at the ends of a member, while really connections are more or less rigid, and so-called *secondary stresses* exist, which are usually neglected; the shape of a frame is taken in its original (undeformed) condition; reactions are assumed to act at the center of bearings; the deformation of parts or settling of foundations may greatly modify the stresses. Indeed, the computations throughout are based on assumptions that are untrue. Nevertheless, computations are often made with great attempted accuracy. In view of the fact that the factor of safety is arbitrary, and that whether it should be, for instance, three or four is a

[1] GRIFFIN and CLARKE, "Loads and Strains of Bridges," *Trans. Am. Soc. C. E.*, vol. II, p. 93.

matter of opinion and judgment, it may well be that great nicety in computation is an unnecessary refinement. But obviously, inaccuracies in computation must be allowed for by some factor. A real stress may be 50 per cent greater than that calculated.

There is no way of eliminating this uncertainty. Secondary stresses are rarely computed; the computations are very long and tedious, and are themselves based on untrue assumptions. The best way of treating this element is to design so that secondary stresses shall be as small as practicable, and that all circumstances affecting the stress, within practical limits, shall be allowed for. It will, however, always be necessary to make some considerable arbitrary allowance to cover inaccuracies of computation.

Temperature stresses of considerable amount may exist in any material, despite any precautions.

Initial stresses, due to processes of manufacture, and particularly cold straightening of bent pieces, may exist, and may be considerable. With the improvements and greater care in the manufacture of steel these stresses, however, need not cause any anxiety. In reinforced concrete, on the contrary, the stresses due to shrinkage in drying out exist practically always and may be large.

6(2) Increase of Loads.—In the computations of stress, the loads are assumed, and they may change materially in a few years. A structure should not have so small a margin that even a considerable increase of load would make it dangerous. The weight of locomotives has increased greatly in recent years. In Massachusetts, the increase on the Boston and Albany Railroad was from 30 tons in 1873 to 60 tons in 1887, and to 118 tons in 1912;[1] yet many bridges built in 1873 were in use and safe much later than 1887. A room in a building, used as a schoolroom, may be later used as a storeroom or library carrying much heavier loads.

It is clearly impossible and undesirable to provide always for extreme contingencies which may never occur, such as cyclones, earthquakes, or other "Acts of God," but some margin for increased loads is essential; and in localities where earthquakes or cyclones may occur, forms of construction should be selected that are least likely to be injured. A margin may be definitely provided, by so designing that a given increase in live stress shall still leave a safe margin below the safe limit. The specifications of the A.R.E.A. in 1920 provided not only that the dead stress in a member added to the live stress plus impact shall not cause more than a specified unit stress, but also:

> In proportioning web members of trusses, use two-thirds of the dead-load stress, plus one and one-sixth times the live-load stress, including impact, where this sum is greater than the sum of the dead-load stress and the live-load stress, including impact.

[1] *Rept.* of Massachussetts Railroad Commission, 1912.

The meaning of this, including always impact in live-load stress, is as follows:

Dead plus live causes not over allowed unit stress

$\frac{2}{3}$ dead plus $1\frac{1}{6}$ live causes not over allowed unit stress

Multiplying the latter expression by 1.5, since full dead-load stress always acts, we obtain:

Dead plus 1.75 live causes not over 1.5 unit stress. In other words, an increase of 75 per cent in the live-load stress must not result in an increase of over 50 per cent in the specified unit stress. As the latter is 16,000 pounds per square inch for tension in steel, this would still leave a margin of about 25 per cent below the elastic limit. This is an excellent provision, but the writer believes that it should be extended to cover all members, and not restricted to web members. (This paragraph has since been changed.)

The specifications of the Committee of the A.S.C.E., in 1923 for steel railway bridges covered this point by the following requirement:[1]

> *Allowance for Increase of Live Load.*—Wherever an additional counter would be required, or reversal of stress would be caused, or heavier designing would result, a member and its details shall be designed for a live load 50 per cent greater than that (specified), with an allowance of 50 per cent increase in unit stresses.

This is not so good as the A.R.E.A. specifications; it may be all right for counters and for reversal of stress, but except in these cases it would not generally result in heavier designing, because an increase of 50 per cent in the live stress would not result in so great an increase in the total stress unless the dead stress were of the opposite kind, in which case it might.

With a provision such as that of the A.R.E.A., the factor of increase of loads may be considered as definitely met.

Structures designed without some such provision for overload will generally have very unequal margin against overload in the different parts. The importance of such provision may be seen from the following analysis:[2]

Let S = live-load stress

KS = dead-load stress

Then total stress $= S + KS$, and this is used with the standard allowed unit stress of 16,000 pounds. If now 26,000 is the limit of useful strength, each part may, before becoming unsafe, have the total stress increased to $\frac{26}{16}S(1 + K)$. But the dead-load stress is unchanged; hence the live load may be increased to $\frac{26}{16}S(1 + K) - KS = \frac{26}{16}S + \frac{10}{16}KS$, or the permissible overload is $\frac{10}{16}S(1 + K)$.

[1] The subject of repetition or reversal of stress forms the subject of Chap. XXII of "Strength of Materials," where it is discussed in detail. Only the results which affect dimensioning will be mentioned here.

[2] Communication from C. F. Loweth.

If $K = 0$, overload is 62.5 per cent.

If $K = \frac{1}{2}$, overload is 93.75 per cent.

If $K = 1$, overload is 125 per cent.

For the chords of a long-span bridge, the allowance for overload may be 125 per cent or over, while for the stringers and floor beams it will be much less.

The last revision of the A.R.E.A. specification in August, 1925, provides:

Web members shall be so proportioned that an increase of live load which will increase the total unit stresses in the chords 50 per cent will not produce total unit stresses in the web members more than 50 per cent greater than the designing stresses.

This is the same as the A.S.C.E. requirement.

7(3) Effect of Repetition of Stress, or Reversal.—It has long been recognized that it is easier to break a piece of almost any material by repeating the stress, and especially by alternating it from tension to compression (reversal), than it is to break it by one application of a load. We break a wire by bending it back and forth, not by pulling on it; that is, we reverse the stress in the extreme fibers. Although Fairbairn, in England, made a few experiments in 1860 and 1861 on a plate girder of about 20-foot span, the first engineer to investigate this question in detail was the Prussian engineer, A. Wöhler, who made a series of experiments covering 11 years, which were published in 1870 and constitute one of the most important engineering investigations ever made.

The law discovered by Wöhler was stated by him as follows, applying to iron and steel:

Fracture of the material may be caused by variations of load repeated many times, between limits, neither limit reaching the ultimate (static load). The difference between these limits, or the *range of stress*, determines the failure.

The absolute magnitude of the limiting stresses is only of importance in that, with increasing stress, the range necessary to produce fracture becomes less.

These results may be stated in other words as follows:

1. If a load is many times repeated, fracture will occur with a smaller stress than the ultimate for a single application.

2. With a fixed minimum stress, the number of repetitions is greater, the smaller the maximum stress. Thus with a minimum stress of zero, the number of repetitions necessary to break wrought-iron axles by bending (in one direction only) was as follows:

For maximum of 57,000 pounds per square inch,[1] 169,750 repetitions
For maximum of 46,724 pounds per square inch, 481,950 repetitions
For maximum of 37,379 pounds per square inch, 4,035,400 repetitions
For maximum of 31,149 pounds per square inch, >48,200,000 repetitions

[1] Calculated by the formula for flexure.

3. With a fixed maximum, the number of repetitions for fracture is greater, the greater the minimum. Thus, for cast spring steel, with a maximum of 103,800, the number of repetitions was as follows:
For minimum of 17,236, 62,000 repetitions
For minimum of 34,675, 149,800 repetitions
For minimum of 51,925, 400,050 repetitions
For minimum of 60,533, $>$19,673,300 repetitions[1]

4. If the maximum is less than a certain value, with stress of the same kind, fracture will never occur.

5. This limiting stress is greater, the larger the minimum. Thus with wrought iron:
With minimum of 16,600 compression, tension limit was 16,600
With minimum of 0 compression, tension limit was 37,380
With minimum of 24,900 tension, tension limit was 45,700
With minimum of 46,700 tension, tension limit was 46,700

These results have been confirmed by later experiments, which are discussed in a chapter in "Strength of Materials."

It thus appeared that the safe ultimate strength, under repeated loads, depended upon the ratio of the minimum to the maximum stress; and, consequently, if the allowable stress is to be taken equal to the ultimate divided by a constant factor of safety, the allowable stress should vary, depending on that ratio. Various proposals were made for doing this, some of which are discussed in "Strength of Materials," and the outcome was the adoption of formulae, suggested by Weyrauch, based upon a discussion of Wöhler's results by Launhardt, as follows: If u is the allowable stress when for stress always of the same kind, there is indefinite repetition between a certain maximum (maximum B) and zero, that is, when a maximum B is applied and entirely removed each time, then:

For wrought iron, stresses of the same kind, either tension or compression, allowable stress

$$= f = u\left(1 + \frac{1}{2}\frac{\text{minimum } B}{\text{maximum } B}\right) \tag{1}$$

For stresses of opposite kinds, maximum B being the maximum numerically, and maximum B' the maximum numerically of the opposite kind,

$$f = u\left(1 - \frac{1}{2}\frac{\text{maximum } B'}{\text{maximum } B}\right) \tag{2}$$

For steel, stresses of the same kind, using u' instead of u,

$$f = u'\left(1 + \frac{9}{11}\frac{\text{minimum } B}{\text{maximum } B}\right) \tag{3}$$

[1] Not broken after these repetitions.

For stresses of opposite kinds,

$$f = u'\left(1 - \frac{5}{11}\frac{\text{maximum } B'}{\text{maximum } B}\right) \tag{4}$$

Weyrauch suggested, $u = 9{,}950$, $u' = 15{,}650$.

Equations (1) and (2) form a continuous curve from minimum tension = maximum tension to maximum compression = maximum tension; but Eqs. (3) and (4) do not form a continuous curve.

These formulae were discussed, in 1885, by Joseph M. Wilson,[1] Bridge Engineer of the Pennsylvania Railroad. In the specifications which he adopted for that railroad, Mr. Wilson, dealing only with wrought-iron bridges, used Eq. (2) for reversed stresses, but modified Eq. (1) in the endeavor to allow for impact. The logical way to do this would be to increase that part of maximum B which is due to the live load; but Mr. Wilson took one-half of B and then, to make up for this, increased u, which has nothing to do with impact, adopting the formula

$$f = u\left(1 + \frac{\text{minimum } B}{\text{maximum } B}\right)$$

with the following values of u:

For double-rolled iron in tension (links or rods), $u = 9{,}000$ pounds per square inch.

For rolled iron in tension (plates or shapes), $u = 8{,}500$ pounds per square inch.

For rolled iron in compression, $u = 7{,}500$ pounds per square inch. If used for compression, the above formulae give only the stress allowable for a short column, and must be reduced by a column formula for long columns.

Similar specifications were adopted by other railroad companies.

It is to be noted that in Eqs. (1) or (3), if the minimum stress is zero, $f = u$ or u'; that is, u or u' is (as defined above) the allowable stress for a load which is entirely removed after each application. Also, if minimum = maximum, that is for a constant static load:

For iron, $f = \frac{3}{2}u$ = static allowable.

For steel, $f = \frac{20}{11}u'$ = static allowable.

In other words, for iron, the allowable stress when the load varies between a maximum and zero is two-thirds that for a purely static load; and in steel eleven-twentieths that for a purely static load. (Compare the discussion in Chap. XXII of "Strength of Materials.")

For reversed stresses, when maximum $B' = 0$, $f = u$ or u', of course. If maximum B' = maximum B, that is, for equal stresses of opposite kinds:

[1] Wilson, J. M., "On Specifications for Strength of Iron Bridges," *Trans. Am. Soc. C. E.*, vol. XV, p. 389.

For iron $f = \frac{1}{2}u$.

For steel $f = \frac{6}{11}u'$.

8. Winkler, from a discussion of Wöhler's results, proposed the following formulae:

For wrought iron, principally in tension,

$$f = \frac{B_{max} - 0.45B_{min}}{0.55K} \tag{5}$$

For wrought iron, principally in compression,

$$f = \frac{B_{max} - 0.4_{min}}{0.6K_1} \tag{6}$$

For steel principally in tension,

$$f = \frac{B_{max} - 0.56B_{min}}{0.44K} \tag{5a}$$

For steel principally in compression,

$$f = \frac{B_{max} - 0.63B_{min}}{0.37K_1} \tag{6a}$$

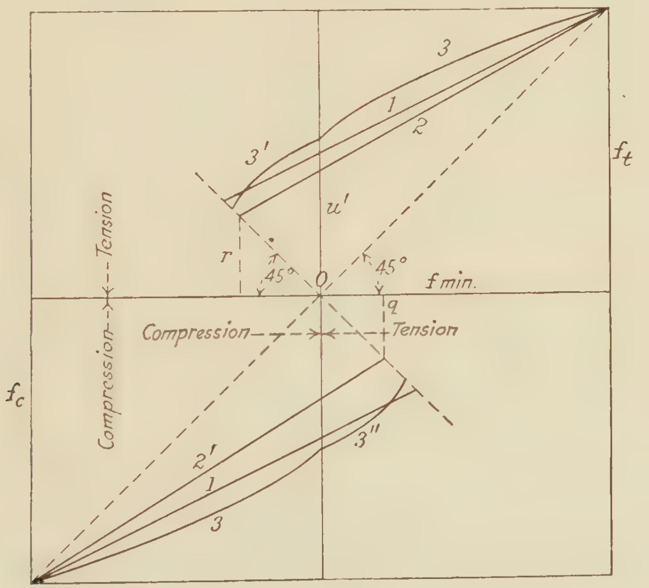


FIG. 187.

In these formulae, K and K_1 are the allowable stresses for a purely quiescent load, in tension and compression respectively. If we assume $K = 20{,}000$, $K_1 = 18{,}000$ for steel, and if for the case of tension $B_{max} = B_0$ (dead) $+ B_1$ (live tension) and $B_{min} = B_0$ (dead) $- B_2$ (live compression), we should then have

steel, in tension principally, $f = \dfrac{B_0 + 2.27B_1 + 1.27B_2}{20{,}000}$

This amounts, therefore, to multiplying the live stresses by factors greater than unity, adding to the dead, and using the allowable dead unit stress for the total.

The allowance for repetition is thus of exactly the same form as the allowance for impact, although the two things are entirely distinct.

Several curves, for steel and wrought iron, are plotted in Figs. 187 and 188. Curves 1 assume $u = \frac{1}{2}f_t$. Curves 2 and 2′ are Winkler's curves for the cases where the maximum stress is tension and where the maximum stress is compression. Curves 3, 3′, and 3″ represent the Weyrauch-Launhardt formulae. It will be noted that, for steel, curves 3 and 3′ and curves 3 and 3″ are discontinuous, as may be seen from the

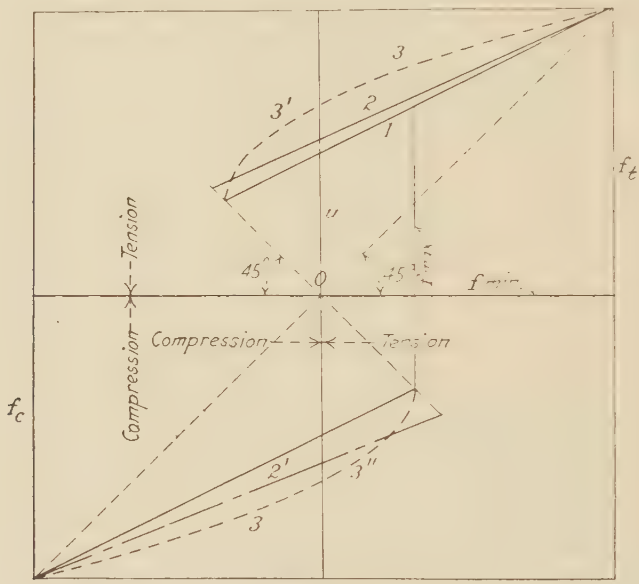


Fig. 188.

equations. Also Winkler's lines are not quite consistent; for the values r of the equal reversed stresses are not equal in 2 and 2′.

9. Formulae for repetition, except in some cases for reversed stresses, have altogether gone out of use, and, indeed, were used for only a few years. The reason for this is as follows. The experiments of Wöhler, Bauschinger, and others have shown pretty conclusively, although there are some engineers who do not quite agree to this, that for stresses of the same kind, a maximum equal to or below the original elastic limit will never cause failure. This holds for structural steel as received, and for harder steels if annealed. Now it is generally conceded that for structures or machinery it is not desirable to allow a stress exceeding the elastic limit or the yield point, on account of the permanent set which occurs

above the elastic limit, and which, in structural steel, is large at the yield point. It follows that if the maximum stress is kept below the elastic limit for stresses of the same kind, repetition has no injurious effect, and may be neglected; and it is unnecessary to use the above formulae. This amounts to basing the factor of safety, not on the static ultimate, but upon the yield point. Many engineers have long urged that the factor should be based on the elastic limit. If the ultimate did not depend on the range, it would make no difference on which it was based; for if there is a factor on the elastic limit there is nearly or quite double that factor on the ultimate, since the elastic limit is generally specified to be not less than half the static ultimate. But since the ultimate under repetition depends on the range, the allowable stress must vary with the range if the factor is based on the ultimate for repetition, but need not vary if the factor is based on the elastic limit.

For reversed stresses the case is different, for in this case the ultimate consistent with any number of reversals, or the so-called "endurance limit," is below the original elastic limit. Here, therefore, the effect of reversal must be considered. This may be done in several ways:

1. The factor of safety may be used on the endurance limit as shown by tests; that is, the allowable stress may be taken as the endurance limit divided by the factor (see Chap. XXII of "Strength of Materials" for values of the endurance limit).

2. A formula similar to Launhardt's may be used. Thus, the latest (1922) specifications of the Public Utility Commissioners of Massachusetts, contain the following provision:

If a piece is exposed to both tension and compression, it must be proportioned to resist the maximum of each kind; but the unit stresses used shall be less than those used for stress of one kind, and shall be determined by multiplying the allowable unit stresses (for stress of one kind) by the quantity

$$1 - \frac{\text{minimum stress}}{\text{twice the maximum stress}}$$

This is exactly Eq. (4) using the same coefficient as for wrought iron ($\frac{1}{2}$ instead of $\frac{5}{11}$).

It is to be noted that according to this specification u or u' is assumed as the static allowable; that is, no difference is allowed for reversals of stresses of the same kind.

3. A common method is to add to each stress a certain fraction of the smaller, and then proportion by the usual unit stresses. Thus the specifications of the A.R.E.A. for August, 1925, provide:

Members subject to reversal of stress under the passage of the live load shall be proportioned as follows:

Determine the resultant tensile stress and the resultant compressive stress and increase each by 50 per cent of the smaller; then proportion the member so that it will be capable of resisting either increased resultant stress. The connections shall be proportioned for the sum of the resultant stresses.

According to Eq. (4), the ultimate for equal reversed stresses is six-elevenths of the ultimate for the worst case of stresses of the same kind, that is, the case where the minimum is zero. For iron, the ratio is ½. For this worst case of stresses of the same kind, repetition need not be considered, as above shown, if the maximum is below the elastic limit. If curve 1 is assumed as correct, the value r of equal reversed stresses is 10 on the diagram, while u or u' is 15; hence adding 50 per cent of the smaller stress and proportioning for stress of one kind would in each case be the same as proportioning for the actual curve. According to Launhardt's formula for steel, $r = 9$, and $u' = 16.5$, so that adding 50 per cent would not be quite enough; to add 80 per cent would be just about right to conform to this formula; and some specifications have required this percentage to be added.

If the line whose ordinates show the allowable stress for reversed stresses is assumed to be a straight line running from u (Fig. 189) when minimum = 0 to r when minimum = maximum (reversed), the value of the allowable stress f will be, if K is a constant equal to the tangent of α, if S is the total stress,

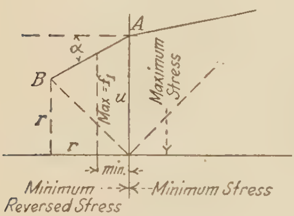


FIG. 189.

$$f_1 = u - Kf_{\min} = u - K\frac{S_{\min}}{A}$$

or

$$Af_1 + KS_{\min} = Au \quad (S_{\min} \text{ is taken numerically})$$

but

$$Af_1 = S_{\max}$$

Hence

$$A = \frac{S_{\max} + KS_{\min}}{u}$$

The method of adding to the maximum a percentage of the minimum, therefore, depends upon the assumption that AB is a straight line.

If $K = 0.5$, as specified by the A.R.E.A., $\tan \alpha = 0.5$ or $r = \frac{2}{3}u$, as it is in lines 1 of Figs. 187 and 188.

It must be remembered that Wöhler's experiments were made over 60 years ago, and that the quality of the materials he used is not accurately known. For modern materials, formulae and methods would have to be adjusted. No doubt light will be thrown on the subject by the investigations now being carried on by the Engineering Foundation, under the direction of Prof. H. F. Moore.

There is no doubt, however, that pieces exposed to reversed stresses may be properly proportioned by the method (c) under discussion, if the percentage of the smaller stress to be added is properly chosen. The percentage specified by the A.R.E.A. will meet reasonable requirements, though the writer would prefer to increase it somewhat.

The smaller the number of repetitions before fracture, the larger the maximum stress may be. If it were desired to design a piece to resist

comparatively few repetitions, the allowable stress might be much higher than if it were designed to resist an indefinite number. It has been suggested that the desired number of repetitions be estimated, and the allowable stress based on the stress corresponding to that number, by means of an S-N diagram for the material used, that is, a diagram in which the maximum stress is plotted against the number of repetitions of that stress necessary for fracture. This practice has been little used, if at all, and would never be used for structures.

10(**4**) **Variations of material** and defects of workmanship must be allowed for, since no material is uniform and no workmanship is perfect. This factor is reduced as far as possible by specifications prescribing the quality of material and the character of workmanship.

(**5**) **Defects of material** are almost certain to exist, and must be allowed for. This factor is reduced by careful inspection, and specifications prescribing methods of manufacture, as of steel, and rules for grading, as of timber. Still, there may be defects that escape detection, and even permitted defects, as in timber, cannot be accurately allowed for.

(**6**) **Deterioration** is certain to occur to some extent, its amount depending on circumstances, care of maintenance, accessibility of all parts for painting, etc. Country highway bridges, and even city bridges, are less likely to be well maintained than railroad bridges. Bridges over steam railroad tracks are particularly likely to deteriorate, owing to the steam, smoke, and cinders to which they are exposed. In the Niagara Falls arch bridge,[1] built in 1896-1897, an inspection in 1918 showed that some metal originally $\frac{1}{2}$ inch thick had been reduced in places to $\frac{1}{16}$ inch, although the loads were 30 to 70 per cent heavier than those for which the structure was designed. Plate girders in Boston, over railroad tracks, and therefore exposed to smoke, steam, and cinders, have been found with holes entirely through the webs, and angles reduced almost to knife edges.

11(**7**) This factor should have been numbered (c), since it has to do with stress produced, but it is numbered last, as it requires considerable discussion. It involves all the elements which make the effect of a live load different from the effect of a dead load, namely:

(1) Effect of duration of application of load, or time effect.

(2) Effect of sudden application, or falling of a load on the structure.

(3) Effect of motion of load, or dynamic effect.

(4) Effect of shocks, vibration, blows, etc., incident to motion, which increase the loads and stresses.

Elements 2, 3, and 4 are grouped together, under the name *impact*, and are discussed in the following chapter.

(1) *Effect of Time.*—In respect to effect of time, dead or constant loads are more destructive than live loads. We have seen that a load above the elastic limit, if allowed to remain, will cause a gradually increas-

[1] *Trans. Am. Soc. C. E.*, vol. LXXXIII, 1924.

ing *strain* up to a certain limit; and a load less than what is called the ultimate, if allowed to remain, will finally cause fracture. Different materials act differently; those which are inelastic or viscous, flowing slowly under a constant load, such as non-ferrous metals, offer an *increased* resistance to rapid distortion or sudden loads.

Wood is very susceptible to the effect of time, and a load far less than is required to break a piece in the usual way, when the test occupies but a few minutes, will ultimately break it if allowed to remain. Thurston tested some yellow pine specimens which carried an average of 375 pounds, but constant loads of 350, 300, and even 250 pounds (about 95, 80, and 60 per cent of the ultimate) broke exactly similar pieces. Those loaded to 350 pounds broke within 43 hours; those loaded to 300 pounds within 3½ days, 5 days, and a little more than a month; those loaded to 250 pounds broke at 250 days, 1 year, and 15 months.[1] Hence a factor of safety of 1.05 was not a factor of safety at all for a load applied for several days; a factor of 1.25 was no factor at all for a time of a month, and even a factor of 1.5 was no factor for a constant load. The deflection under the usual test was 1.8 inches; under 350 pounds it averaged 2.4 inches; under 300 pounds, the maximum deflection was 3 inches; and under 250 pounds it was less. "A factor of safety of 2 would possibly have permitted indefinite endurance under static load." Professor Thurston concludes, reasonably, that the factor of safety for wood should be 5 under static loads, and as high as 8 or 10 for much ordinary work.

(2) *Sudden application*, (3) *dynamic effect*, and (4) *shocks and blows* are all included under the head of Impact, and are fully treated in the next chapter. For present purposes a brief statement will suffice.

A load suddenly applied, but not falling upon a structure, will produce double the strain, and double the corresponding stress, that will be produced by the same load gradually applied, provided such doubled stress is not above the elastic limit. If it is above the elastic limit, the resulting *strain* will be much more than doubled, while the stress will be in general less than doubled, and may be found approximately from the stress-strain diagram.

A load not only suddenly applied, but falling a certain distance before striking the structure, will produce a still greater impact, which may be calculated as shown in the next chapter.

The application of a live load is also accompanied by shocks and blows of various kinds and due to various causes, such as uneven roadway or track, vibration of the moving vehicles, etc. The motion of the load also produces certain dynamic effects.

The net result is that on account of these factors the real stress in a piece will be greater than that computed by assuming the loads at rest. This increase is generally allowed for by adding to the static live-load

[1] THURSTON, "Materials of Engineering," Part I, pp. 114–118.

stress a certain varying percentage of that stress, as described in the next chapter.

It is worthy of note that to add a percentage for impact is really to use a formula of precisely the form of the Launhardt formula for repeated stress. For, if L and D are the live and dead stresses, c the factor for impact, A the area of a piece, and f_1 the unit stress allowed for the sum of L and D, and f_d the allowable stress for a dead load alone,

$$A = \frac{D + L + cL}{f_d} = \frac{D + L}{f_1}$$

$$f_1 = f_d \frac{D + L}{D + L + cL} = \frac{f_d}{1 + c}\left(\frac{D + L + cL + cD}{D + L + cL}\right) = u\left(1 + \frac{cD}{D + L + cL}\right)$$

$$= u\left(1 + c\,\frac{\text{minimum}}{\text{maximum}}\right)$$

since $\frac{f_d}{1 + c} = u =$ allowed stress when minimum $= 0$. *It is therefore easy to confuse an allowance for impact with the effect of repetition.*

12. The preceding discussion has shown that factors (2) and (3) of Art. 4 may be eliminated, and factor (7) allowed for by adding a percentage of the live-load stress, determined by judgment or experiments. Factors (1), (4), (5) and (6) must still be allowed for by a factor of safety, which will depend upon the material and the circumstances of each case. Based on all these considerations, engineers have in many cases agreed upon certain allowable stresses. Data regarding these have been given in the chapter on Materials. A few standard requirements may now be added.

13. Allowable (or Working) Stresses.—(*a*) *The Unit Stresses Specified by the A.R.E.A for Railroad Bridges* (1925).[1]

	Pounds per Square Inch
Axial tension, net section	16,000
Axial compression, gross section	$15{,}000 - 50\frac{l}{r}$
but not to exceed	12,500
l = the length of the member in inches.	
r = the least radius of gyration of the member in inches.	
Tension in extreme fibers of rolled shapes, built sections and girders, net section	16,000
Tension in extreme fibers of pins	24,000
Shear in plate-girder webs, gross section	10,000
Shear in power-driven rivets and pins	12,000
Bearing on power-driven rivets, pins, outstanding legs of stiffener angles, and other steel parts in contact	24,000

The above mentioned values for shear and bearing shall be reduced 25 per cent for countersunk rivets, hand-driven rivets, floor-connection rivets, and turned bolts.

Bearing on expansion rollers, per linear inch	$600d$
d = the diameter of rollers in inches.	

[1] These specifications are now (1927) undergoing revision, and it is probable that the basic unit stresses will be increased to the figures in the next article.

	Pounds per Square Inch
Bearing on granite masonry	800
Bearing on sandstone and limestone masonry	400
Bearing on concrete masonry	600

For cast steel in shoes and bearings, the above-mentioned unit stresses shall apply

The gross area of the compression flanges of plate girders *and rolled beams* shall not be less than the gross area of the tension flanges, but the stress per square inch shall not exceed

$$16{,}000 - 150\frac{l}{b},$$

in which

l = the length of the unsupported flange between lateral connections or knee braces.
b = the flange width.

The ratio of length to least radius of gyration shall not exceed the following:

100 for main compression members.
120 for wind and sway bracing.
140 for single lacing, and for double lacing not riveted at intersections.
170 for double lacing riveted at intersection.
200 for riveted tension members.

(*b*) *The Final Report of the Special Committee of the A.S.C.E. on Specifications for Bridge Design and Construction* (1923).

Unit Stresses

201. The unit stresses to be used in proportioning the several parts of the structure shall be as follows:

Allowable Stresses for Structural and Rivet Steel

	Kips[1] per Square Inch
Tension	16.0
Compression (one diameter)	16.0

Compression on columns:

$$p = \frac{16.0}{1 + \dfrac{l^2}{13{,}500r^2}}$$

in which p = allowable unit stress,
l = length of member, in inches,
r = least radius of gyration of member, in inches,

but not to exceed the value for $\frac{l}{r} = 40$.

Bending in extreme fibers of rolled shapes, built sections, and girders, net section	16.0
Bending in extreme fibers of pins	24.0
Shear in plate-girder and I-beam webs, net sections	12.0
Shear in pins and power-driven rivets	12.0
Shear in turned bolts and hand-driven rivets	10.0
Bearing on pins, power-driven rivets, outstanding legs of stiffener angles, and other steel parts in contact	24.0
Bearing on turned bolts and hand-driven rivets	20.0

Bearing on countersunk rivets. Only one-half the countersink shall be computed as bearing surface.

[1] 1 kip = 1,000 pounds.

Bearing on rollers per linear inch	$0.6d$

in which

d = diameter of roller, in inches.

For cast-steel shoes and pedestals, the allowable unit stress for structural steel will apply.

For members composed of steel of greater strength than structural grade, the allowable stresses may be increased in proportion to the higher yield point of the stronger steel, provided the yield point is not more than 70 per cent of the ultimate strength. In the column formula, the fractional portion of the denominator should be increased in the same ratio.

202. Limiting Length of Members.—The length of main compression members shall not exceed one hundred times their least radius of gyration, and those for wind and sway bracing one hundred and twenty times their least radius of gyration.

The length of riveted tension members shall not exceed two hundred times their least radius of gyration.

324. Compression Flanges.—The gross section of compression flanges of plate girders or I-beams shall not be less than the gross section of the tension flanges, but the stress per square inch shall not exceed

$$16{,}000 - 150\frac{l}{b}$$

in which

l = length of unsupported flange, between lateral connections or knee braces, in inches.

b = width of flange in inches.

203. Allowable Fiber Stress on Wooden Cross-ties.—The maximum wheel load with 100 per cent impact, will be distributed over three ties:

	Kips per Square Inch
White oak and dense yellow pine	2.0
Dense Douglas fir	1.5
White pine, ordinary yellow pine, and spruce	1.2

204. Allowable Pressure on Masonry.

	Kips per Square Inch
Granite masonry	0.8
Limestone and sandstone (good quality)	0.4
Concrete (1:2:4)	0.6

14(*c*) A Joint Committee of the A.S.C.E. and the A.R.E.A. is now at work in the endeavor to unify the two preceding specifications. A progress report was submitted January, 1927, in which the following unit stresses are suggested:

	Pounds per Square Inch
Axial tension, net section	20,000
Axial compression, gross section	$\dfrac{20{,}000}{1 + \dfrac{l^2}{18{,}000\,r^2}}$
but not to exceed	17,000

in which

l = length of member, in inches.

r = least radius of gyration of member, in inches.

Tension in extreme fibers of pins	30,000
Shear in plate-girder webs, gross section	12,500
Shear in power-driven[1] rivets and pins	15,000
Bearing on power-driven[1] rivets, pins, outstanding legs of stiffener angles, and other steel parts in contact	30,000
Bearing on turned bolts and hand-driven rivets	25,000
Bearing on expansion rollers, per linear inch	$750d$

in which

d = the diameter of rollers, in inches.

Allowable Pressure on Masonry

	Pounds per Square Inch
Bearing on granite masonry	1,000
Bearing on sandstone and limestone masonry	500
Bearing on concrete masonry, 2,500 pounds at 28 days	750

Provision for Overload

Web members shall be so proportioned that an increase of the live load by 30 per cent will not produce unit stresses in the members more than 30 per cent greater than those specified under Unit Stresses.

The Committee discarded the 300-foot maximum limit of span length fixed by the A.R.E.A. specifications, as it was the intention to make these specifications applicable without restriction as to span lengths.

(*d*) *The Unit Stresses Specified by the Department of Railways and Canals of Canada* (1908, *effective* 1922).

83. The maximum stresses in any part of a structure shall not exceed the following amounts in pounds per square inch, except the unit tensile stress for eyebars, in bottom chords, main diagonals, and lateral rods, which may be seventeen thousand (17,000) pounds per square inch.

82. No compression member shall have a length exceeding forty-five times its least width, nor an unsupported length in any direction exceeding one hundred times its least radius of gyration, about an axis perpendicular to that direction, excepting wind bracing and lateral struts, which may have an unsupported length of one hundred and twenty times the least radius of gyration.

84. Permissible Unit Stress in Tension or Compression on Medium Steel, 16,000.—Unit stresses for columns are to be reduced by the following formulae:

I. For square bearing or fixed ends:

$$\text{Permissible stress} \div 1 + \frac{l^2}{18{,}000\, r^2}$$

II. One square bearing and one pin end:

$$\text{Permissible stress} \div 1 + \frac{l^2}{12{,}000\, r^2}$$

III. Pin bearings:

$$\text{Permissible stress} \div 1 + \frac{l^2}{9{,}000\, r^2}$$

Formula III. Shall be used for all railway posts.

[1] Rivets driven and bucked by pneumatically or electrically driven hammers are considered power driven.

Shearing

85. On shop-driven rivets in reamed or drilled holes.................... 10,000
On pins.. 12,500
On webs or girders, gross area.................................. 10,000

Bending

86. Bending on outer fibers of pins................................ 25,000
Bending, on extreme fibers of rolled shapes, built sections and girders; net sections.. 16,000

Bearing

87. On diameter of shop-driven rivets in reamed or drilled holes measured on the projection of semi-intrados on a diametric plane......... 20,000
Pins (measured as above).. 22,000

88. Field rivets shall be increased twenty-five (25) per cent above the number determined by the above stresses if driven by hand, and ten (10) per cent if driven by power. Turned bolts will be taken at the unit stress of field rivets, power driven.

89. Per lineal inch of rollers or rockers, cast iron.................... $750\sqrt{D}$
Per lineal inch of rollers or rockers, steel........................ $1{,}200\sqrt{D}$
D represents the diameter of the roller in inches.

90. Steel and bronze discs of pivots of swing bridges, 3,000.

91. Bedplates, etc., upon first-class masonry; and upon Portland cement concrete, not less than 1 month old:

	Pounds per Square Inch
Sandstone	300
Concrete	400
Sound limestone	400
Granite	500

92.

Timber

Maximum stresses per square inch	Tension	Compression			Extreme fiber stress	Shearing	
	With grain	End bearings	Columns under 15 diameters	Across grain		With grain	Across grain
White oak	1,600	1,100	800	500	1,000	250	1,100
Georgia longleaf yellow pine	1,500	1,000	950	300	1,200	200	1,400
Douglas, Oregon and yellow fir	1,500	1,000	950	300	1,200	200	1,400
Canadian white and red pine	1,100	700	600	170	800	100	600
Spruce and eastern fir	1,100	700	600	170	800	100	800
Hemlock	1,000	600	550	150	700	90	600

93. Columns and struts, with flat ends, over 15 diameters.

l = length of pillar in inches.
d = diameter in inches.

Southern yellow pine and Douglas fir	White oak	White pine and spruce
$\dfrac{1,200}{1 + \dfrac{l^2}{1,000d^2}}$	$\dfrac{1,000}{1 + \dfrac{l^2}{1,000d^2}}$	$\dfrac{800}{1 + \dfrac{l^2}{1,000d^2}}$

94. These permissible stresses for timber are static; when combined with a live load use impact formula and increase the above stresses by fifty (50) per cent.

Unit Stresses for Electric Railway and Highway Bridges

95. Twelve and one-half (12½) per cent in excess of the preceding stresses and limiting lengths for bridges carrying electric railways.

Twenty-five (25) per cent in excess of the preceding unit stresses and limiting lengths for highway bridges.

Coefficient of Friction

96. Wrought iron or steel on itself	0.15
Wrought iron or steel on cast iron	0.20
Wrought iron or steel on masonry	0.25
Wheels on rails	0.20
Masonry on itself	0.50

(*e*) *The Unit Stresses Specified by the Public Utilities Commission of Massachusetts for Bridges Carrying Electric Railways* (1922).

17. The total maximum stress in any piece from the vertical loads and impact shall not cause greater stresses per square inch than the following:

a. On Timber

Longleaved yellow pine:	Pounds
In tension	1,500
In bending	1,600
In bending on ties	1,200
In shearing along the grain	120
In compression across the grain	400
In local bearing along the grain	1,500
In compression on columns	$1,000 - 10\dfrac{l}{d}$

in which l = length of column, in inches and d = least transverse dimension, in inches.

For spruce or chestnut, use two-thirds of the figures for yellow pine.

For Douglas fir use nine-tenths of the figures for yellow pine.

For white oak use the same as for yellow pine in tension, shearing, bearing, and compression across the grain; and two-thirds of that for yellow pine in bending and on columns.

b. On Structural Steel

	Pounds per Square Inch
Tension, on net section	16,000
Shearing (as on plate girder webs and on pins), on net section	12,000
Compression (rolled beam flanges) (see 55)	16,000
Compression (plate-girder flanges) (see 59)	14,000
Compression (in other members and parts)	$\dfrac{12{,}000}{1 + \dfrac{1}{20{,}000}\dfrac{l^2}{r^2}}$

in which l = length of piece, in inches, from center to center of connection; r = radius of gyration of the section, in inches, taken around the axis about which bending is supposed to occur (the least radius of gyration when the piece is unsupported) (see also 25, 55, 59).

	Pounds per Square Inch
Bearing, on pins, average on an area equal to thickness of piece multiplied by diameter of pin	22,000
Bending, on pins, when the forces are considered to act at the centers of the bearing surfaces	24,000
On rollers, per linear inch, d being the diameter of the roller in inches	$600d$
On rollers of drawbridges, per linear inch	$400d$

c. On Rivet Steel

	Pounds per Square Inch
Shop rivets, power driven:	
Shearing	11,000
Bearing, on area equal to thickness of piece times original diameter of rivet	22,000

For field-driven rivets the stresses shall be three-quarters of those given above if the rivets are hand driven, and nine-tenths of those given above if power driven.

The allowable stresses on wrought iron shall not exceed three-fourths of those allowed on structural steel.

The working unit stresses used in reinforced concrete construction shall be those recommended in the *Report* of the Special Committee on Concrete and Reinforced Concrete published in the *Transactions* of the A.S.C.E., Vol. LXXXI.

25. The length of a compression member between supports in any direction shall not exceed one hundred times its radius of gyration about an axis perpendicular to that direction; except for lateral and sway bracing, for which the limit may be 120.

55. In proportioning rolled beam stringers, the allowable compressive stress shall be 16,000 pounds only when the ratio of unsupported length to width of flange does not exceed 20; when this ratio r is greater than 20, the allowable compressive stress shall be $19{,}200 - 160r$. Neither separators nor a tie floor shall be considered to render the flange of a stringer "supported," even if the ties are notched.

59. In proportioning the flanges of plate girders, one-eighth of the gross area of the web may be considered available in each flange. The allowable stress in the compression flange of a plate girder, when the ratio r of unsupported length to width exceeds

20, shall be 16,400 − 120r (see 17). Ties resting upon the flange shall not be considered as supporting it laterally. Gusset plates at floor-beam connections will generally be considered as supporting the flange; but in some cases angles will be required to stiffen these plates, and in other cases the support may not be considered complete, depending upon circumstances. Generally the gross area of both flanges shall be the same.

(f) *Unit Stresses Adopted by the American Institute of Steel Construction for Steel in Buildings* (*also Adopted* 1926 *by a Joint Committee of the A.S.C.E. and the A.I.S.C.*[1]).—All parts of the structure shall be so proportioned that the sum of the maximum static stresses in pounds per square inch shall not exceed the following:

Tension: Rolled steel, on net section 18,000

Compression: Rolled steel, on short lengths or where lateral deflection is prevented 18,000

$$\text{On gross section of columns } \frac{18{,}000}{1 + \dfrac{l^2}{18{,}000r^2}}$$

with a maximum of 15,000

in which l is the unsupported length of the column, and r is the least radius of gyration of the section, both in inches.

For main compression members, the ratio l/r shall not exceed 120, and for bracing and other secondary members, 200.

Bending: On extreme fibers of rolled shapes, and built-up sections, net section, if lateral deflection is prevented 18,000

When the unsupported length l exceeds fifteen times the width b of the compression flange, the stress in pounds per square inch in the latter shall not exceed

$$\frac{20{,}000}{1 + \dfrac{l^2}{2{,}000b^2}}$$

The laterally unsupported length of beams and girders shall not exceed forty times the width b of the compression flange.

On extreme fibers of pins, when the forces are assumed as acting at the center of gravity of the pieces 27,000

Shearing: On pins 13,500

On power-driven rivets 13,500

On turned bolts in reamed holes with a clearance of not more than $\frac{1}{50}$ in 13,500

On hand-driven rivets 10,000

On unfinished bolts 10,000

On the gross area of webs of beams and girders where h, the height between flanges, in inches, is not more than sixty times t, the thickness of the web in inches 12,000

On the gross area of the webs of beams and girders, if the web is not stiffened, where h, the height between flanges, in inches, more than is sixty times t, the thickness of the web, the maximum shear per square inch, S/A shall not exceed

$$\frac{18{,}000}{1 + \dfrac{h^2}{7{,}200t^2}}$$

in which S is the total shear, and A is gross area of web in square inches.

[1] See *Proc. Am. Soc. C. E.*, p. 139, March, 1927.

	Double Shear	Single Shear
Bearing: On pins	30,000	24,000
On power-driven rivets	30,000	24,000
On turned bolts in reamed holes	30,000	24,000
On hand-driven rivets	20,000	16,000
On unfinished bolts	20,000	16,000

Expansion rollers per lineal inch, six hundred times the diameter of the roller in inches.

Combined stresses: For combined stresses due to wind and other loads, the permissible working stress may be increased 33⅓ per cent, provided the section thus found is not less than that required by the dead and live loads alone.

For members carrying wind stresses only, the permissible working stresses may be increased 33⅓ per cent.

For concrete and reinforced concrete, the so-called Joint Committee, representing five engineering associations, has prepared standard specifications (Aug. 14, 1924) in which unit stresses are specified in detail, with a full discussion of the subject. These specifications may be obtained from the Portland Cement Association.

As there are many points in this specification with which the writer is not in agreement, the unit stresses will not be given here, but will be discussed in the chapter on Reinforced Concrete, in the next volume of this work.

CHAPTER XIX

IMPACT

1. The stresses usually computed in structures are *static stresses;* that is, the loads and the structure are supposed to be at rest. As a matter of fact, the live load is generally in motion, sometimes at a high velocity, and the structure itself may be moving or vibrating. If the live loads are applied gradually, increasing from zero to their maximum value at a rate so slow that the deformation of the structure can follow without vibration, the condition will be static.

Really the live loads are moving, or may be applied suddenly, or may fall upon the structure. These modes of application cause vibration of the structure, and a maximum strain or deflection greater than would be produced by the static loads; and the static stress corresponding to this increased deformation exceeds the static stress computed. The difference or excess is the *impact* caused by the moving loads.

Impact, then, means the effect of moving loads in causing an increase of strain, and so of corresponding stress, above what they would cause if stationary.

If a material particle is moving with a uniform velocity in a straight line, the forces acting on it are balanced. If it is moving with an acceleration, the forces will be balanced if a force is added equal to its mass (W/g) multiplied by its acceleration and acting in a direction opposite to that of the acceleration (d'Alembert's principle; see Art. 11, Chap. II of "Strength of Materials").

2. Effect of a Live Load.—A live load is distinguished from a dead load in three ways: (1) It is removed and reapplied, that is, repeated. (2) It may cause reversal of stress. (3) It causes vibration and increased deformation and corresponding stress; that is, it causes, or may cause, impact.

The first two effects are not impact. Repetition may be allowed for if necessary, and is discussed in Chap. XXII of "Strength of Materials." Reversal of stress is computed and allowed for in the design, as described in the same chapter.

Impact may be due to four causes: (1) suddenness of application, or the effect of speed of the moving load; (2) the fact that a load may fall a short distance upon the structure; (3) centrifugal force, or dynamic effect, as when a load passes over a deflected beam; and (4) the effect of springs, vibration, defective rolling stock, and other causes, which change the

distribution of the load, produce blows, or cause other effects not existing when the load is quiescent.

3. *A load instantaneously applied causes double the strain that is caused by the same load gradually applied, provided the elastic limit is not exceeded.*

Consider the stress-strain diagram of a tension piece 1 square inch in section, in Fig. 190. A load OP, if applied starting at zero and increasing so slowly that there is no vibration, causes a strain or elongation OS. The triangle $OP'S$ represents the work done in stretching the piece, or the *elastic work of elongation.* The load begins with zero, and increases gradually and uniformly till it attains the value $OP = P$, and its average value is $P/2$. If e is the elongation OS, the elastic work is $Pe/2$, which is the area of the triangle $OP'S$.

Now suppose the load to be suddenly applied, but without being dropped. This may be conceived by supposing the weight P to be on

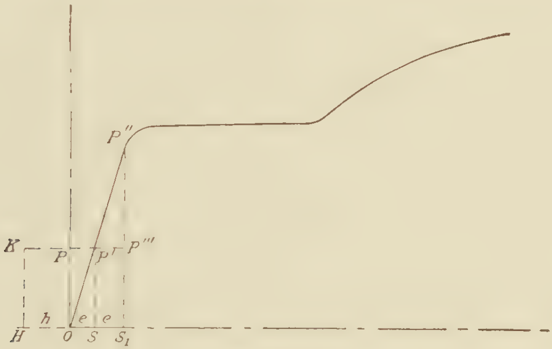


Fig. 190.

the scale pan at the end of the vertical tension bar, just touching it but not resting upon it, and then supposing it suddenly released so that its whole weight is applied at once. In this case, when the elongation is OS, the load P has done work represented by the rectangle $OPP'S$, while the elastic work of elongation is the triangle $OP'S$, as before. Hence there is accumulated work represented by the triangle OPP', which must be expended, and will cause further stretching of the bar. When the stretch has reached OS_1, the load P has done work equal to $OPP'''S_1$, and the elastic work is $OP''S_1$. These are equal when $P'P'''P'' = OPP'$, or, if P'' is below the elastic limit, when $P''S_1 = 2P'S$, and $OS_1 = 2OS$. When the strain reaches the value OS_1 it will no longer increase but will decrease, and the bar will vibrate longitudinally, finally becoming quiescent with a strain OS, corresponding to the load P. The excess energy OPP' which had been done by the load when the strain first reached OS, in excess of the elastic work corresponding to P, will be finally expended in internal friction and dissipated as heat. A small

amount will be so dissipated at the first elongation, which will not be quite twice OS.

Since $P''S_1 = 2PS$, it is generally stated that a suddenly applied load produces double the stress that the same load would cause if gradually applied, though what is shown above is that it causes double the *strain*. We have seen that there may be a strain in any direction with no stress in that direction, if there is a stress at right angles. Here, however, there is no stress at right angles; hence it appears that the longitudinal stress must correspond to the strain OS_1, or must equal $2P$. At all events, this is assumed as true.

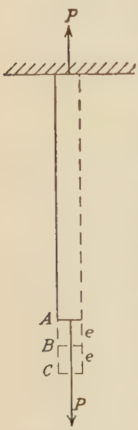


FIG. 191.

A similar result is reached if d'Alembert's principle is used. Figure 191 represents a bar to which a load P is applied. AB is the stretch that would result if the load were gradually increased from zero to P. If the load is suddenly applied it will stretch the bar to C, and it will then vibrate above and below B, finally coming to rest at that point. When the load is first applied, it exceeds the elastic resistance, which is proportional to the stretch; indeed, the elastic resistance at first is zero, since there is no stretch. This must be so if stress varies as strain. Hence the load moves with a downward acceleration a, which at the beginning will be the acceleration of gravity g, since there is no elastic resistance. As the bar stretches, elastic resistance develops and increases until at B it equals P. From A to B there is downward acceleration, and to reduce the case to one of statics, an *upward* force must be applied equal to $\frac{Pa}{g}$, and the stress in the bar is

$$S = P - \frac{P}{g}a$$

As we have seen, a is a variable, greatest $(= g)$ at A, and zero at B; hence, at A, $S = 0$; and, at B, $S = P$. Below B, the elastic resistance exceeds P, the downward motion is decelerated (or accelerated upward), and to produce a statical condition a *downward* force must be applied equal to $\frac{Pa}{g}$, so that the stress in the bar is

$$S = P + \frac{P}{g}a$$

The upward acceleration increases from B to C, and is a maximum at C, when the elastic resistance is greatest. At this point it is g, acting upward, and the stress in the bar is $2P$. The resultant force on P at this point is thus an *upward* force equal to its weight. The upward acceleration continues back to B, where it is zero, and there is again downward acceleration up to A and back to B.

On account of molecular friction and dissipation of energy, the stretch will probably not be quite $AC = 2AB$, but nearly so; and the load will not return quite to A.

The weight of the bar is neglected in the above.

It is obvious that wherever there is a downward acceleration the stress in the bar is less than P, and wherever there is an upward acceleration it is greater than P.

The question at once suggests itself, how suddenly must the load be applied in order to produce impact? There are degrees of suddenness. If the application is instantaneous, there will always be impact; but it may not be instantaneous, and yet so quickly applied that it may cause some impact, though less than if instantaneous. The only answer that the writer can give to the above question is, that *there will be some impact if the load is applied more rapidly than the stretch can follow it.* If an

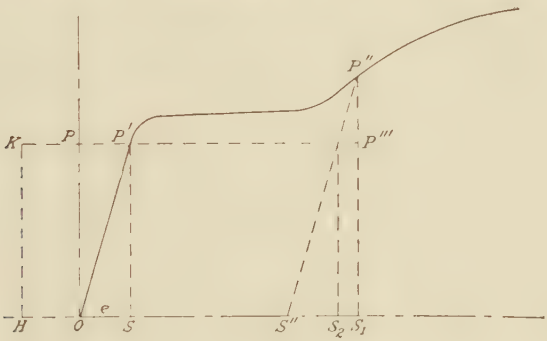


FIG. 192.

infinitesimal increment of load produces its corresponding stretch, without vibration, before the next increment is applied, there will be no impact. In other words, there will be no impact if the load is applied so slowly as to produce no vibration. This is really little more than saying that there will be no impact if there is no impact; but at all events the relationships have been indicated. What the limiting speed of application or increase of load should be, in order to produce no impact, depends upon the material, and is not known. Some materials respond in stretch to a load more quickly than others. For steel, there is no doubt that a load may be increased from zero to its full value quite rapidly without causing impact, though numerical figures cannot be given.

Referring again to Fig. 190, if $P''S_1$ exceeds the elastic limit, it will not be double $P'S$. In that case ($P'S$ greater than half the elastic limit) the only way is to find from the stress-strain diagram a point P'' such that the area $P'P'''P''$ equals the area OPP' (Fig. 192). In this case the strain will be much more than doubled, while the stress will generally

be much less than doubled. In this case, the bar will not vibrate between O and OS_1 and will not finally come to rest at a point such that the elongation is $\frac{1}{2}OS_1$, as it would (approximately) if $OP < \frac{1}{2}$ elastic limit. In stretching to S_1 (Fig. 192) the bar will have been given a permanent set, and the material will have flowed plastically, the plastic work of elongation being represented approximately by the area $OP'P''S''$, if $P''S''$ is parallel to OP'. The plastic or permanent elongation will be approximately OS'', and the elastic elongation will be S_1S''. When the bar comes to rest with an elongation OS_1, if the elastic limit has been raised to S_1P'' this is the elastic stress in the bar, and is greater than the load. The bar will vibrate about the elongation OS_2, and will finally come to rest with that elongation. The phenomenon will be a little complicated by the precise elastic condition of the bar, and by the presence of a hysteresis loop,[1] if there is one, but the above is the writer's conception of the action.

4. Effect of a Load Falling.—The effect of a load falling from a height h upon the end of the bar, involving not only suddenness of application but an accumulated kinetic energy Ph at the instant of application, may be found in the same manner. Here OH (Fig. 192) is laid off equal to h, to the same scale as the elongations. The rectangle OK is Ph; and the point P'' is found by making the area $P'P'''P''$ equal to the area $OHKP'$. Obviously, a small value of h causes the total strain OS_1 and a corresponding stress S_1P'', which may be much greater than OP, since OS is very small.

If the point P'' is below the elastic limit, the relation between OP and S_1P'' may be easily found (Fig. 190); for, calling $S_1P'' = P_1$, and $OS_1 = e_1$,

$$P(h + e_1) = \frac{1}{2}P_1e_1$$

$$\frac{P}{P_1} = \frac{e}{e_1};\ Pe_1 = P_1e$$

from which

$$e_1 = e + \sqrt{e(2h + e)}$$

$$P_1 = P\left(1 + \sqrt{\frac{2h + e}{e}}\right) \tag{1}$$

If $h = 0$; $e_1 = 2e$; $P_1 = 2P$; as already found.

Otherwise, and more generally, if a weight W drop from a height h axially upon a bar of length l and area A, and if e_1 is the maximum strain and P_1 the static stress corresponding to e_1, then if the limit of elasticity is not exceeded,

$$W(h + e_1) = \frac{1}{2}P_1e_1;\ e_1 = \frac{P_1l}{EA}$$

[1] See Chap. IV of "Strength of Materials" for explanation of a hysteresis loop.

from which

$$P_1 = W\left(1 + \sqrt{1 + \frac{2hEA}{Wl}}\right) \qquad (1a)$$

which is the same as Eq. (1) since $e = Wl/EA$.

It is sometimes stated that when a weight falls upon a piece, the resulting stress $P''S_1$ will be such that the work of deformation up to that point (area of $OP''S_1$) will equal the kinetic energy of the weight at the moment of striking. This overlooks the fact that during deformation the weight does additional work $OPP'''S_1$. The stress will obviously be greater than the above statement would indicate.

If, at the time a load is suddenly applied to a bar like Fig. 191, there is a quiescent dead load W applied at A, that dead weight will move with the weight P suddenly applied, and will at each point have the same acceleration.

It may seem, therefore, that the effect of the sudden application of P is to increase the dead stress as well as the live stress, and that the impact percentage should be a percentage of the total load, and not of the live load alone. This, however, is never done, and the following point of view indicates that it should not be done: If there is no dead load W, the load P must accelerate only a mass P/g, and below the median point (B in Fig. 191) a downward force must be added equal to $\frac{P}{g}a$, making the total stress $P + \frac{P}{g}a$, as in Art. 3; when a has its maximum value g, the stress is $2P$; but if there is a dead load W, the load P must accelerate a larger mass $\frac{P + W}{g}$; hence the acceleration a' produced will be less than a, and below the median point the downward force to be added will be $\frac{P + W}{g}a'$ and the total stress will be $P + W + \frac{P + W}{g}a'$; and since a' is less than g at its maximum, this stress will not reach $2(P + W)$; that is, the total stress will not be doubled if there is a dead load W.

Considering, as we have thus far, a tension bar 1 square inch in area, let us find the value of P which would make P_1 equal to the elastic limit if falling a height of 1 inch on a bar 5 feet long. Here $h = 1''$; $P_1 =$ 30,000 pounds; $e = \frac{30{,}000 \times 60}{30{,}000{,}000} = \frac{6}{100}$ inch. Hence $P_1 = 6.86P$, or $P = 4{,}370$ pounds. This shows the large effect of even a slight drop, and proves the necessity of having a smooth track on a railroad bridge, and the necessity of making some allowance for the effect of moving loads on any bridge if the floor is not smooth.

Similar considerations apply in the case of the other kinds of stress, such as flexure, shearing, and compression.

Sometimes a falling weight should be considered in the design. In the case of a clock tower, with a floor just below the clock, the writer considered it desirable to see that the floor below would not be injured if the clock weight should fall from a given height. This can be done by means of principles that have already been explained. Thus, suppose that the weight is above the center of the beam AB (Fig. 193), which in turn is carried by two beams into which it is framed. If the falling weight is W, the work it does in falling a height h, if d is the maximum deflection of the beam AB, and d_1 that of the beams CD, is

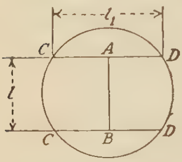


FIG. 193.

$$\text{work} = W(h + d + d_1) \tag{2}$$

This must equal the elastic work done by all three beams up to the maximum deflection, assuming the load gradually applied, and the question is, what weight P should be assumed as applied at the center of AB. It has been shown in "Strength of Materials" that the elastic work of flexure done by a transverse load is $\int \frac{M^2 dx}{2EI}$. The moment diagram for beam AB is a triangle with center ordinate $Pl/4$, if l is the length of AB, and the integral above is the moment of the area of the moment diagram about its base, divided by EI. Hence

$$\text{Elastic work of flexure of } AB = \frac{1}{EI} \cdot \frac{l}{2} \cdot \frac{Pl}{4} \cdot \frac{Pl}{12} = \frac{P^2 l^3}{96EI}$$

$$\text{Elastic work of flexure of 2 beams } CD = \frac{P^2 l_1^3}{192EI_1}$$

Hence

$$W(h + d + d_1) = Wh + W \cdot \frac{Pl^3}{48EI} + W \frac{Pl_1^3}{96EI_1} = \frac{P^2 l^3}{96EI} + \frac{P^2 l_1^3}{192EI_1}$$

from which if $\frac{l^3}{96EI} + \frac{l_1^3}{192EI_1}$ be called a,

$$P = W\left(1 + \sqrt{1 + \frac{h}{Wa}}\right) \tag{3}$$

which is in similar form to Eq. (1a).

The formulae and methods explained in this paragraph are based upon the stress-strain diagram drawn for a load gradually applied, as is usual. It must not be supposed, however, that they are exact. *The stress-strain diagram for a sudden load, or for a load with initial energy, will not be the same as for a gradually applied load.* There is no inherent reason why they should be the same, and in fact it is found that the work necessary to fracture a piece by impact is often quite different from the work necessary to fracture it by a gradually applied load, being sometimes

greater and sometimes less. Results on this point are contradictory, and additional investigation is necessary. Langenberg[1] gives results of tests of notched pieces of low-carbon steel broken in tension, and says that the elongation when broken by impact was 30 per cent, and when broken by a static test, 175 per cent: and he says that the energy absorbed in a static test was decidedly greater than that absorbed in an impact test. This is what would be expected, for in an impact test the material would not have time to elongate. Cornu-Thenard, however, found in one class of steel the energy absorbed in a static test to be 81 per cent of that absorbed by the same material under a dynamic test, and in another class of steel 108 per cent. Other experiments are quoted[2] stating that "in every case the elongation and reduction of area are greater in the specimen broken under impact tension." The writer believes the last statement open to doubt. It is certainly opposed to common experience. The conditions of the tests may not have been the same, or may not have been practical.[3]

5. Methods of Allowing for Impact.—Impact may be allowed for either by adding to the static live stress a certain fraction of that stress, or by using a smaller unit stress in designing the sections. The former method is more scientific, and is generally used. The additional or impact stress is generally stated as a percentage of the live static stress. Thus, for a load instantaneously applied, 100 per cent would be added to cover this factor; for a load falling, more than 100 per cent; generally, considerably less than 100 per cent unless the conditions are exceptional. For railroad bridges, since most of the impact arises from the engine and little from the train, it has been suggested by Lloyd Jones that impact might reasonably be allowed for as an engine excess.

In addition to suddenness of application, it must be remembered that the dynamic effect, as well as the effect of shock and vibration, must also be allowed for. Sudden application is only one element.

6. In some cases, as in some hoisting apparatus, the maximum load on a structure may be suddenly applied, and the static stresses should in consequence be doubled. The engineer must use his judgment as to these cases. The case may involve even greater impact than sudden application. There may be an initial energy. In ordinary bridges, the maximum load is never instantaneously applied, though it may be applied so quickly that it is wise to consider it instantaneous. On a railroad or highway bridge, an open rail joint or a defective pavement may cause one wheel load to drop a short distance, but this wheel load causes but a small fraction of the total stress in most members. A railroad train takes an appreciable time to cover the entire span. Clearly the degree of suddenness will depend upon the longitudinal distance which must be

[1] *Trans. A.S.T.M.*, p. 128, 1922.

[2] *Trans. A.S.T.M.*, p. 45, 1922.

[3] For much information on this whole subject, see "Resumé of Impact Testing," in *Trans. A.S.T.M.*, 1922.

loaded in order to produce the maximum stress in a member, the so-called "loaded length." The smaller this is, the greater the degree of suddenness of application, and therefore the greater the impact so far as concerns this factor. For the chords of a simple span, supported at each end, the loaded length is the entire span; for a web member it is a fraction of the span, strictly speaking, the distance from one end to the neutral point of the appropriate panel; for a floor beam, it is two panel lengths; for a stringer it is one panel length. A train moving at 50 miles an hour moves about 73 feet per second, and therefore requires but ⅓ second to cover a span of 25 feet, and this comes very near to being instantaneous application, and in such a case it is best to double the static stress, *i.e.*, to add 100 per cent for impact to cover suddenness.

The percentage to be added, so far as degree of suddenness of application is concerned, will thus vary from 100 per cent down to nothing for a loaded length such that the stress is applied to the member no faster than the stretch can follow. This lower limit is not accurately known, and it may well be that even ⅓ second is not faster than the material could follow.

7. Other Influences.—But it is not suddenness of application, alone, which causes impact, though in many cases it may be the main element. The condition of track, and of equipment, flat or excentric wheels, the smoothness and character of a pavement, centrifugal force on curves, and the character of the loads all have an influence as stated in Art. 2. A steam locomotive has the crank and connecting rods attached to the driving wheels, of course, at one side of the center. To balance them, counterweights are cast between the spokes at the opposite side of the center, but the balancing is often not perfect, so that with every revolution of the wheel a blow is struck on the rail. This is called the hammer blow of the locomotive, and causes some impact, but only, of course, amounting to some fraction of the weight on the driver. Also, when one driver is striking this blow, the other driver on the same axle is tending upward, or reducing its load, so that the two effects tend to counterbalance each other. Lateral vibration of a train also tends to throw the weight of an axle more on one wheel than on the other, while vertical vibration of the train alternately increases and decreases the load.

According to the experiments of von Weber, in Germany, reported by him in 1869, the increase or decrease of wheel pressure of a six-wheel locomotive reached 100 per cent. It will obviously be different for different types of locomotive. The reports of the Committee of the A.S.C.E. on Stresses in Railroad Track give results of measurements of the increase in stress in rail at given speeds as compared with a speed of 5 miles per hour, which is practically a static stress. In one type of engine the maximum under one driver was 190 per cent, at a speed of 45 miles per hour.[1]

[1] *Trans. Am. Soc. C. E.*, p. 348, March, 1923.

Impact due to the deflection of the structure and the fact that, if a beam or truss is originally horizontal, the load will move in a vertical curve, with a centrifugal force on the structure, is discussed in Art. 12. It is not generally necessary to take it into account, and it may be partially overcome if the structure is cambered or curved upward when carrying its own weight only.

Impact is always accompanied by vibration. Just as the impact due to sudden application of a load on a tension bar causes it to vibrate about the position that it would take under the static load, so any excess of stress above the static load causes vibration about the position that would be assumed under that load. It may therefore be said that if there is vibration, there is impact, and *vice versa.*

All these other influences act in a manner somewhat similar to sudden application; that is, the effect depends upon the loaded length.

Impact is also affected by the character of the floor. If the floor is elastic, as with wooden ties, and rails closer than the spacing of the stringers, the impact to the main trusses or girders will be damped by the elasticity of the floor. If the floor is rigid, the whole impact will be carried to the trusses or girders. This amounts to saying that, if the floor is elastic, a part of the work of sudden application or initial energy will be absorbed by the floor and will not reach the main structure.

While, generally speaking, the impact due to sudden application is greater the shorter the span, there is a limit to this. A very short span may have less impact than a longer span. And while, generally speaking, the impact is greater than the velocity, there is a limit to this also. A short span may have less impact than a longer span, and the impact at high velocity may be less than at a lower velocity; just as thin ice may be skated over safely at high speed, which at low speed would give way. A train at high speed might almost "jump" a short span. Thus the experiments of the A.R.E.A. gave the greatest observed impact (133 per cent) on a span of 60 feet.

8. Ambiguity Regarding the Term Impact.—As shown in Art. 2, the term impact properly includes the dynamic effect of the live load, suddenness, shocks, etc. The effect of repetition of load is not properly impact.

In the dimensioning of structures, however, it is customary to add the dead stress in a member to the live stress, and also to add a percentage of the live to cover not only what is strictly impact, but also the effect of repetition, and this entire percentage is termed *impact*, and will be so understood hereafter unless the term is qualified. It has been sometimes assumed that the impact may affect the dead-load stress also, but as the percentages found by experiment are always expressed as percentages of the live load, it is best to continue this practice.

The effect of repetition is discussed in the previous chapter, and reasons are shown why it is not generally considered, except for reversed stresses.

9. Formulae for Impact.—The methods of dimensioning have been discussed in the previous chapter. It is here only necessary to consider the formulae for impact. The stress caused by impact is always considered as a fraction (sometimes greater than unity but generally less) of that caused by the live load, and covers all the elements enumerated in Art. 2. The allowance for impact may be made by specifying the percentages of the live stress to be added for the different specified members, or by using a formula which applies to all. The impact stress is designated as I (which should not be confused with moment of inertia).

The Massachusetts Board of Public Utilities specifies, for electric railway bridges, or highway bridges over which electric cars run:

	Percentage of Live Added for Impact
For the auto truck specified:	
For stringers, floor beams, hangers, and truss members receiving their whole load from one panel point only	50
For wood flooring and wood stringers	0
For all other live loads:	
For floor beams and stringers	25
For floor-beam hangers	40
For all counters	40

For other members in trusses, and for main girders, the percentage shall be 26⅔ minus one-twelfth the loaded length in feet, with a maximum of 25 and a minimum of 10 per cent. The greater allowance for counters is in agreement with the suggestions made in this book. For other truss members, the requirements conform to the formula

$$\text{Percentage of impact} = \frac{100I}{L} = \frac{320 - l}{12} \tag{4}$$

I being the impact, L the live stress, and l the *loaded length* (not the span).

The earliest impact formula known to the writer was that of the Pencoyd Bridge Company in the eighties:

$$I = L\left(0.7 + \frac{5}{l}\right) \tag{5}$$

Another formula, used by the New England Railroad about the same time was

$$I = L\frac{5}{\sqrt{l}} \tag{6}$$

Shortly after, Fred Thompson, Bridge Engineer of the Southern Railway, suggested and used in the specifications of that road the formula

$$I = L \cdot \frac{300}{300 + l} \tag{7}$$

This was immediately adopted by C. C. Schneider and used in the specifications of the Pencoyd Bridge Company, and has been generally adopted since. It was used in the A.R.E.A. specifications until the last revision, when it was replaced by the formula

$$I = L\frac{300}{300 + \frac{l^2}{100}} \tag{8}$$

H. B. Seaman, in his specifications for bridges and subways,[1] used the formula

$$I \text{ (in percentage)} = 125 - \frac{1}{8}\sqrt{2000l - l^2} \tag{9}$$

Mr. Seaman has recently suggested the simple formula

$$I = L \cdot \frac{400 - \frac{1}{2}l}{400 + l} \tag{9a}$$

In deciding upon a formula for impact, it must first be decided at what span the percentage shall be zero, and whether a greater percentage than 100 shall be allowed for very short spans. Some experiments have shown impact exceeding 100 per cent, but it is generally considered that it is unnecessary to allow more than 100 per cent when $L = 0$. These two things having been decided, a formula should be chosen which will give a reasonable variation, not differing greatly from current practice, which has been proved to be safe. It is useless to attempt to get a formula which will agree with experiments, because experiments are so variable, and many experimental results are unreliable, due to causes that have been mentioned.

The following table gives the percentages according to three formulae. Equation (7) gives too great percentages for very long spans. The other two formulae are preferable.

VALUES OF IMPACT FOR SINGLE TRACK

Loaded length l	Impact per cent		
	By Eq. (8)	By Eq. (7)	By Eq. (9a)
25	98	92	91
50	93	86	83
100	75	75	70
200	42.7	60	50
300	25	50	35.7
400	15.8	29	25
500	10.8	37.5	16.7
600	8.33	33	10
800	4.5	27	0
1,000	3.0	23	0

[1] *Trans. Am. Soc. C. E.*, vol. LXXV, p. 326.

The Department of Railways and Canals, of Canada, specifies, for steam and electric railway bridges in which the loaded length is less than 80 feet,

$$I = L\frac{80 - l}{200} \tag{10}$$

which would make the *percentage* of L to be added,

$$\text{Percentage of impact} = \frac{100I}{L} = \frac{80 - l}{2} \tag{11}$$

This formula is similar in form to Eq. (4), with different constants. There is no allowance when l exceeds 80 feet, except that it is provided that to counteract the effect of impact and vibrations, the stresses, produced by the specified live *and dead* loads, shall be increased according to certain formulae. These allowances, however, are not for what we have called impact, since both live and dead stresses are increased. They are, more accurately, for repetition of load, and are discussed in Chap. XVIII. This discussion, however, indicates that a truly scientific treatment might require that the range of stress should be considered in fixing the unit stresses allowed; and the Canadian specifications are in accord with this, though they allow no strictly live-load impact when the loaded length exceeds 80 feet.

G. Lindenthal, in the specifications for the New York Connecting Railroad, uses still another formula for impact:

$$I = L \cdot \frac{L}{L + D} \cdot \frac{1{,}200 + a}{600 + 4a} \tag{12}$$

in which a is the length of *train* in feet behind the locomotive tender for the position giving maximum stress.

L = maximum live stress.

D = maximum dead stress.

This formula is based on an allowance of 200 per cent for the locomotive and tender, and 25 per cent for the cars, which has much to justify it. The value given by the formula is reduced 10 per cent for double track, 20 per cent for triple track, and 30 per cent for four tracks. This formula combines in one the effects of impact and repetition, which are separated in the Canadian specifications.

We have seen, however, that unless stresses are reversed, there is no need of allowing for repetition.

Lindenthal's formula introduces the factor $\frac{L}{L + D}$; in other words, it makes the *percentage of the live stress* which is added for impact depend upon the ratio of dead to live, decreasing as this ratio increases. For short spans and light floors the percentage is large; for long spans and

heavy ballasted floors it is smaller. The same idea is strongly urged by Mr. Steinman,[1] who proposed the formula

$$I = \frac{200}{l + 100} \cdot \frac{L}{L + D} \tag{13}$$

L and D being the live and dead stresses respectively, and l the loaded length. Mr. Steinman urges that the inclusion of the factor $\frac{L}{L + D}$ is correct in principle, and argues that to include it "makes automatic correction for the greater inertia resistance of the heavier sections." Presumably his value in Eq. (13) must be multiplied by L to give the actual impact, and as a heavy section may be due mainly to a high live stress or to a high dead stress, the formula would take account of inertia in different ways in the two cases. He also urges that it makes the largest allowance in members in which the live stress is large compared with the dead, and so contributes to the longevity of the structure under increasing loads: but this may be done equally well in the manner described in Chap. XVIII, Art. 6. He urges that it will conform to the principle that impact should be less on bridges with solid ballasted floors than on those with open decks; but this principle, as will be shown, is not a generally correct one.

For reasons partly suggested above, there seems to the writer no good reason for including the factor $\frac{L}{L + D}$. Impact properly represents the additional stress produced by the sudden application, shocks, etc. of the live load, and it should not necessarily decrease as the dead load increases. It has nothing to do with longevity, which should be adequately secured by other means; nor is it particularly concerned with inertia of the parts. The shocks due to uneven track, blows due to unbalanced drivers, etc., with the same track and drivers, will be the same whatever the dead load. The more rigid the body receiving a blow, the greater the force developed. An elastic floor will to some extent cushion and reduce the impact on the trusses or girders, but it is elasticity and not weight that produces this effect. The idea of including the factor $\frac{L}{L + D}$ seems to the writer to arise from confusing impact with repetition and other things.

Many engineers, however, seem to believe that impact should be less with a ballasted floor than with an open floor.[2] One engineer states that the English tests, *barring erratic results*, prove that this is true. The writer can find no such proof in the English tests,[3] and, indeed, the report states, specifically:

[1] *Trans. Am. Soc. C. E.*, p. 891, April, 1922; and p. 1461, August, 1922.

[2] *Trans. Am. Soc. C. E.*, pp. 914 and 906, April, 1922.

[3] *Eng. News-Record*, Aug. 20, 1921.

High results were, in certain cases, obtained on ballasted bridges, and it does not appear therefore, at the moment, to be advisable to make special concessions in respect of this type.

Of course, almost any conclusions can be derived from a series of figures or tests, if all figures which are deemed erratic, or inconsistent with such conclusions, are excluded.

C. W. Anderson, in his paper[1] "On Impact Coefficients for Railway Girders," suggested, for railways in India, the formula

$$I = L \cdot \frac{50}{50 + l}$$

In this paper, however, the effect of impact is confused with that of repetition of stress, and even with the uncertainties due to errors in computation.

As a result of experiments made in England in 1920, a subcommittee on bridge specifications, in a *Report*[2] to the Ministry of Transport, suggests the following formulae for railroad bridges:

$$I = L \cdot \frac{75}{50 + l} \tag{14}$$

$$I = L \cdot \frac{120}{90 + l} \tag{15}$$

The second of these is stated more nearly to represent the result of the tests.

It should be observed that, owing to differing types of locomotives and rolling stock, the impact on railway bridges in England is probably quite different from that in America.

Also, the impact on highway bridges may be very different from what it is on railway bridges. On machinery, cranes, etc., it may be different still, and on buildings it may be practically zero.

10. There is some difference in practice in applying these formulae. Suppose a cross-girder of an elevated railroad is supported at each end, and supports the longitudinal girders of two tracks, or four lines of girders in all. Let the longitudinal girders have spans of 50 feet, and suppose the flange section of the cross-girder at the middle of its length is being designed. The section will have its greatest stress when both adjoining spans are loaded over their entire lengths. Shall l be taken as 100 or as 200 feet? The practice of the New York Transit Commission is to take it as 200 feet, or the loaded length of all tracks which contribute. The writer does not think this consistent, and would take it as 100 feet. There is some reason, however, for the practice of the commission, arising from the fact that both tracks and both spans would probably

[1] *Proc. Inst. C. E.*, vol. CC, pp. 178–286, 1915, and vol. CCI, pp. 301–305, 1915.

[2] See *Eng. News-Record*, Oct. 20, 1921.

not be loaded at the same time with the heaviest load, and by taking $l = 200$, the value of the impact would be diminished.

11. When Impact Allowance is Unnecessary.—If the stress in a member reaches it by being transmitted through other members, the impact is reduced or damped by the elasticity of those members just as the impact of a car is damped by the springs and that of an automobile by the shock absorbers, and depends more upon the *unsprung weight* than upon the *total* weight. The impact from a locomotive or highway vehicle is felt mainly by the floor, to which it is directly applied, less by the stringers which carry the floor, still less by the floor beams which carry the stringers, and still less by the truss members.

Moreover, it is well recognized that if a stress is due to several causes, such as vertical loads, impact, wind pressure, tractive force, etc., it is not necessary to assume the maximum of each to act simultaneously. A single stress, if it does not occur often, may safely be much larger than if it occurs with every application of the load.

For these reasons, it is not customary to allow impact on piers, abutments, or foundations, and it may sometimes be safely omitted in certain members. It is not allowed for in finding stress due to lateral forces or longitudinal traction. The specifications of the A.R.E.A. expressly provide that "impact shall not be added to stresses produced by longitudinal, centrifugal, and lateral or wind forces."

12. Impact Due to Centrifugal Force on a Beam.—If a single load moves over a straight horizontal beam, it deflects the beam into a curve which is concave upward, and the centrifugal force causes an additional load on the beam.

Let a wheel carrying a weight P move with a velocity v over a beam of span l, supported at each end. The pressure of the wheel on the beam is not necessarily P. There are two reasons for this. In the *first* place, if a part of the weight P comes to the wheel through springs, and if the motion of the load is accompanied by vibration, the springs will damp the vibration just as do springs or shock absorbers on an automobile, and the pressure on the beam is correspondingly modified. In the second place, if the weight has a vertical acceleration, as it always will have, the production of this acceleration will require part of the weight to produce it, and only the remainder will be the pressure on the beam. This remainder may be greater or less than the weight, depending upon the direction of the acceleration. If the acceleration is downward, P is decreased; if it is upward, P is increased.

Hence the pressure on the beam will equal P only when there is no vertical acceleration and no effect of springs.

Let us assume P to be a load without springs (unsprung load), so that all we have to consider is vertical acceleration. The weight will move in a curve which is concave upward, and there will be a centrifugal force

acting downward, and therefore an acceleration acting upward. Let Q be the actual pressure of the load on the beam (Fig. 194). When this load is at a distance x from the right support, it produces a deflection, positive downward, at any point distant z from that support, if $z > x_1$ equal to

$$u_1 = \frac{Qx(l - z)}{6lEI}(2lz - z^2 - x^2) \tag{16}$$

If $z = x_1$ the deflection, at the load, is

$$u_1 = \frac{Qx^2(l - x)^2}{3lEI} \tag{16a}$$

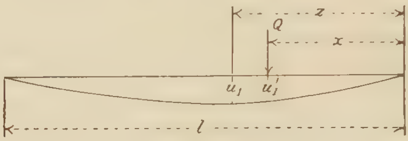


FIG. 194.

The actual deflection u will not equal u_1 or u This is because Q is really a variable; for it is the actual pressure on the beam, or the load P plus the centrifugal force; and that centrifugal force varies with the radius of curvature of the path pursued by the load, and hence varies with x. Q will only be constant (sensibly) when the speed is very slow, in which case Eq. (16a) represents the path pursued by the load.

Generally, we may express the deflection at a distance z from the right support as

$$u = u_1 + u_2 \sin \omega t \tag{17}$$

if t represents time.

The last term represents the law of simple harmonic vibration, that is, the motion along OY (Fig. 195) of the projection of a point P which is traveling in a circle of radius r with a uniform velocity v. If t is the time, measured from the time the point P is at N, and if ω is the angular velocity, $v = \omega r$,

$$\alpha = \omega t;\ y = r \sin \alpha = r \sin \omega t$$

The radius r is the *amplitude* of vibration, or the extreme vibration on either side of the mean point O.

The *period* of a complete vibration around the circle is $T = 2\pi/\omega$; the *frequency*, or number of revolutions per second, is $N = 1/T = \omega/2\pi$.

In the right-hand part of Fig. 195, time is plotted along OX, and displacement along OY, and the equation of the curve of vibration is $y = r \sin \omega t$, or the curve of sines. The velocity of the point P at the position shown, along OY, is $v \cos \alpha = \omega r \cos \alpha$, and its acceleration along OY is $\frac{d(V \cos \alpha)}{dt} = -\frac{\omega r \sin \alpha \cdot d\alpha}{dt} = -r\omega^2 \sin \alpha = -\omega^2 y$. This is negative

while P is moving from N around to N', and positive from N' to N; that is, it is always an acceleration toward O, as is easily seen by remembering that the radial acceleration of P is always toward O.

Let m be the mass at the point where Q acts. This will be P/g (for unsprung weight, as assumed) *if the weight of the beam is neglected.* The *downward* acceleration of this mass is

$$a = \frac{d^2u}{dt^2}$$

and to produce this a downward force $ma = \frac{P}{g}a$ is required. If the acceleration is downward (a positive), this force must be subtracted from P to obtain Q. Generally

$$Q = P - \frac{P}{g}a$$

Neglecting u_2, we must differentiate Eq. (16a). Observe that

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = v\frac{du}{dx} \tag{18}$$

$$\frac{d^2u}{dt^2} = v^2 \cdot \frac{d^2u}{dx^2} \tag{19}$$

Hence, making $x = l/2$,

$$\frac{d^2u_1'}{dx^2} = -\frac{Ql}{3EI} \tag{20}$$

$$Q = P + \frac{QPv^2l}{3gEI} \tag{21}$$

This is really no more than using the formula for centrifugal force

$$F = \frac{mv^2}{r} = mv^2\frac{d^2u_1'}{dx^2}$$

Hence, when the load is at the center of the span,

$$Q = \frac{P}{1 - \dfrac{Pv^2l}{3gEI}} \tag{22}$$

The deflection at the center, under a static load P, is

$$\delta = \frac{Pl^3}{48EI} = \frac{Pl}{3EI} \cdot \frac{l^2}{16}$$

Hence

$$Q = \frac{P}{1 - \dfrac{16\delta v^2}{gl^2}} \tag{23}$$

Q is greater than P. The relative increase is

$$\frac{Q-P}{P} = \frac{1}{\dfrac{gl^2}{16\delta v^2} - 1} \tag{24}$$

But it is not correct to neglect u_2. Zimmermann[1] gives an elaborate analysis, which differs from the above in the following respects: (1) He uses the exact value for the radius of curvature. (2) He allows for the fact that Q varies with x. He gives the following approximate formula for the relative increase of deflection and stress:

$$\frac{Q-P}{P} = \frac{1}{\dfrac{gl^2}{16\delta v^2} - 3} \tag{25}$$

which gives larger values than Eq. (24). Assuming $v = 70$ feet per second, $l = 25$ feet, $g = 32.2$ feet per second per second, and $l/\delta = 1{,}200$,[2] the results are as follows:

From Eq. (24): 8.83 per cent

From Eq. (25): 10.7 per cent

Referring to Eq. (17), we have seen that t is the time, measured from the time the particle P (Fig. 195) passes N; but y will be zero at intervals

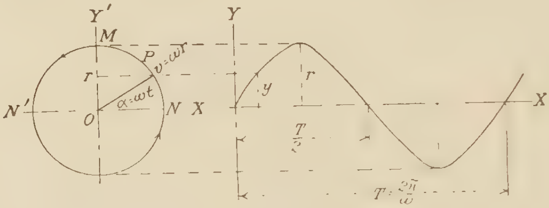


FIG. 195.

of $T/2$ after this starting time. When the load enters the span ($x = 0$), t may be taken as zero, and obviously $u = 0$ and $du/dt = 0$, u referring now to the deflection at a point distant z from the right support, given by Eq. (16). Lloyd Jones, in his parliamentary report on the vibration of bridges, finds u_2 from the equation

$$\frac{du_1}{dt} = 0 = u_2\omega + \frac{Qv(l-z)(2l-z)z}{6lEI} \tag{26}$$

substituting

$$\delta = \frac{Ql^3}{48EI}$$

[1] "Die Schwingungen eines Trägers mit bewegter Last," Ernst & Sohn, Berlin, 1896.

[2] For a central load, $\delta = Pl^3/48EI$; f = maximum fiber stress $= Plh/8I$, if h is the depth of the beam. With $f = 16{,}000$, $l/h = 10$, the value of $l/\delta = 1{,}125$.

and N_1 = number of vibrations during passage of Q over beam

$$= \frac{1}{T}\frac{l}{v} = \frac{\omega l}{2\pi v}$$

we find

$$u_2 = \text{amplitude} = -\frac{4z\delta}{\pi l N_1}\left(1 - \frac{z}{l}\right)\left(2 - \frac{z}{l}\right) \quad (27)$$

or for $z = l/2$

$$\frac{u_2}{\delta} = -\frac{3}{2\pi N_1} = -\frac{1}{2N_1}\text{(nearly)} \quad (28)$$

The minus sign is of no significance, as the amplitude varies from one side to the other of the mean position. The relative increase of deflection and stress due to u_2 is therefore $1/2N_1$. But N_1 is unknown, so that this cannot be computed.

The effect of inertia of the beam is not considered in any of the above. Gardiner[1] attempts to allow for this by assuming an equivalent mass concentrated at the center, which mass he estimates as two-thirds the total mass of the beam.

The writer considers these investigations of little practical value, because they do not conform to facts. Some omit the effect of inertia of the beam itself. All assume a single load moving over a perfectly straight beam, while in practice there are many loads, and often camber, and the dead load itself causes deflecton. If there are several loads, the vibrations caused by each would generally interefere with each other, though they might possibly synchronize. Gardiner concludes that the effect of speeds used in practice is small, while Zimmermann's elaborate study leads him to the conclusion that the maximum relative increase (due to speed alone) is about 14 per cent. Inequalities of track and rolling stock, flat or unbalanced wheels, etc., are all neglected, and are much more important than centrifugal force.

13. Experiments on Impact.—The best way to find out the extent of impact is to measure it. For this purpose, in order to avoid the necessity of measuring small elongations and to obtain a clear idea of the vibration produced, it is best to use an autographic machine in which the stretch in a given distance is registered on a paper wound about a revolving cylinder. The vibrations and the *percentage* of impact may be clearly seen on such a diagram.

The best experiments on this subject are those made by Prof. F. E. Turneaure, of the University of Wisconsin, for the A.R.E.A., published in *Bulletin* 125, July, 1910. This is one of the most admirable series of experiments ever made.[2] In work of this kind, skill, experience, and

[1] "Notes on Impact," *Trans. Am. Soc. C. E.*, pp. 437–454, 1921. With discussion.

[2] See also, TURNEAURE, "Some Experiments on Bridges under Moving Train Loads," *Trans. Am. Soc. C. E.*, vol. XLI, p. 410, 1898.

proper instruments are necessary. With some instruments there may be vibrations due to the inertia of moving parts of the instruments themselves, which will entirely vitiate the results. In some cases, the vibrations of the individual bars of the structure may entirely mask the action of the structure as a whole.

It is widely believed that speed has a great effect upon impact. Professor Turneaure's experiments indicated that the effect of speed was small. He says:

> On the contrary, they show in nearly every case very constant values at all speeds . . . and the conclusion seems warranted that the effect of speed alone on bridges of spans exceeding 40 feet in length is of no consequence whatever.

This refers to the effect of speed alone, or quickness of application of load, as distinguished from vibration. Vibration no doubt increases with speed, and as this vibration is due in part to rapidity of application, it seems impossible to separate vibration from speed, and it must be concluded that total impact increases with the speed, though the law connecting them is unknown.

The principal cause of vibration, in a railroad bridge, appears to be the engine, and the effect of the engine is mainly due to unbalanced driving wheels and to what is called "nosing" or lateral movements, which entirely alter the distribution of load on the two wheels of the same axle, sometimes doubling one of them. Anyone who has ridden on an engine at high speed will realize its vibration. There may also be a cumulative vibration due to the synchronizing of various effects. In other words, vibration may be caused not only by sudden application, or by a falling weight, or by a severe blow, or by the sudden release of a strain, but also by the repetition of a number of small impulses, properly timed. A company of soldiers in step may cause much greater vibration of a bridge than a much greater load applied at irregular intervals. A dog trotting across a suspension bridge has been known to throw it into violent vibration. It has even been suggested (I will not say claimed) that a fiddler might "fiddle a bridge down" by continually striking a note or notes with which the bridge would vibrate in unison, just as he may make a piano string vibrate sympathetically by striking certain notes on his violin. In a railroad bridge, the elasticity of the floor, the relation of circumference of drivers to panel length, the speed, the springs, the relation of panel length to car length, and other relations all affect cumulative vibration. If the panel length is nearly half a car length, the vibration produced by cars seems relatively large. The vibration will increase with the speed; but the *time of vibration* appears to be independent of the speed. M. Rabut,[1] who has been the principal experimenter on this subject in Europe, found that increase in speed

[1] *La Genie Civil*, vol. XXII, p. 88, 1892.

scarcely affects the mean deflection, that it increases vibration, but that this increase, for long spans, is only a small percentage of the deflection. Owing to cumulative effects and sympathetic vibration, there is a certain "critical speed" at which the effect of the revolution of the drivers will cause a maximum vibration.

The following is a summary of the principal results and conclusions reached by Professor Turneaure:[1]

1. Speeds less than about 25 miles per hour are not likely to result in much vibration.

2. The increase in deflection due to vibrations, caused by locomotives running at speeds of 40 to 50 miles per hour, is likely to be 40 or 50 per cent for girder spans of less than 50 feet in length.

3. This percentage decreases rapidly for longer spans, becoming about 25 per cent as a maximum for 75-foot spans.

4. Owing to cumulative effect, the percentage is likely to be a maximum of 20 or 25, for spans from 75 to 150 feet, or more, in length, but the experiments indicate no increase in percentage for increase in span.

5. The relative increase in chord stress is about the same as in deflection; that in center diagonal is somewhat more than in the deflection; and in hip vertical it corresponds more nearly to that in girders of 40- to 50-foot span lengths.

6. The effect of speed of application of the load on mean deflection was of no consequence in the spans tested (the increase in deflection from live load being due to vibration), although theory points to an appreciable increase from this cause in very short spans without camber.

7. Secondary stresses are likely to be high in small girders with shelf angles, and in some parts of trusses, and the discrepancy between observed and computed stress may be greater from this cause than from the dynamic effect of moving loads.

The following summary gives the results of the tests made by the Committee of the A.R.E.A.:[2]

1. With track in good condition the chief cause of impact was found to be the unbalanced drivers of the locomotive. Such inequalities of track as existed on the structures tested were of little influence on impact on girder flanges and main truss members of spans exceeding 60 to 75 feet in length.

2. When the rate of rotation of the locomotive drivers corresponds to the rate of vibration of the loaded structure, cumulative vibration is caused, which is the principal factor in producing impact in long spans. The speed of the train which produces this cumulative vibration is called the "critical speed." A speed in excess of the critical speed, as well as a speed below the critical speed, will cause vibrations of less amplitude than those caused at or near the critical speed.

3. The longer the span length the slower is the critical speed, and therefore the maximum impact on long spans will occur at slower speeds than on short spans.

4. For short spans, such that the critical speed is not reached by the moving train, the impact percentage tends to be constant so far as the effect of the counterbalance is concerned, but the effect of rough track and wheels becomes of greater importance for such spans.

[1] *Trans. Am. Soc. C. E.*, pp. 814–815, 1898.
[2] *Bull.* 125, pp. 32–33.

5. The impact as determined by extensometer measurements on flanges and chord members of trusses is somewhat greater than the percentages determined from measurements of deflection, but both values follow the same general law.

6. The maximum impact on web members (excepting hip verticals) occurs under the same conditions which cause maximum impact on chord members, and the percentages of impact for the two classes of members are practically the same.

7. The impact on stringers is about the same as on plate-girder spans of the same length, and the impact on floor beams and hip verticals is about the same as on plate girders of a span length equal to two panels.

8. The maximum impact percentage as determined by these tests is closely given by the formula[1]

$$I = \frac{100}{1 + \frac{l^2}{20,000}}$$

in which I = impact percentage and l = span length in feet.

9. The effect of differences of design was most noticeable with respect to differences in the bridge floors. An elastic floor, such as furnished by long ties supported on widely spaced stringers, or a ballasted floor, gave smoother curves than were obtained with more rigid floors. The results clearly indicated a cushioning effect with respect to impact due to open joints, rough wheels, and similar causes. This cushioning effect was noticed on stringers, floor beams, hip verticals, and short-span girders.

10. The effect of design upon impact percentage for main truss members was not sufficiently marked to enable conclusions to be drawn. The impact percentage here considered refers to variations in the axial stresses in the members, and does not relate to vibrations of members themselves.

11. The impact due to the rapid application of a load, assuming smooth track and balanced loads, is found to be, from both theoretical and experimental grounds, of no practical importance.

12. The impact caused by balanced compound and electric locomotives was very small, and the vibrations caused under the loads were not cumulative.

13. The effect of rough and flat wheels was distinctly noticeable on floor beams, but not on truss members. Large impact was, however, caused in several cases by heavily loaded freight cars moving at high speeds.

Previous to Professor Turneaure, Prof. S. W. Robinson,[2] of the Ohio State University, in his work as consulting engineer for the State Railroad Commission, had made a number of measurements of vibration. Professor Malverd A. Howe[3] had also made some measurements, but at speeds so low (only one over 25 miles per hour) that the impact was small. A bibliography of the subject accompanies *Bulletin* 125 of the A.R.E.A.

Recent experiments have been made in England and also in India.

14. Impact under Electric Traction.—Since mere speed of application, independent of vibration, produces little impact, and since the principal

[1] In his discussion of Mr. Anderson's paper before the Institution of Civil Engineers (vol. CC, p. 270, 1915), Professor Turneaure stated that he was then inclined to think that this value was too small, and was inclined to suggest

$$I = \frac{100}{1 + \frac{l^2}{30,000}}$$

[2] "The Vibration of Bridges," *Trans. Am. Soc. C. E.*, vol. XVI, pp. 42–65, 1887.

[3] "Bridge Deflections," *Jour. Assoc. Eng. Soc.*, vol. XIV, pp. 513–532, 1895.

source of vibration of a railroad bridge is due to the locomotive, owing largely to the reciprocating parts and unbalanced drivers, it should be expected that with a smooth track and with electric traction, in which the tractive power comes from a torque of the motor axle, the impact should be small. It should be particularly small in multiple-unit trains. Tests mentioned by Mr. Gardiner appear to justify these conclusions. But as track is never perfect, and every train has some vibration, and as rolling stock is not perfect, it is always proper to make some allowance for impact.

In 1917, Professor Turneaure made some experiments on bridges on the electrified section of the Chicago, Milwaukee, and St. Paul Railway in Montana.[1] Six bridges were tested, with an electric locomotive of two units, 112 feet in length overall, with a total weight of 635,710 pounds, followed by a sufficient number of loaded cars to cover the structure. The maximum percentages of impact found were as follows:

	Per Cent
60-foot plate girder	5
80-foot plate girder, maximum speed 61 m.p.h	17
100-foot pony truss, maximum speed 68 m.p.h	
Inclined end post	5
Main diagonal	13
Inside edge, top flange, top chord	23
210-foot deck truss span, maximum speed 55 m.p.h	11
240-foot deck truss span, maximum speed 60 m.p.h	17

The tests led to the conclusion that the impact from electric locomotives on structures over 25 feet in length is not more than one-third of that produced by steam locomotives.

15. Speed of Transmission of Stress.[2]—Suppose a bar ab, with cross-section unity, (Fig. 196) to be loaded with a load P at a. This load first stretches the infinitesimal length at a which transmits the load to the next length above, and so on. The stress travels with a speed which is to be determined. Let the unit strain be e, and the velocity v. In 1 second the stress will have travelled to c, a distance $ac = v$. The stretch in that distance will be ev, and the center of gravity of ac will have moved a distance $ev/2$. At the end of this second the center of gravity, and every point of ac, will move downward with a velocity ev, since it partakes of the movement due to the stretch of the parts above c, and this movement is ev in each second. Let w be the weight of the material of the bar per cubic unit, that is, per unit of length. Then the weight $P = eE$ has given the weight wv a velocity ev in 1 second; that is, it has

Fig. 196.

[1] *Proc. A.R.E.A.*, pp. 781–788, 1918.

[2] Compare Merriman, "Mechanics of Materials," Art. 184.

given the mass wv/g an acceleration of ev in one second, and from the formula of mechanics $F = Ma$,

$$eE = \frac{wv}{g} \cdot ev; \; v = \sqrt{\frac{Eg}{w}} \tag{29}$$

This is the formula for the velocity of propagation of stress in elastic materials. E must be expressed in pounds per square foot if v and g are feet per second and w pounds per cubic foot.

Using average values of g, E, and w as shown in the following table, the values of v as given by the formula are shown in the last column, which agree with those given by Merriman.

Velocity of Transmission of Stress in Material

Material	g, feet per second per second	E, pounds per square inch	w, pounds per cubic foot	v, feet per second
Wood	32.16	1,500,000	40	13,200
Stone	32.16	6,000,000	160	13,200
Cast iron	32.16	15,000,000	450	12,400
Wrought iron	32.16	25,000,000	480	15,500
Steel	32.16	30,000,000	490	16,900

CHAPTER XX

EARTH PRESSURE

1. The Cohesion of Earth.—When a trench is dug in earth, the sides will frequently stand without caving even if vertical, so long as the depth is not too great. It may even overhang. It does not need a wall to sustain it. This is due to the cohesion. The cohesion differs with the material. Clean sand and gravel have little; hard clay has a great deal. If the earth is excavated and thrown in a pile, it loses some or all of its cohesion, and will stand without caving only when the surface slope is less than some definite angle, varying with the material, and depending upon friction alone. For this reason, a cut will generally stand at a steeper slope than an embankment. If the excavated earth is compacted by ramming or rolling, and wetting, as it is in an earth dam, the cohesion may be partially or nearly wholly restored. Moisture affects the cohesion and the slope at which earth will stand, increasing it in some cases up to a certain point and above that point reducing it. If there is enough moisture, the material may be almost fluid and will assume a level surface, like soft mud. Cohesion is also reduced by freezing and thawing, and by vibration, so that a cut which will stand at a steep slope when first excavated will cave later and assume a flatter slope.

If the normal pressure on a given area A in a mass of earth be N, the friction on that area, if the earth is just at the point of sliding on that area, will be fN, or will be independent of the area and dependent only upon the total normal pressure, f being a constant called the coefficient of friction. The cohesion on that area, if just at the point of sliding, is often assumed to be cA, or dependent only upon the area and the constant coefficient of cohesion, and independent of the normal pressure. The coefficient of cohesion will vary greatly with the character of the material. As a matter of fact, the coefficient of cohesion varies also with the compactness of the material, and as this depends upon the pressure, the coefficient actually does depend upon the pressure N. Thus the cohesion per unit area c will generally increase with the depth below the surface of the earth.

If the earth is not just on the point of sliding on the surface A, the actual friction and the actual cohesion will be less than fN and cA, respectively.

If ab is a surface of earth (Fig. 197), the weight W of a particle at the surface may be resolved into a component normal to the surface $N = W$

cos α, and a component parallel to the surface, $S = W \sin \alpha$. The former presses the particle against the earth below; the latter tends to make it slide down the slope, and is resisted by *friction* and *cohesion*. If there is no cohesion, it is resisted only by friction, and when the slope reaches a limiting angle φ', the particle will slide. The friction depends simply upon N and upon the character of the surface, and equals fN. When the limiting angle φ' is reached, if there is no cohesion, the friction, $S = N \tan \varphi'$, so that $f = \tan \varphi'$. This limiting angle is called the *angle of repose;* and if there is no cohesion, its tangent equals the coefficient of friction. If the earth stands at a slope steeper than the value of φ' for material without cohesion, then cohesion is assisting friction.[1]

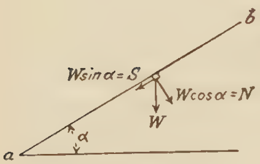


FIG. 197.

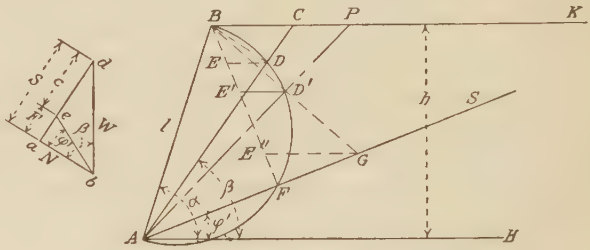


FIG. 198.

In Fig. 198 let BK be the horizontal earth surface one unit wide perpendicular to the paper, and AB the slope of a cut. Let AS be a surface making the angle φ' with the horizontal AH. Let AC be any plane above AS; draw BF perpendicular to AS, BD perpendicular to AC, and ED horizontal. The weight W which tends to slide on AC is that of the triangle ABC, or $W = \frac{1}{2}w \cdot AC \cdot BD$, if w is the weight of a cubic unit of earth. Lay this off as db, and draw ad parallel to AC and ab perpendicular to AC. The normal component of W is $ab = N$; the tangential or sliding component is $ad = S$. The friction F is $N \tan \varphi' = ae$, so that ed is the cohesion C. The angle $dbe = \beta - \varphi' = CAF = FBD$; and the angle edb equals the angle BDE, since db is perpendicular to ED and ed to BD. The triangles edb and EDB are similar.

Since $W = \frac{1}{2}w \cdot AC \cdot BD$, it may be said that BD represents W (by multiplying it by $\frac{1}{2}w \cdot AC$); and in the same way ED represents C, or $C = \frac{1}{2}w \cdot AC \cdot ED$; and the actual cohesion per unit area of the plane AC is $\frac{1}{2}w \cdot ED$; or

$$\text{actual cohesion per unit area of } AC = \frac{1}{2}w \cdot ED \qquad (1)$$

This assumed that the cohesion is the same throughout, and depends only upon the area and not at all upon the pressure. This would prob-

[1] In this chapter, φ' is called the angle of repose if there were no cohesion, or $f = \tan \varphi'$; while φ is the actual angle of repose of the material, and probably in almost all cases includes the effect of some cohesion.

ably be true if the earth were homogeneous. But it is not always homogeneous, and probably pressure makes it more compact and increases the coefficient of cohesion, at least in some kinds of earth, which are compressible. If $K = c/w$,

$$\text{actual } K = \frac{1}{2} \cdot ED \tag{2}$$

This cohesion will be different for different planes AC. For any plane, the cohesion is a measure of the extent to which cohesion must assist friction in order to prevent sliding on that plane. For AF the coefficient is zero, since there is no tendency to slide on that plane, which is at the angle of repose; here D and F coincide. For AB it is zero, because there is no weight of earth above AB to slide. If we can find the position of AC for which the cohesive resistance must be greatest, that will be the plane on which there is the greatest tendency to slide, since both friction and cohesion will have their limiting values. This position may easily be found. The point F is fixed for a given slope AB, since $FAH = \varphi'$, and BF is perpendicular to AF; while D varies as AC rotates about A. Since BFA and BDA are right angles, the point D will always lie on a circle drawn with AB as a diameter. F will lie on the same circle. ED will be a maximum when D is at the middle of the *arc* BF, or when AC bisects the arc BF, or bisects the angle BAF. If D' is this point, then for the plane AP, if $D'E'$ is parallel to DE,

$$c = \frac{1}{2}w \cdot E'D' = \text{maximum value of cohesion necessary per unit area if the face of the slope is } AB \tag{3}$$

If the force of cohesion of the earth, per square foot, is equal to or greater than $\frac{1}{2}w \cdot E'D'$, the earth will stand with the face AB; otherwise it will cave. If AB is the steepest slope at which the earth will stand at the height h, this value of c is the maximum value of the cohesion per unit area for this material. This maximum value of c may be found by determining by experiment the steepest slope AB at which the earth will stand, at a given height, and calculating c by the above equation.

Draw BD' and produce it to G. Then since BD' is perpendicular to AD', and the angles BAD' and $D'AG$ are equal, it follows that, if GE'' is parallel to DE,

$$AG = AB; \text{ and } D'E' = \frac{1}{2}GE''$$

Hence,

$$K \text{ maximum} = \frac{1}{2}E'D' = \frac{1}{4}GE'' \tag{4}$$

To find K, it is thus only necessary to draw the circle with AB as a diameter, take $AG = AB$, draw BF perpendicular to AG, and draw GE'' horizontal.

If $AB = AG = l$

$$FG = l(1 - \cos(\alpha - \varphi'))$$
$$GE'' = l(1 - \cos(\alpha - \varphi')) \sec \varphi'$$
$$K = \frac{l(1 - \cos(\alpha - \varphi'))}{4 \cos \varphi'} \tag{5}$$

and

$$\text{maximum cohesion per unit area} = c = Kw. \tag{5a}$$

If h' is the maximum height at which the earth will stand vertically, $\alpha = 90°$, and $l = h'$.

$$\therefore \text{maximum } K = \frac{h'}{4}\frac{1 - \sin \varphi'}{\cos \varphi'} = \frac{h'}{4} \tan\left(45° - \frac{\varphi'}{2}\right) \tag{6}$$

and

$$\text{cohesion per unit area} = c = Kw = \frac{wh'}{4} \tan\left(45 - \frac{\varphi'}{2}\right) \tag{6a}$$

2. Parabola of Cohesion.—If there is no cohesion, the limiting or steepest slope will be the angle of repose, and the earth will stand at this slope no matter what the height. But when cohesion is present, the steepness of the maximum slope is dependent upon the height, and the greater the height the flatter is the steepest slope at which the material will stand.

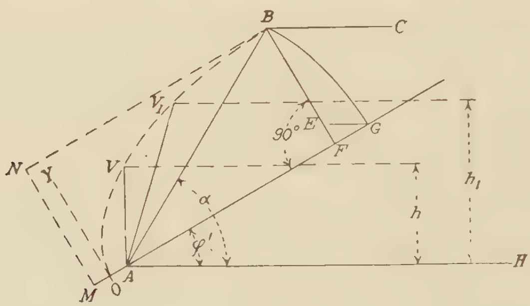


FIG. 199.

In Fig. 199 let ABC be the earth surface, as in Fig. 198, lay off AG at the angle of repose, take $AG = AB$, draw BF perpendicular to AG, and EG horizontal. Produce GA and lay off $AM = FG$. Draw BN parallel to AG and MN at right angles to it. Then

$$BN = FM = AG = AB$$
$$K = \frac{EG}{4} = \frac{FG}{4} \sec \varphi'$$
$$AM = FG = 4K \cos \varphi'$$

Assuming that the earth is homogeneous and the cohesion uniform throughout, the positions of the point M and of the line MN are fixed,

no matter how the slope AB changes in length or in inclination, because AM depends only on K and φ'; that is, no matter where B is, it is always the same distance from A as from the line MN, since $BN = AB$. Hence the locus of B is a parabola, of which A is the focus, MN the directrix, and AG the axis. This parabola passes through O half way between A and M, and its equation referred to rectangular axes OG and OY through O is

$$y^2 = 4 \cdot OA \cdot x = 2 \cdot FG \cdot x = 8K \cdot \cos \varphi' \cdot x$$

In order to draw this parabola, it is necessary either to know K and φ', in which case it can be drawn from its equation or else to have one point upon it, such as B in Fig. 199, AB being the steepest slope at which it will stand up to the height B. When drawn, the parabola gives interesting information. Draw a vertical line through A to meet the parabola at V; then AV is the greatest height at which the earth will stand vertical.

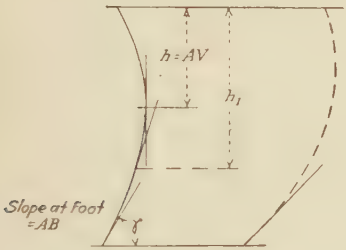


FIG. 200.

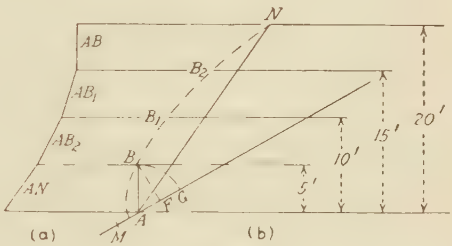


FIG. 201.

If the height is less than AV, it can overhang. For any given height h_1 the steepest slope will be AV_1. A face of a bank may be curved; beginning at the top, it may overhang until the depth is AV, where it must be vertical. As the depth increases, the slope gradually diminishes (Fig. 200). At any depth h_1 in Fig. 200, the slope would be parallel to AV_1 in Fig. 199 for the same h_1 above AH. Theoretically, at the very top the overhang might be horizontal; for if there is cohesion, it is of course true that the earth could resist some tension and act as a beam.

If it were known that the earth would stand vertical for a height of 5 feet, we could draw the Fig. 201*b* by laying off AB vertical and equal to 5′, draw the angle of repose AG, the perpendicular BF, the arc BG, make $AM = FG$, and draw the parabola. Then for a height of 20 feet the slope that could be given would be AN. The bank might be given the shape, as indicated at the left, consisting of a curved or a broken line.[1]

3. Factor of Safety for Cohesion.—If the face of the bank in Fig. 200 is given the shape just indicated, based upon the parabola of cohesion

[1] In the foregoing discussion I have closely followed the clear and excellent presentation of the subject by Prof. Charles Prelini in his book "Earth Slopes, Retaining Walls, and Dams," published by D. Van Nostrand Company. His discussion has been abbreviated though perhaps not improved.

which in turn is based upon the value of K corresponding to the *steepest* slope AB (Fig. 199) at which the earth will stand at the level BC, it will be in the limiting condition at every point; that is, there will be no factor of safety. Any desired factor may be given, say 2, by taking K equal to one-half the previous value. In this case the parabola will pass under B, and the curve of the bank will be flatter, as shown at the right in Fig. 200. At the top it will still be overhanging the vertical, but for a smaller depth.

4. Surcharge.—If the surface of the ground back of a wall is not level but slopes upward from the wall, or if it has a superimposed load,

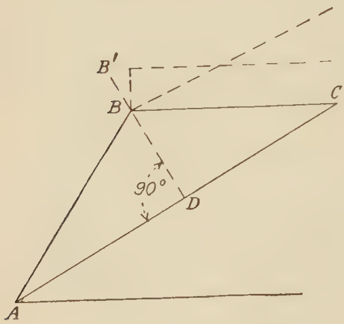


Fig. 202.

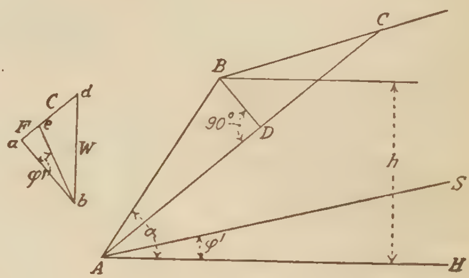


Fig. 203.

this is called a *surcharge*. A superimposed load may be treated by replacing the surcharge by an equivalent depth of earth. Thus, if there is a load of 110 pounds per square foot on the horizontal surface, and w = 110, the surcharge is equivalent to a depth of 1 foot of earth. In this way the surcharge is indicated by one of the dotted lines in Fig. 202. The weight of earth above any plane AC will not now be $\frac{1}{2}w \cdot AC \cdot BD$, but will be greater, and may be expressed by $\frac{1}{2}w \cdot AC \cdot DB'$, and the point B', which varies with AC, may be used instead of B in the construction of Figs. 198 and 199. The effect of the surcharge is to decrease the slope at which a given earth will stand at a given height. In other words, if B' is used instead of B in Fig. 198, $E'D'$ and GE'' will be increased, so that in order that the earth may stand at the slope AB with the surcharge, the cohesion c will be greater than if it stands at the same slope without the surcharge.

If the earth slopes from B, as in Fig. 203, however, the point C may be taken as the intersection of any plane with the sloping surface, not with the horizontal through B. The weight of ABC will then be, as before, $\frac{1}{2}w \cdot AC \cdot BD$, and will be represented by db in the force triangle abd. The plane of rupture will, as before, bisect the angle BAS; but the cohesion de will be greater than it would be if the surface were level, since db will

be greater. If the surface is horizontal with a uniform surcharge (Fig. 204), make the triangles DBA and $DB'A$ equal, and use B' instead of B in Fig. 198; here again the cohesion must be greater than if there is no surcharge, if AB is the steepest slope at which the earth will stand at the height h.

The foregoing principles involve a knowledge of three elements: (1) the density, or weight per cubic foot, of various materials; (2) the friction and angle of repose; (3) the coefficient of cohesion.

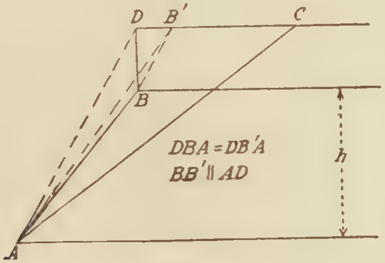


Fig. 204.

5. Weight of Materials.—The weight per cubic foot of any variety of earthy material is obviously subject to large variations, depending upon whether it is loose or compacted, the degree of compacting, the amount of moisture, the variation in size of particles, the percentage of voids, etc.

The weight of ordinary earth is greater in its natural condition in the ground than after it has been excavated, because in excavating it is loosened and is said to swell, so that a cubic foot in excavation will make originally more than a cubic foot in fill. The swelling may be 8 to 10 per cent or, it is said, even 40 per cent. When compacted afterward, as in the construction of an earthen dam or a railroad fill, shrinking occurs, and the original weight of a cubic foot of the earth may be equaled or exceeded, or the final volume will be less than in its original position.

The amount of moisture affects the density. If the original voids are more or less filled with water, the density is of course increased; if the amount of moisture is such that the original voids are increased in size, the density may be further increased or decreased; it will be decreased if the specific gravity of the material is greater than unity.

If the original voids are large, the density may be small compared with that of solid material. If those voids are partly filled by particles of a smaller size, the density will be increased. Screened crushed stone will have a certain weight per cubic unit, but if the fine stone dust is allowed to remain and partly fill the voids, the density will be greater. A sand

having grains of varying size will have a greater density than one having grains of uniform size, even if small size.

For these reasons, the data usually given as to density are necessarily variable. In important cases, the weight should be determined by direct measurement. The usual weights are about as given in the following table:

WEIGHTS OF VARIOUS MATERIALS, FOR FINDING THEIR PRESSURES AGAINST WALLS OR CONTAINERS

Kind of material	Weight per cubic foot in pounds	Angle of repose in degrees
Sand:		
Dry	80–110; average 100	20–35; average 30
Moist	100–110	30–45
Wet	110–120	20–40
Gravel, dry	90–115	20–40; average 30
Loam:		
Dry, loose	70–90	30–40
Moist	70–90	30–40
Saturated	100–120	
Rammed	90–100	
Mud	100–120	
Clay	100	
Broken stone	90–110	30–45
Cinders	30–50	25–40
Bituminous coal	50	35
Anthracite coal	40–55	27
Coke	23–32	30–45
Cement	87	30–40
Slag	100–112	45
Ore	125	45
Wheat	45–50	25
Corn	45–55	27
Malt	33	22
Barley	39	27
Oats	28	27

Obviously there is considerable room for variation in the value to be used.

6. Friction and Angle of Repose.—The following table gives average values for the angle of repose, coefficient of friction, and slope as usually expressed by the ratio of horizontal distance for 1 foot of vertical rise (as given by Prelini). The angles of repose given are not those with absolutely no cohesion, but doubtless include some cohesion, especially in the materials that are not granular.

Kind of soil	Angle of repose, degrees	Coefficient of friction	Slope, horizontal distance for 1 foot vertical
Sand:			
Dry	35	0.7002	1.4281
Moist	40	0.8391	1.1918
Very wet	30	0.5773	1.7320
Siliceous soils:			
Dry	39	0.8098	1.2349
Moist	44	0.9657	1.0355
Vegetable soil:			
Dry	40	0.8391	1.1918
Moist	41	0.8693	1.1504
Clayey soil:			
Dry	42	0.9004	1.1106
Moist	44	0.9657	1.0355
Gravel:			
Round	30	0.5773	1.7320
Sharp	40	0.8391	1.1918

The "American Civil Engineers' Pocket Book" gives the following tables:

Angles of Repose and Weights of Loose Earth

Kind of soil	Angle of repose	Coefficient of friction	Slope, horizontal distance for 1 foot vertical	Weight per cubic foot, pounds
Sand, clean	33° 41′	0.67	1.5	90
Sand and clay	36° 53′	0.75	1.33	100
Clay, dry	36° 53′	0.75	1.33	100
Clay, damp, plastic	26° 34′	0.50	2.0	100
Gravel, clean	36° 53′	0.75	1.33	100
Gravel and clay	36° 53′	0.75	1.33	100
Gravel, sand, and clay	36° 53′	0.75	1.33	100
Soil	36° 53′	0.75	1.33	100
Soft rotten rock	36° 53′	0.75	1.33	110
Hard rotten rock	45°	1.0	1.0	100
Bituminous cinders	45°	1.0	1.0	45
Anthracite cinders	45°	1.0	1.0	30

For Material Excavated and Dumped into Water Behind a Wall

Kind of soil	Angle of repose	Coefficient of friction	Slope, horizontal for 1 foot vertical	Weight per cubic foot, pounds
Sand, clean	26° 34′	0.5	2	60
Sand and clay	18′ 26′	0.33	3	65
Clay	15′ 57′	0.29	3.5	80
Gravel, clean	26° 34′	0.5	2	60
Gravel and clay	18° 26′	0.33	3	65
Gravel, sand, and clay	18° 26′	0.33	3	65
Soil	15° 57′	0.29	3.5	70
Soft rotten rock	45°	1	1	65
Hard rock, riprap	45°	1	1	65
River mud	0	0	Infinite	90

The weights in this table must be the weights in water of the *solid material* in a cubic foot. Take sand, for instance, which in the first table is given as weighing 90 pounds per cubic foot. This loses 30 pounds in water, so that $30 = 62.5 \times$ percentage of solid, and the percentage of solid is 48, and of voids, 52. As the specific gravity of the solid particles in sand is, say, 2.65, each cubic foot, if solid, would weigh $62.5 \times 2.65 = 165.6$ pounds, and the percentage of voids would be $\frac{75.6 \times 100}{165.6} = 45.7$. In water the percentage of voids must be greater than out of water, if the weight given in this table is correct.

The weight of a cubic foot of salt water may be taken as 64 pounds.

7. Determination of Cohesion Coefficient.—The coefficient c may be found in two ways. The first is by finding the steepest slope at which the material will stand at a given height, and finding c from Eq. (5). This requires that φ' for friction alone be known, and it is not known. *It must be remembered that φ' in that equation is not the maximum slope at which the actual material, with cohesion, will stand, but the slope at which it would stand if there were no cohesion.* It is, therefore, difficult to see how this method can be used, except afterward to break up the cohesion by screening or shoveling and then measuring φ'; but even then it will probably have *some* cohesion.

There has been much discussion as to whether friction and cohesion can act at the same time in a material, whether solid or granular. It is claimed by some that friction can act only after cohesion has been destroyed. The fact that cast iron in compression fails by shearing on a plane steeper than the plane of maximum shear seems to prove conclusively that friction and cohesion can act together, and the writer can see

no good reason why such should not be the case. Whether the friction that acts is that of repose or that of motion, or something between, is another matter; probably it is the former. It is well known that these two are different. As soon as a body begins to slide, the coefficient of friction is at once reduced; that is to say, the steepest angle at which a body will remain without motion on an inclined plane is greater than the slope which will just maintain it in a state of uniform motion. The experiments of Galton on railway brakes show a large variation of the coefficient of friction as the speed increases. Very likely the reduction of the coefficient when the body begins to move is due to the fact that previous to motion the resistance included some adhesion between the surfaces, and that the real friction is not much changed. The surface of any body, however smooth, is really somewhat rough, and there are little projections which, when the body is at rest, interlock with similar projections on the surface of the body upon which it rests. When one body is in motion, there is not time for this interlocking to take place, and the

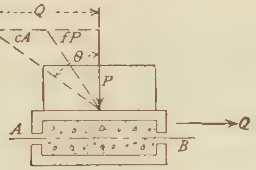


FIG. 205.

motion rubs off the projections to some extent. Hence the friction of rest, independent of adhesion, should be greater than the friction of motion. Cohesion, as the word is here used, may include some adhesion, for we have meant it to include all resistance to motion except what is due to friction.

Assuming, as seems justified, that friction and cohesion may act together, both may be determined, as suggested by Professor Cain,[1] by measuring the force Q necessary just to start a box A to move over a box B, the load on the surface AB (Fig. 205) being P. In this case

$$Q = fP + cA \tag{7}$$

or, if q is the shear per unit area, and p the normal load per unit area,

$$q = fp + c = p \tan \varphi' + c \tag{8}$$

From this equation, if q is measured for two values of p, $\tan \varphi'$ and c may be found; or, if a greater number of measurements are made, the most probable values of φ' and c may be found.

[1] "Cohesion in Earth; the Need for Comprehensive Experiment to Determine the Coefficients of Cohesion," *Trans. Am. Soc. C. E.*, 1916.

Cain quotes experiments by Leygue[1] which gave the following values:

	f	φ'	C, pounds per square foot
Dry sand	0.7	35°	1.47
Wet sand	0.85	40° 22′	8.28
Very wet sand	1.70	59° 30′	6.36
Damp fresh earth	1.63	58° 29′	18.45

These experiments were with very small pressures, of from 7 to 40 pounds per square foot or *less than the weight of a foot of earth.* Later experiments by Jaquinot and Frontard,[2] gave for very moist earth consisting of 60 per cent clay, 32 per cent silica as an impalpable dust, and 8 per cent siliceous sand, weighing 112 pounds per cubic foot, and for values of p varying from 692 to 7,154 pounds per square foot, corresponding to earth columns of 6 to 64 feet in height:

$$f = 0.14;\ \varphi' = 8°;\ c = 395 \text{ to } 448 \text{ pounds per square foot}$$ [3]

Another series of experiments, though not so reliable, gave for the same material, but with "the minimum quantity of water compatible with plasticity," the following results:

$$f = 0.185;\ \varphi' = 10°\ 35';\ c = 443 \text{ pounds per square foot}$$ [4]

These values of f are surprisingly low, but the material was moist or wet clay. It must be remembered, however, that the values of φ' are for material with no cohesion whatever.

Prelini gives the following values, but does not refer to the original sources nor tell anything about the experiments:

Quality of soil	Cohesion in pounds per square foot
Ordinary earth, dry	110.8
Ordinary earth, moist	114.6
Clayey soils, dry	107.3
Clayey soils, moist	190.8

In tests by A. L. Bell,[5] a vertical cylinder 3 inches in diameter was fitted with a piston to which a load was applied. The cylinder was cut

[1] *Annales des Ponts et Chaussées*, 1885.

[2] See Resal, "Poussée des Terres," Paris, 1910.

[3] *Trans. Am. Soc. C. E.*, p. 1321, 1916.

[4] *Ibid.*, p. 1333.

[5] Bell, Arthur Langtry, "The Lateral Pressure and Resistance of Clay and the Supporting Power of Clay Foundations," *Proc. Inst. C. E.*, vol. CXCIX, p. 233, 1914–1915.

across smoothly on two planes, and filled with clay to a point above the upper cut. The central portion was subjected to a horizontal force which was measured, and thus the shearing resistance (friction plus cohesion) along two parallel planes was determined. Mr. Bell gives the following, as average values:

Description of clay	Angle of friction φ', degrees	Cohesion per square foot, pounds
Very soft puddle clay	0	400
Soft puddle clay	3	600
Moderately firm clay	5	1,000
Stiff clay	7	1,400
Very stiff boulder clay	16	3,200

For dry sand there was no cohesion found, but the value of f in Eq. (8) "was much less than the tangent of the natural slope," which seems contradictory.

8. Combined Effect of Friction and Cohesion.—This combined effect cannot be represented by a constant angle that the pressure on any plane must make with the normal to that plane; for, from Eq. (7), if θ be that angle,

$$\tan\theta = \frac{Q}{P} = f + \frac{c}{p} = \tan\varphi' + \frac{c}{p} \tag{9}$$

where p = intensity of normal stress. Hence θ is always greater than φ', but it varies with p. The greater p is, the smaller θ.

The coefficient c, which is the cohesion per unit area, may also vary with p, increasing as p increases.

9. Practical Conclusions.—As a certain cohesion exists in most materials used for filling, the foregoing principles are important for the engineer. The following conclusions may be drawn:

1. There is much uncertainty regarding the values of the cohesion in different material, and further experiments are desirable, as Professor Cain urges.

2. Where cohesion exists, the surface of rupture, when a bank caves, will not be a plane, but a concave surface; and it may be vertical, or overhang, at the top. This may often be observed where steam shovels "undercut" a bank.

3. Where cohesion exists, there is no pressure on a wall for a certain distance down from the top, which may be determined if the coefficient of cohesion is known.

4. If the material is homogeneous, it is the lower portion of a bank that collapses first, and the slope should be flattest at the bottom.

5. A fill should be given a flatter slope than a cut, in general.

6. For very deep cuts, it may be desirable to apply the principles that have been explained, and by giving the bank a curved surface, or making it with steps of different slopes, a considerable saving in excavation may be made, as compared with giving it a uniform slope. Any desired factor of safety may be used for the cohesion. Differences in character of material, lack of homogeneity, and climate must all be considered.

7. The cohesion of earth in place probably often increases with the depth, owing to the greater pressure and compactness, and also owing to the fact that freezing and thawing may partially or wholly destroy the cohesion at the surface and for some distance below, or as far as the frost penetrates. It is, therefore, quite possible that the earth pressure against the bracing of a trench may be greater at the top than lower down. This has been found to be true in some instances (see Arts. 30 and 31).

8. Owing to the uncertainty regarding the amount of cohesion, and the fact that it may be reduced by moisture and other causes, it has generally been neglected in the theory of earth pressure. For foundations in some kinds of material, like clay, which may have much cohesion, it is, or may be, important to consider it. For retaining walls, the earth behind which is filled in and the cohesion broken up, cohesion is properly neglected. The earth behind such walls should be drained at the bottom by putting there a layer of gravel or other porous material, and with "weep holes" through the wall at intervals, so that the earth behind may not become saturated and the angle of repose much reduced, which would increase the pressure.

9(*a*) **Soils.**—Soils are materials with which the engineer must deal, and he should study them like other materials. They are rock, gravel, sand, clay, and mixtures of these. Gravel and sand differ mainly in the size of the particles, and have no cohesion unless cemented together by clay or other material. Clay is composed of very fine particles, perhaps one-fiftieth the size of an average sand grain. Clay, when moist, contains water held by capillary attraction in the fine pores, and is nearly impervious to water unless mixed with sand. Moist clay, if heated, will give up the water it holds, and will shrink sometimes as much as 20 per cent, cracking as it shrinks. Its most striking characteristic is plasticity, or change of form under pressure. Its tensile strength may be 100 or 200 pounds per square inch. By virtue of this cohesion, a bank of clay will stand vertically at first, or even overhang. It is not safe, however, to rely upon its standing long without bracing, and when the side of a clay trench caves, a crack opens at a distance back from the edge equal to about half the depth, and it caves suddenly. Yellow clay, which contains iron oxide, will stand longer than blue clay, which is treacherous.

The grains of sand or gravel do not absorb water to any perceptible extent, and do not shrink if water is driven off. If water is added to sand, it merely fills the void spaces, and by lubricating the surfaces, causes the sand to become more compact. Wet sand is thus more compact than dry sand.

Quicksand is not a material, but a *condition*, and may be produced in any granular material. As defined by Allen Hazen:[1]

> Quicksand is an even-grained sand (generally extremely fine and containing no clay), containing for the time more water than would normally be contained in its voids, and, therefore, with its grains held a little distance apart, so that they flow upon each other readily.

It may be coarse or fine. It may contain some clay, but if so, it is quicksand notwithstanding the clay, and not because of it. The excess of water under pressure may wash out the sand; but if prevented from washing out, a foundation may be built upon it.[2]

THEORY OF PRESSURE OF EARTH WITHOUT COHESION

10. The problem in earth pressure, as in the problem of stress in any other body, is to find the intensity of pressure and its direction on any plane at any point. Having this, the total pressure on any plane surface can be found.

A mass of granular earth, if unsupported, will assume a surface slope whose angle with the horizontal, along the line of greatest declivity, depends upon the character of the material. The steepest slope at which it will stand is called the *angle of repose*, and is designated by φ'. It may stand at any angle α less than φ'.

From the previous discussion of cohesion, it will be observed that φ' is properly the angle at which the earth would stand if it had no cohesion, or, in other words, it is the angle whose tangent is the coefficient of friction. For the present we neglect cohesion, so that φ' is the actual angle of steepest slope. By doing this, cohesion is really taken account of, in an approximate manner.

It is easiest to assume that the earth is a granular mass, with no cohesion, in order to determine the pressures. In this case, it is a necessary condition that on no plane can the pressure make an angle with the normal to the plane greater than φ'; otherwise there would be sliding on the plane, for the steepest slope of the surface is φ', and if the earth has this slope, the pressure upon a plane parallel to the surface, which pressure is vertical, makes an angle φ' with the normal to the plane (Fig. 206). If the earth is not a granular mass, the cohesion acts like that of a solid

[1] *Trans. Am. Soc. C. E.*, vol. XLIII, p. 582.

[2] Gow, Col. Charles R., "Characteristics of Quicksand and Other Soils Met in Ordinary Excavation," *Jour. Boston Soc. C. E.*, February, 1924.

body, and in this case the slope of the surface will be greater than φ', *i.e.*, steeper. It may even be 90° or over; that is, the sides of a trench may stand vertically without support, or even overhang, as already shown. The cohesion, however, is likely to be diminished by the action of water, by freezing and thawing, and by vibration, so that in practice no cohesion is assumed to exist permanently, though it is often relied upon temporarily.

There is but one method of studying the stress in any body, namely, to take a section and consider the equilibrium of the part on one side of the section. If the section ab (Fig. 207) be taken through a hill of earth, all that the principles of statics tell us is that the resultant pressure on the plane, P', is vertical, equal, and opposite to the weight of earth above ab. We know nothing of the distribution of P', and it need not be vertical at all points. It is, therefore, necessary to take some special case, and the assumption is made that the surface is plane and of indefinite extent. In this case, the total pressure on any plane parallel to the surface must be

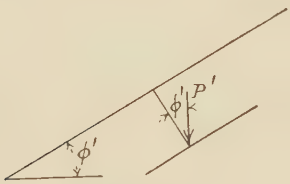


Fig. 206.

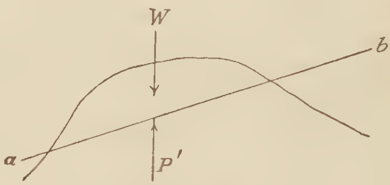


Fig. 207.

vertical; and if the mass is unlimited in extent, it must have the same direction and the same intensity at all points; hence it must be vertical at all points (on a plane parallel to the surface), and the weight on any area of that plane must be that of the earth vertically above that area. Let ab (Fig. 208) be any small surface of the plane, with an area unity, at a depth of h vertically below the surface. Then the horizontal area bc of the prism vertically above ab is $\cos \alpha$, and the weight of the prism is $wh \cos \alpha$, if w is the weight of a cubic unit of the material. The pressure per unit area on a plane parallel to the surface at a depth h vertically is therefore

$$p' = wh \cos \alpha = wd \tag{10}$$

if d is the *normal* distance of the plane from the surface.

If, now, the mass of earth be imagined divided by the plane ab, not parallel to the surface, and the left-hand portion removed (Fig. 209), then a pressure applied on each elementary area of ab of the same magnitude, and in the same direction as the pressure exerted on that area before the portion to the left of the plane was removed, will clearly hold in equilibrium the earth to the right of the plane. These are not the only forces, how-

ever, which will hold the earth in equilibrium. In fact, the pressure on ab necessary to maintain equilibrium may vary within considerable limits. First, let this pressure be just sufficient to prevent the earth above from slipping down; evidently if P is the *total* pressure on ab, and W the weight of any prism abc (Fig. 209), the pressure on the plane bc will be the resultant of these, or R, and the angle β which this makes below the normal to bc must be less than φ, the angle of repose; otherwise the earth would slide down on bc. This must be true for any plane bc, going through b, and there will be one of these planes on which we may say the tendency to slide will be greatest, that is, which will require the largest value of P in order that β may be less than φ. This value of P is the minimum it can have and hold the earth to the right of ab in equilibrium; and it is the pressure which any retaining wall, whose back is ab, must be capable of resisting It is called the *active earth pressure,* because the earth acts actively and the wall simply prevents it from sliding down.

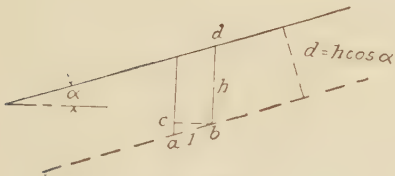


FIG. 208.

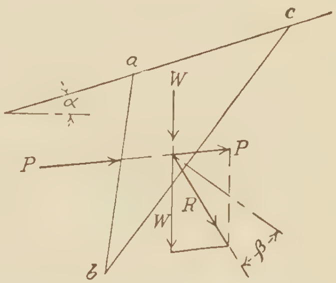


FIG. 209.

If we now increase P by any means, as by letting an arch abut against the wall or in any way whatsoever, the angle β is diminished on every plane passing through b. If P is increased sufficiently, the resultant will be thrown to the right of the normal, until finally, on *some* plane bc', passing through b, the resultant will make an angle $\beta = \varphi$ *above* the normal, in which case the earth will be forced to slide *upward* on the plane on which this condition of things first occurs. This will give the *maximum* value of P, and it is called the *passive earth pressure,* because we now have an active force exerted to force the earth up, instead of passively resisting its tendency to slide down. Between these two limits P may vary. In the case of retaining walls we have to do with active earth pressure, but in other cases we may have to consider the passive earth pressure. These two limiting values are all that we ever have to consider; in one case the earth is just on the point of sliding downward, and in the other it is on the point of being forced upward; in *either,* there is one plane, and but one (passing through b), on which the pressure is inclined at an angle φ with the normal to the plane.

There are three theories of earth pressure to be considered. All agree in neglecting cohesion. The first two agree in assuming a plane upper surface of indefinite extent; these two agree with each other, one being analytical and the other graphical, but they do not agree with the third. All three will now be explained and compared.

11. Rankine's Theory.—This theory assumes the earth to be a purely granular mass with a plane upper surface of indefinite extent. Equation (10) gives then the intensity of pressure p' on any plane parallel to the surface at a vertical depth h below the surface. The stress on any plane at any point is necessarily compression. This stress is found by using the simple principles of equilibrium or internal stress in solid bodies, with the aid of the single principle that the obliquity of the stress on any plane passing through a point (the angle between that stress and the normal to the plane) must not exceed φ, while on *one plane* it is exactly φ.

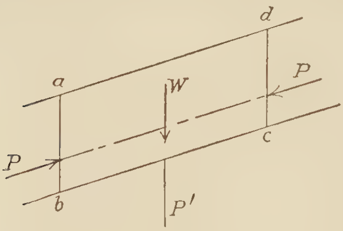


FIG. 210.

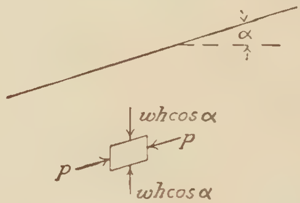


FIG. 211.

In Fig. 210 let *ad* be the surface, *bc* parallel thereto, and *ab* and *dc* vertical. As shown above, the pressure on *bc* must be vertical and equal and opposite to the weight of *abcd*. This may also be shown as follows: The mass being unlimited, the pressures P on *ab* and *dc* must be equal, opposite, and parallel; hence P' and W must be equal and opposite; and since this must be true for any small vertical prism between a and d, W and P' must lie in the same line. From this it necessarily follows that the two forces P are not only parallel, but are in the same line; otherwise the particle would not be in equilibrium. Hence the principle: *The pressure on any vertical plane is parallel to the surface.* Hence at any point in such a mass of earth as is considered (Fig. 211), the pressure of intensity $wh \cos \alpha$ on a plane parallel to the surface, and the pressure of intensity p on a vertical plane, are a pair of conjugate pressures, or the direction of the pressure on either plane is parallel to the other (see Chap. V of the volume on "Strength of Materials"). The former of these ($wh \cos \alpha$) is given; the latter is to be found from the condition that the maximum obliquity is φ. Hence (see Eq. (28), Chap. V):

$$p \gtrless wh \cos \alpha \frac{\cos \alpha \mp \sqrt{\cos^2 \alpha - \cos^2 \varphi}}{\cos \alpha \pm \sqrt{\cos^2 \alpha - \cos^2 \varphi}} \tag{11}$$

That is, p may have any value between these two extremes consistent with equilibrium. The two extremes are when the earth is just on the point of sliding up or down on some plane. Equation (11) will presently be demonstrated again.

If the surface is horizontal,

$$p = wh\frac{1 \mp \sin \varphi}{1 \pm \sin \varphi} \tag{12}$$

If the surface is inclined at the angle of repose, $\alpha = \varphi$, and there is but one possible value of p, namely,

$$p = wh \cos \varphi = wd \tag{13}$$

The intensity of pressure on any plane varies directly with the depth. Hence, on the back of a retaining wall the pressure is zero at the surface, and increases uniformly with the depth; if the back of the wall is plane, the pressure at all points is parallel. Hence the resultant pressure on any portion of the wall, from the top down, is applied at two-thirds of the distance down from the top.

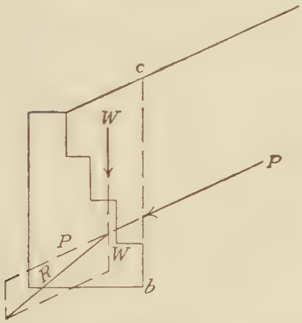


FIG. 212.

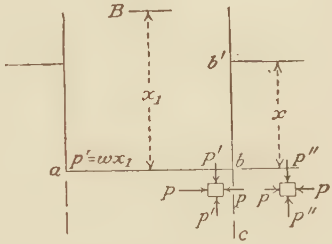


FIG. 213.

The above formulae give the intensity at any point of a vertical plane. The total pressure on a strip 1 foot wide from the surface down to a vertical depth h will be

$$P = \frac{wh^2}{2} \cos \alpha \frac{\cos \alpha \mp \sqrt{\cos^2 \alpha - \cos^2 \varphi}}{\cos \alpha \pm \sqrt{\cos^2 \alpha - \cos^2 \varphi}} \tag{11a}$$

If the back of the wall is not vertical (Fig. 212), find first by this formula the pressure P on bc and compound it with the weight W of the earth between bc and the wall. This applies whatever the form of the back of the wall, and requires finding only the weight and center of gravity of a volume of earth 1 foot thick between bc and the wall. For retaining walls (active earth pressure) the upper signs in Eq. (11a) are used.

Passive earth pressure has to be considered principally in cases where the thrust of an arch is taken by a mass of earth, or where a weight itself

is so great that it is likely to force up the earth on either side of it. Thus, let a building B rest on the foundation ab (Fig. 213) and suppose the load to be uniformly distributed over ab. Reduce the weight of the building to an equivalent height of earth, and let x_1 be the reduced height; then the intensity of pressure on ab is p' or wx_1. Now on the plane $b'bc$ there is a horizontal thrust. The earth to the right of this plane sustains this thrust just as a retaining wall would, and the thrust at the foundation level is the active thrust due to the load wx_1, or

$$p = wx_1\frac{1 - \sin\varphi}{1 + \sin\varphi}$$

If this thrust is too large, it will force up the earth to the right of b'; hence it cannot be larger than the passive pressure due to the height x, or

$$wx\frac{1 + \sin\varphi}{1 - \sin\varphi}$$

Hence

$$p = wx_1\frac{1 - \sin}{1 + \sin} \leq wx\frac{1 + \sin\varphi}{1 - \sin\varphi}$$

$$\therefore x \geq x_1\left(\frac{1 - \sin\varphi}{1 + \sin\varphi}\right)^2 \tag{14}$$

or

$$x_1 \leq x\left(\frac{1 + \sin\varphi}{1 - \sin\varphi}\right)^2$$

or

$$\text{maximum } p' \leq wx\left(\frac{1 + \sin\varphi}{1 - \sin\varphi}\right)^2 \tag{15}$$

This is the maximum intensity of pressure possible on a foundation at a depth x, with angle of repose φ.

Similarly, the minimum pressure p' on a foundation will be given by the equation, for level surface,

$$\text{minimum } p' = wx\left(\frac{1 - \sin\varphi}{1 + \sin\varphi}\right)^2 \tag{15a}$$

To put this somewhat differently, to find the maximum p' consistent with given $p'' = wx$, p must first be made as large as possible in proportion to p'', or it must be $p''\frac{1 + \sin\varphi}{1 - \sin\varphi}$, and then p' must be made as large as possible in proportion to p, or

$$\text{maximum } p' = p\frac{1 + \sin\varphi}{1 - \sin\varphi} = p''\left(\frac{1 + \sin\varphi}{1 - \sin\varphi}\right)^2$$

and similarly for minimum p'.

It will be noticed that Rankine's theory makes no assumption regarding the angle which the pressure on the back of a wall makes with that back, but leaves this to be found. Other theories assume either that the pressure acts normal to the back, or makes an angle φ with that normal. There is no reason for assuming it to act always normal to the back. It is argued, however, that when the wall tips, the earth *must* slide on the back of the wall, that is, make an angle φ with normal, downward. This tangential component adds to the stability of the wall, and doubtless when the wall does tip, the pressure really acts in this direction. We shall further consider this matter, however, after studying the other theories of earth pressure.

The student should remember Eq. (11), and the fundamental principles. He will then be able to solve any problem of earth pressure.

12. Mohr's Theory.—This theory, first suggested by Prof. O. C. Mohr of Dresden, is founded on the same principles as Rankine's, but is graphical in character. It involves the "circle of stress" discussed in Chap. V of "Strength of Materials." It will first be given as presented by Mohr, and afterward its analogy to the circle of stress will be pointed out.

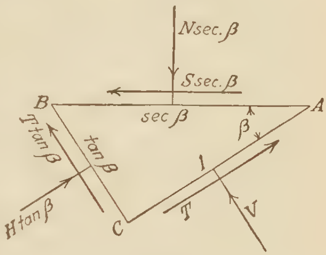


Fig. 214.

Consider an infinitely small triangular prism with a length perpendicular to the paper equal to unity, and although all the other dimensions are infinitely small, let us take AC (Fig. 214) parallel to the surface and equal to 1, BC perpendicular to AC, and the angle $CAB = \beta$. The areas of the sides of this prism are

$$AC = 1; \quad BC = \tan \beta; \quad BA = \sec \beta$$

If the stresses on these slides are resolved normally and tangentially, there are six forces acting on this prism, in the plane of the paper, as follows: on AC, V and T; on BC, $H \tan \beta$ and $T \tan \beta$; on AB, $N \sec \beta$ and $S \sec \beta$. V, T, H, N, and S are intensities.

These forces must be in equilibrium, and consequently they must form a closed polygon. Such a polygon is $EFGUVWE$ in Fig. 215, in which

$$EF = T;\ FG = V;\ GU = H \tan \beta;\ UV = N \sec \beta;\ VW = S \sec \beta;\ WE = T \tan \beta$$

Supposing these forces to be laid down correctly, we can investigate some of their properties which will enable us to draw the figure.

1. E and G are in the same vertical line; GE is the intensity of the pressure on AC, which must be vertical.

2. Produce GF and UV to K. Then $GK = GU \cot\beta = H \tan\beta \cot\beta = H$.

3. Produce EF and WV to I. Then $EI = EW \cot\beta = T \tan\beta \cot\beta = T = EF$.

It follows that the points E, F, G, K, and I are independent of β. If those points are given in the case of the plane AB, they will be the same for any other.

4. The angles IFK and IVK are each 90°. Hence a circle can be drawn through I, F, and K, which will contain all the points V for every plane; IK is the diameter and M the center of the circle.

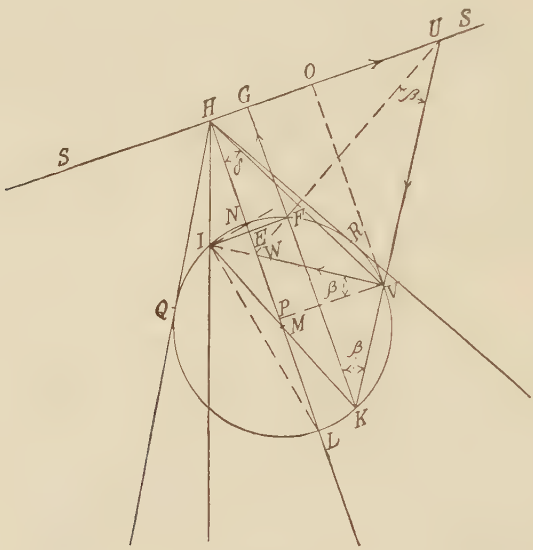


Fig. 215.

5. Draw VO parallel to FG. Then $VO = VU \cos\beta = N \sec\beta \cos\beta = N$.

6. Draw VP parallel to EF to meet EW produced. Then $VP = VW \cos\beta = S \sec\beta \cos\beta = S$.

7. It follows from the above that if once the circle IFK is drawn, then in order to find the intensity of the pressure on any plane AB, we have only to draw IV parallel to AB, and VO and VP are the normal and tangential intensities of stress on this plane.

Draw $HEWPML$ and HI. The latter is vertical, since $IE = EF$ and $EH = FG$. HV is the intensity of the stress on AB; the angle PHV is δ, or the angle which that stress makes with the normal to AB.

8. As the plane AB is changed or rotated about A, the point V moves around the circumference of the circle. The intensity of stress on AB has for its maximum value HL, on a plane parallel to IL, and for

its minimum value HN on a plane parallel to IN. On these planes the stress is normal; hence the stresses are the "principal stresses." The planes IL and IN are at right angles. Hence the *planes on which the principal stresses act are at right angles to each other*, as we know must be the case.

9. The angle δ has its maximum value for a plane parallel to IR, HR being tangent to the circle; as well as for a second plane parallel to IQ, HQ being also tangent to the circle.

10. The line WU is parallel to the actual stress on the plane AB, by construction, since UV and VW are the two components.

If the circle is drawn properly, the stress on any plane may be found. The angle $\delta = VHM$, can never exceed φ. The maximum value of δ for

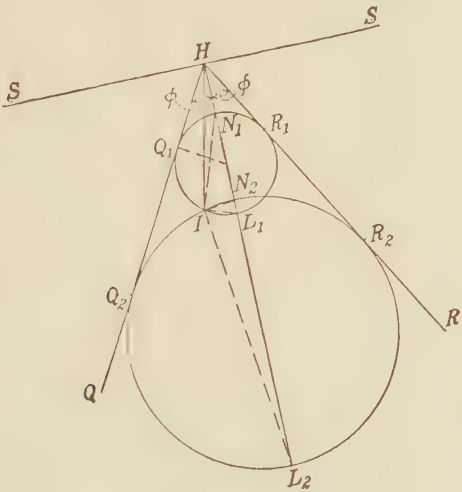


FIG. 216.

the circle is RHM. Hence if from H two lines are drawn (Fig. 216), making angles φ on either side of HE, the circle must lie within those lines HR and HQ. Any circle within those lines will correspond to a possible state of equilibrium. The limiting case, where the earth is just on the point of sliding, will be when the circle is tangent to HR and HQ.

Remembering that HI *represents* the intensity of stress on a plane parallel to the surface, we may assume that the distance HI, to some appropriate scale, represents the length of a column of earth having one unit cross-section, whose weight is the pressure on a plane parallel to the surface at the point considered; *i.e.*, the diagram gives stresses, represented by weights of columns of earth, at a point whose normal distance from the surface is HI (see Eq. 10). Two circles may be drawn through I tangent to HR and HQ (Fig. 216). These represent the two limiting cases of

active and passive pressure, when the earth is just on the point of sliding on some plane.

The larger circle represents the case where the maximum principal pressure HL (Fig. 215) is increased until the limiting condition is reached. This we call the *passive* earth pressure; it occurs when a thrust or pressure, as of an arch or building, is exerted on the earth until it is just ready to give way. The smaller circle represents the case where the minimum principal pressure HN is decreased until the limiting condition is reached. This we call the *active* earth pressure; it occurs when the thrust of a mass of earth is resisted, as by a retaining wall, and is, therefore, the most usual in the practice of the engineer. For the passive earth pressure the maximum pressure is HL_2 (Fig. 216) on a plane parallel to IL_2, and

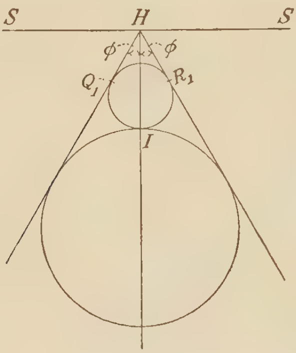


Fig. 217.

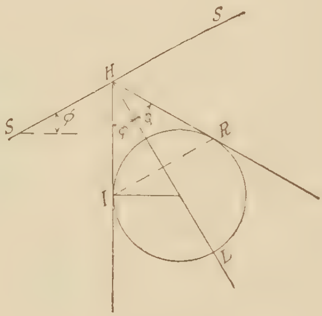


Fig. 218.

the minimum pressure HN_2 on a plane parallel to IN_2; for the active earth pressure the maximum pressure is HL_1 on a plane parallel to IL_1, and the minimum pressure HN_1, on a plane parallel to IN_1: *all at a point whose normal distance from the surface is HI.*

The maximum obliquity occurs on planes parallel to IR_2 and IQ_2 in the first case; IR_1 and IQ_1 in the second.

In the special case of a horizontal surface the force $FG = V$ is vertical; hence $EF = EI = 0$, and the two circles are tangent (Fig. 217). If the surface is inclined at the angle of repose φ, HQ is vertical and coincides with HI; hence only a single condition of equilibrium is possible (Fig. 218).

H is taken in the surface SS. The figure applies to a point at a normal distance HI from the surface, and the intensity of stress on any plane, say parallel to IV in Fig. 215, is HV measured to the scale of distance multiplied by the weight of a cubic unit of the material.

13. To Find the Total Pressure on Any Plane.—Let AB (Fig. 219) represent the back of a retaining wall, on which it is desired to find the active pressure. SS is the surface. From any point H in SS draw a normal HL, and two lines HR and HQ, making angles of φ with HL.

Draw *any circle* tangent to HR and HQ, and draw the vertical HI meeting the circle at the lower intersection I. The diagram then holds for the point G on the wall, obtained by drawing the arc ID with H as a center, and DG parallel to SS. Draw IV parallel to AB; then $HV \times w$ is the intensity of pressure at G. The total pressure on AB may be obtained as w times the area of the triangle ABC if HV is laid off $= GV'$ *normal* to AB, and AV' drawn to C, with BC parallel to GV'. (The triangle will not represent the total pressure unless GV' is normal to AB.) The area of the triangle is to the scale of distance for which the wall AB is drawn.

To find the real direction of the pressure, prolong IV till it meets HL at W; draw the diameter IK, and KV to its intersection with SS at U.

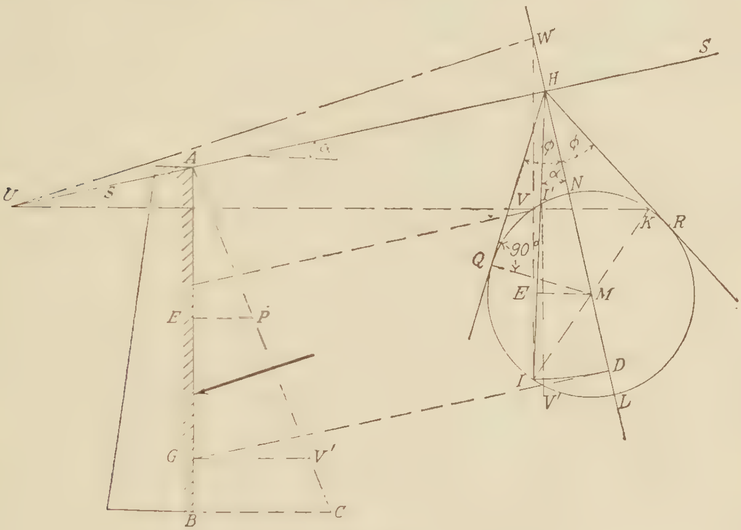


Fig. 219.

Then the stress acts parallel to UW (see Fig. 215). The intensity of pressure at any point of the wall E is $w \times EP$ if EP is drawn parallel to GV'.

If the back of the wall is not plane, but composed of a number of planes, the pressure on each may be found; or the pressure may be found on a vertical plane through B, and this combined with the weight of the prism of earth between it and the wall.

If the surface of the earth is loaded, reduce the load to the specific gravity of the earth, and let h be the height representing it (Fig. 220). The problem is therefore to find the pressure on a plane ab, as though the surface of the earth were $S'S'$. It will be represented by a trapezoid $aa'b'b$.

That the pressure on a plane varies directly as the distance below the surface is easily seen when we remember that the pressure on a plane

parallel to the surface follows that law, and that consequently the other pressures must do the same, inasmuch as Fig. 215 holds for any point, only the scale of forces changing. The resultant pressure on a plane wall extending to the surface is, therefore, applied two-thirds of the distance from the top to the bottom.

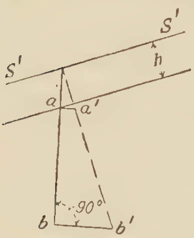


FIG. 220.

This method is direct and easy. The writer uses it a great deal, both for active and passive pressure. For the latter, the same circle may be used, but it then applies to a point at a normal distance HI' from the surface. Thus, to find the passive pressure on AB (Fig. 219) draw through I' a line parallel to AB, to V', and HV' will represent the intensity at a distance HI' normal from the surface.

14. Demonstration of Eq. 11.—Equation (11) may easily be demonstrated from the figures, and though it has been done in Chap. V, of "Strength of Materials," it may be repeated here.

In Fig. 219, the stress intensity p' on a plane parallel to the surface is represented by HI; and that on a vertical plane, p, by HI'. Start with the obvious identity, which can be seen if the radical is expanded.

$$p = \frac{p' + p}{2} - \sqrt{\frac{(p' + p)^2}{2} - p'p}$$

From the center of the circle M, draw a perpendicular to $HI'I$ at E. Then

$$\frac{p' + p}{2} = \frac{HI' + HI}{2} = HE = HM \cos \alpha$$

$$p'p = HI \cdot HI' = \overline{HQ}^2 = \overline{HM}^2 \cos^2 \varphi$$

Hence

$$p = HM(\cos \varphi - \sqrt{\cos^2 \alpha - \cos^2 \phi})$$

Similarly, there is the identity

$$p' = \frac{p' + p}{2} + \sqrt{\frac{(p' + p^2)}{2} - p'p}$$

$$= HM(\cos \alpha + \sqrt{\cos^2 \alpha - \cos^2 \varphi})$$

Hence

$$\frac{p}{p'} = \frac{\cos \alpha - \sqrt{\cos^2 \alpha - \cos^2 \varphi}}{\cos \alpha + \sqrt{\cos^2 \alpha - \cos^2 \varphi}} \tag{11}$$

This is the ratio of the smaller to the larger of two conjugate stresses whose common obliquity is α. For passive pressure the right-hand member would give p'/p.

15. Identity of Mohr's Method with the Circle of Stress.—By referring to Art. 20 of Chap. V of "Strength of Materials" the student will see

that the solution above given is merely a particular case of the circle of stress. It may be understood from the above, however, even if Chap. V has not been studied.

16. Limits of Applicability of Rankine's Method.—The two methods that have been explained are not applicable in all cases. For a granular material without cohesion and having the assumed angle of repose, with a plane upper surface of indefinite or *unlimited* extent, they give accurately the intensity that would exist in the mass on any plane at any point. But this is not necessarily the pressure on a retaining wall. If the top surface is not plane, the results are inapplicable, though often an irregular top surface may be replaced by a plane for which the results would be approximately the same. But even with a plane upper surface, we have seen that the pressure on any other plane may have various values compatible with equilibrium. The theories give the two limiting

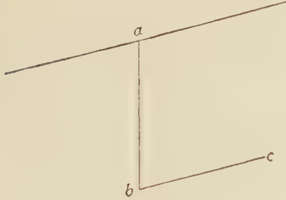


Fig. 221.

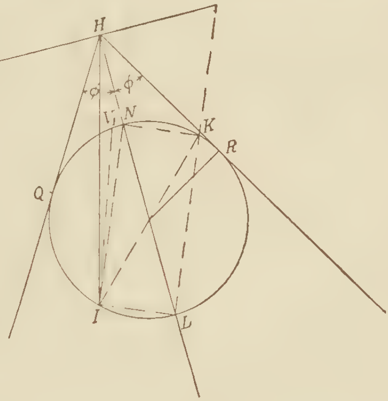


Fig. 222.

values, when the earth is just on the point of sliding, or the greatest and least values compatible with equilibrium. In the case of a retaining wall, it must be proportioned to withstand the least pressure that will keep the earth from sliding. But suppose the earth slopes *downward* from the back of the wall. Can it be supposed that the pressure on the plane *ab* (Fig. 221), when the earth to the left is removed and the wall *ab* is supporting the earth to the right, is the same as when the earth to the right is removed and the wall *ab* is supporting the earth to the left? In other words, is the pressure on a wall when the earth slopes downward the same as when it slopes upward at the same angle? Clearly not. Again, in the unlimited earth the pressure on a plane parallel to the surface *bc* would be the weight of earth above it; but if *bc* were the back of a wall lying upon the earth, this would not be true. A similar result would follow if the wall were inclined backward at a sufficient angle, though steeper than *bc*. Referring to Fig. 222, the maximum principal

pressure is IL on the plane IL, and is normal to that plane, or in the direction KL; and the minimum principal pressure is HN, on the plane IN, acting along KN, or having an upward component. On a plane IH the pressure is parallel to the surface. On some plane IV_1 between IH and IN the pressure will be horizontal; on all planes to the right of this through I there will be an upward component, and on all planes to the left there will be a downward component. It is held by some that the active pressure (which is what we are considering) *can have no upward component*, so that the theory would not be applicable if the wall slopes at an angle nearer the horizontal than IV_1. By others it is held that the theory is inapplicable *if the direction of the maximum principal pressure does not lie within the earth;* that is, it does not apply if the wall slopes backward at a steeper angle than IN.

If the theory is inapplicable, according to the criterion which may be chosen, then either the theory yet to be explained may be used, or the vertical upward component of the pressure may be neglected, though these procedures will lead to different results.

17. Surcharge.—When the earth back of a retaining wall is loaded by external loads, or when the surface slopes up from the wall, the wall is said to be *surcharged*. If the earth slopes, the pressure is given by the formulae which have been discussed. If there is a superimposed load on the earth surface, from buildings, railroad tracks, merchandise, or otherwise, suppose this load is p pounds per square foot. This may be supposed replaced by an equivalent load of earth having a vertical height $s = p/w$. Then the earth pressure on the wall is considered, by the previous theories, to be the same as would be exerted on the wall if the level of the earth were at a height s above its actual level, as in Fig. 223.

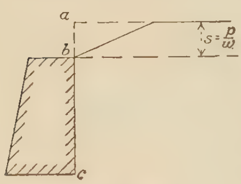


FIG. 223.

18. Direction of Thrust on Walls.—Before we proceed to the third theory of earth pressure, we must discuss the direction of the thrust. By Rankine's theory, this direction is parallel to the surface if the back of the wall is vertical, and it is determined from the principles of statics if the back is not vertical. Its direction depends upon the inclination of the back of the wall. Now if experiments are made on earth pressure, by measuring the thrust that will just correspond to the *tipping* of walls of different heights, it is found that Rankine's theory gives pressures much greater, sometimes double, the actual pressures. This is explained mainly by cohesion. The wall cannot tip unless the earth slides down on its back surface, unless that back surface is quite flat, in which case the earth may be ruptured along a plane passing through the foot and more nearly vertical than the back of the wall, and there will be sliding of earth along this plane instead of along the back of the wall, the earth

in front of this plane going with the wall. This case, however, is rare, since the back is generally nearly vertical or stepped. As to the angle which the pressure makes with the normal to the back of the wall, when tipping, it is either that corresponding to sliding earth on wall *or* of earth on earth, whichever is the lesser of the two. For if the coefficient of friction of earth on earth is less than that of earth on wall, a thin layer of earth will adhere to the wall, and the rest of the earth will slide on that. Experiments, therefore, will not agree with Rankine's theory, and it is not to be expected that they will. At the same time, in the case of an actual wall, which does *not* tip, it seems reasonable to suppose that the pressures would act as in Rankine's theory if the material were granular. If the wall is not strong enough to sustain this pressure, it will tend to tip, but this will introduce at once the tangential component along the back, and the wall may stand nevertheless. The fact, then, that a wall would theoretically tip under the pressure given by Rankine's theory, does not prove that it will tip in practice. But a wall should not have so small a margin of safety; it should be designed to have no tension at any point, or with the line of resistance inside the middle third. In this case it would generally sustain Rankine's thrust without tipping or tending to tip (except so far as the compression of the masonry or foundation causes some rotation, which, however, takes place at first, and is then constant, so that it has no effect); and it is therefore fair to suppose that the pressures may exist as Rankine's theory gives them (except for cohesion). At any rate, a wall designed by Rankine's theory will be safe, and on the safe side. If a wall is designed to take account of the tangential force on the back due to slipping of the earth, and if it is made so that under this theory it will just stand, then it will have no margin of stability whatever; if it is designed under this theory to have no tension on any joint, then it will really stand, but the actual pressure on the back may be greater than assumed and actually there may be tension on some joints. Tension on a masonry joint may result in disintegration of the mortar, entrance of water, and destructive effects due to frost.

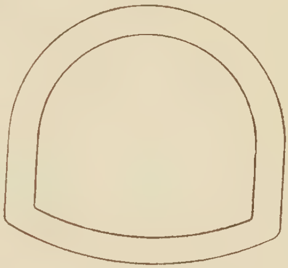


Fig. 224.

When we speak of the wall "just tipping," we mean not actually tipping, but being *on the point of tipping*. There may be no actual motion, but only "impending" motion.

As to the direction of thrust, consider the case of a tunnel arch (Fig. 224). If on each side the thrust be assumed to act downward at the angle φ with the normal, then when we reach the crown from either side, we shall have two different directions for the thrust. This seems to show that the assumption that the thrust always makes the angle φ with the

normal to the back is not correct, in this case at least. Nevertheless, the theory we are about to give makes this supposition.

19. Theory of the Sliding Prism, or Old Theory (Fig. 225).—Let ab be the back of a wall, and let the mass of earth behind it have any shape. Draw the vertical ab' and af at the angle of repose. Let $a1$ be any plane through a. Then the forces acting on the prism $ab1$ are (1) its weight W, which can be completely found; (2) the earth pressure E from the wall on the earth, which is assumed to make the angle with the normal equal to the angle of friction of earth on earth (φ) or earth on wall (φ_1), whichever is the smaller; (3) the pressure R on $a1$. These three forces must be in equilibrium. Generally φ is taken equal to φ_1. The only condition is that R must not make an angle with the normal to $a1$ greater than φ, and for the limiting condition of active pressure it must make just the angle φ and must be directed as shown. The magnitude and line of action of W can be found. The problem then is to find the value of E which will

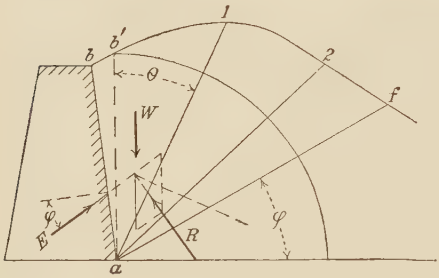


Fig. 225.

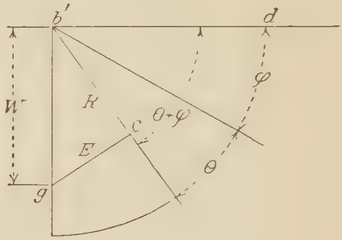


Fig. 226.

make the direction of R as shown. In other words, in the triangle of forces we have the magnitude and direction and line of action of W, and the directions of E and R. R must make the angle φ with the normal to $a1$ but E might be assumed in any desired direction, though by this theory it is assumed to act at an angle φ with the back of the wall. For active pressure the earth would be sliding downward on $a1$; for passive pressure it would be sliding upward. If θ is the angle of $a1$ with the vertical, then R makes an angle $\theta + \varphi$ with the horizontal, for active pressure. In Fig. 225 describe an arc with a as a center and radius ab', and in Fig. 226 describe an arc with b' as a center and a radius $b'd = ab'$ in Fig. 225. Then in Fig. 226 the angles may be laid off by making the chords equal to those in Fig. 225. Lay off $b'g$ to represent W, draw gc parallel to the assumed thrust E, and since $b'c$ is parallel to R, gc represents E. In other words, E must be equal to or greater than gc, in order that the earth may not slide down on the plane $a1$. The same must be true for any other plane $a2$; hence if we find E for every plane passing through a, the actual pressure on the wall, which we may call E_0, must be *at least* equal to the

maximum of these values. This method applies to any form of surface or to any surcharge if reduced to an equivalent depth of earth.

A plane surface of rupture has been assumed passing through a. If there is any cohesion, the surface will be a curve, as shown in Art. 2. The present theory gives no means of treating the case of a curved surface, since the pressure on it would make different angles at different points.

20. Figure 227 shows an example of the construction for finding E_0. Extend the back of the wall to meet the surface at C, and draw Ag vertically. To find the weights of the prisms between AC and the lines $A1$,

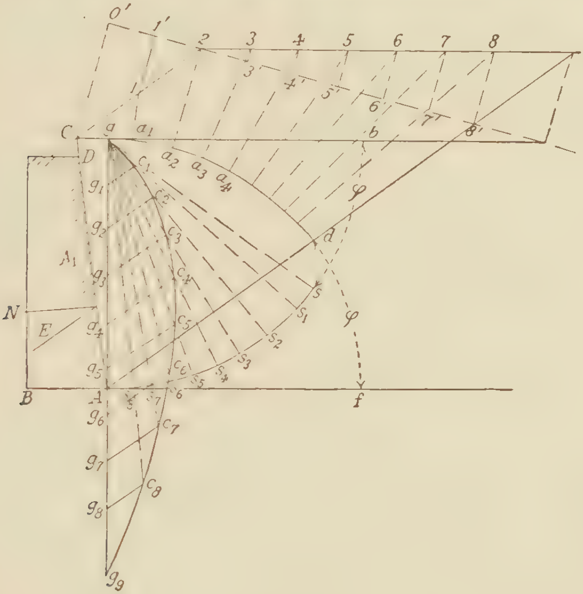


FIG. 227.

$A2$, etc., reduce them all to equivalent prisms having a common base $A2$, by drawing $0'8'$ perpendicular to $A2$, and from C, 1, 3, etc. lines parallel to $A2$. Then the prism $AC2$ has an area $\frac{1}{2} \times A2 \times 0'2$, or the weight of any prism like $AC6$ is proportional to $0'6'$. Lay off $gg_1 = 0'1'$, $gg_2 = 0'2$, and so on. Draw the arc gf with center at A and radius Ag; and the arc Ab with the same radius and center at g. Lay off the angle $bgs = \varphi$ by making $bs = fd$. Lay off the arcs $ss_1 = ga_1$, $ss_2 = ga_2$, etc. Then the line gs_2, for instance, is the direction of a force on $A2$ which makes an angle φ downward from the normal to $A2$. Draw through g_1, g_2, etc. lines parallel to the assumed direction of E, to meet the lines gs_1, gs_2, etc., at c_1, c_2, etc. Draw a curve through c_1, c_2, etc. The maximum ordinate y to this curve parallel to E is the maximum value E_0. To find it in pounds,

$$E_0 = \frac{1}{2} w \cdot A2 \cdot y$$

in which w is the weight of a cubic foot of the earth. The plane on which E_0 acts is the one corresponding to y, and it is the plane on which the earth has the greatest tendency to slide. The wedge between it and the wall is the "wedge of greatest thrust."

21. Point of Application of E_0.—In the above figure, the thrust on the portion CA_1 of the wall would not be proportional to the square of CA_1, for the diagram, if A_1 were the base instead of A, would not be similar to the diagram above; in other words, the intensity would not vary with the depth, and the resultant pressure on the wall would not be applied at two-thirds the depth. If the ground were plane from D, however, the diagram for A_1 as base would be similar to that for A as base; and as the weights of the prisms would vary as the squares of similar sides, the total pressure from the top down to any point A_1 would vary as $\overline{CA}_1^2$, and the resultant would act two-thirds of the way down.

The passive earth pressure may be found in a similar manner, in this case the direction of R being inclined on the other side of the normal to the plane of rupture, or it would make an angle with the horizontal of $\theta - \varphi$ instead of $\theta + \varphi$. The points g_1, etc. would be unchanged, but the direction of E would be on the other side of the normal to the back of the wall since the earth would be sliding upward on it. The value of ge to be used would be the *minimum* value.

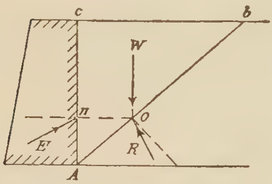


FIG. 228.

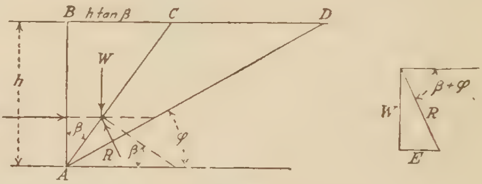


FIG. 229.

22. Error in the Theory of the Sliding Prism.—This theory supposes the pressure on the wall, and that on the plane of rupture, to make angles of φ with the normal. The weight acts vertically and these three forces *must* be in equilibrium unless the wall is actually tipping. W alone is completely given, and also the directions of the two pressures, which are found by merely drawing the triangle of forces. But these three forces *must* meet in a point. If the surface is plane from D (Fig. 227), the theory necessarily leads to the conclusion that E and R act at one-third the distance up from A, and if drawn through these points in the directions assumed, they will generally not meet W in the same point, as shown in Fig. 228, where W and R meet on Ab, and E would have to be horizontal to go through this point, which is the direction in which it does act by Rankine's theory.

This error, which underlies the whole theory, and which renders its results entirely fallacious, is not generally mentioned in the books. It

shows the importance, before developing a theory, of seeing that all the necessary conditions are fulfilled.

If the direction of E is taken as it really is according to Rankine's theory, the results of the two theories will agree perfectly. In such case, there is no objection to considering different prisms of earth, and finding the value of E which will just prevent any of them from sliding down. Thus, if the wall is vertical and the surface horizontal (Fig. 229), the weight W will meet AC at the point of application of R, and E must be horizontal, as it is by Rankine's theory. If the earth is just at the point of sliding, R acts at an angle $\beta + \varphi$ with the horizontal.

$$W = \frac{wh^2}{2} \tan \beta$$

$$E = W \tan (90 - \beta - \varphi) = \frac{wh^2}{2} \tan \beta \tan (90 - \beta - \varphi)$$

For maximum value of E,

$$\frac{dE}{d\beta} = 0 = \frac{wh^2}{2} [\tan (90 - \beta - \varphi) \sec^2 \beta - \tan \beta \sec^2 (90 - \beta - \varphi)]$$

which leads to

$$\tan (90 - \beta - \varphi) \sec^2 \beta = \tan \beta \sec^2 (90 - \beta - \varphi)$$

which is fulfilled by

$$\beta = 90 - \beta - \varphi$$

so that *in this case the plane of rupture bisects the angle BAD.*

23. Comparison of Theories.—Rankine's and Mohr's theories are identical, except that one is analytical and the other is graphical. They both suppose a granular mass of indefinite extent, without cohesion, and the direction of E is deduced without reference to the tipping of the wall. The theory of the sliding prism supposes a granular mass, but not necessarily of indefinite extent, though without cohesion; and it further supposes a plane surface of rupture, and the direction of E making the angle φ with the normal to the wall. Since this theory agrees with Rankine's in the cases where the direction of E is the same, it follows that the assumption of a plane surface of rupture is not a matter of difference between the theories. Evidently, then, the entire difference between the two theories is in the direction of the thrust E. We have seen that according to the last theory, the direction assumed is incompatible with equilibrium, since E, W, and R do not meet in a point. This theory is therefore erroneous. It will, however, often agree best with experiments, by which the pressure is measured when a wall begins to tip, in which case there is not, of course, a condition of equilibrium. The Rankine theory will generally give a larger pressure on a wall than the old theory, because, before a wall can actually tip, the friction will be brought into play. This is another illustration of the fact that a structure will stand if it is possible for it to do so.

24. Center of Gravity of a Trapezoid.—In problems of earth pressure, the pressure intensity varies as the depth (Fig. 230). At a depth of h_1 it is a constant times h_1, or ch_1; and at a depth h_2 it is ch_2. The pressure on the rectangular surface ab, one unit wide perpendicular to the paper, will act at the c.g. of the area $abdc$. The height of the point of application above the base x will be found from the following equation, if $abdc$ be divided into two triangles.

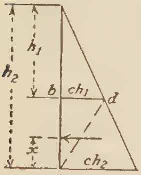


FIG. 230.

$$\frac{ch_1(h_2 - h_1)^2}{3} + \frac{ch_2(h_2 - h_1)^2}{6} = \frac{cx(h_2 - h_1)(h_1 + h_2)}{2}$$

from which

$$x = \frac{h_2}{3} - \frac{2}{3}\frac{h_1^2}{h_1 + h_2} \tag{16}$$

This formula need not be used. The pressure may always be taken as the resultant of two forces, one being $ch_1(h_2 - h_1)$, acting at the center of ab, and the other being $\frac{c(h_2 - h_1)^2}{2}$, acting at one-third of ab above b.

25. Graphical Solution by Theory of the Sliding Prism.—Consider a slice 1 foot thick perpendicular to the paper (Fig. 231). AB is the

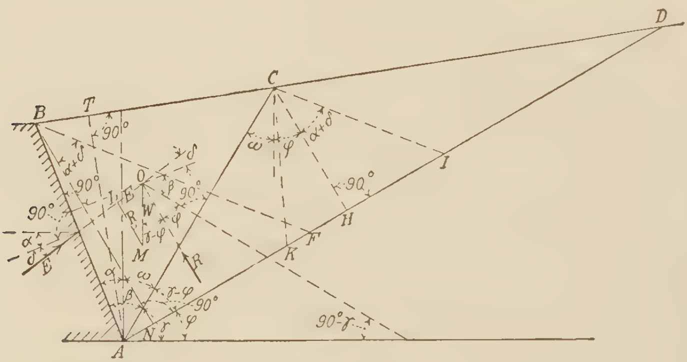


FIG. 231.

back of the wall, here shown as inclining forward. Let AC be any plane of rupture, making an angle β with AB, and γ with the horizontal. AD is a slope making the angle of repose with the horizontal.

Figure 231 is drawn for the earth surface plane (Fig. 232) for two plane surfaces, $B'GD$; often GD will be horizontal, as GD'. The same construction applies to both cases if the triangle $AB'G$ in Fig. 232 is replaced by the equal triangle ABG, $B'B$ being parallel to AG.

W is the weight of the prism ABC, equal to w, the weight of a cubic unit of earth, multiplied by the area of the triangle ABC. The forces in equilibrium are W, the pressure E from the back of the wall on the

earth, and the pressure R from the plane AC on the earth. We assume E to make any angle δ with the normal to the back of the wall, δ being less than φ; and we assume that R makes the angle φ with the normal to AC, since the earth is supposed just on the point of sliding on that plane.

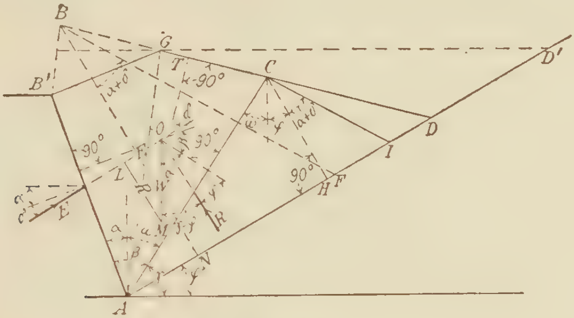


FIG. 232.

Lay off $W = OM$, and draw OL parallel to E, and ML parallel to R. Then OL represents E, and is to be found. From the triangle OML,

$$\frac{E}{W} = \frac{\sin OML}{\sin OLM} = \frac{\sin (\gamma - \varphi)}{\sin (\beta + \delta + \varphi)} \tag{17}$$

We must now suppose AC to change, and must find the maximum value of E, which is the pressure that the wall must exert in order that no sliding may occur on any plane. Draw CI so that

$$ACI = \beta + \delta + \varphi = \alpha + \omega + \delta + \varphi \tag{18}$$

This is easily done by drawing CH perpendicular to AD; the angle between CH and the vertical is φ, and the angle between AC and the vertical is w; so that we have only to lay off $\alpha + \delta$ which is the angle between E and the horizontal, above CH. Draw AT normal to BC. Then

$$W = \frac{1}{2} w \cdot AT \cdot BC$$

and from Eqs. (17) and (18)

$$E = \frac{1}{2} w \cdot AT \cdot BC \, \frac{CI}{AI}$$

for the angle $CAI = \gamma - \varphi$; and $ACI = \beta + \delta + \varphi$.
Draw BF parallel to CI. Then the triangles BFD and CID are similar, and

$$\frac{BC}{BD} = \frac{FI}{FD}; \; BC = BD\,\frac{FI}{FD}$$

$$\frac{CI}{ID} = \frac{BF}{FD}; \; CI = ID\,\frac{BF}{FD}$$

Hence

$$E = \frac{1}{2} w \cdot \frac{AT \cdot BD \cdot BF}{FD^2} \cdot \frac{FI \cdot ID}{AI}$$

The point F is independent of C; for the angles AIC and AFB are equal and

$$\begin{aligned} AIC &= 180° - \gamma + \varphi - \omega - \varphi - \alpha - \delta \\ &= 180° - (\gamma + \omega) - \alpha - \delta = 90° - \alpha - \delta \end{aligned}$$

and is thus independent of γ. Or, more simply, CI makes the angle $\varphi + \alpha + \delta$ with the vertical, which does not depend upon γ. As AC is varied in position, therefore, the only part of this expression that changes is $\frac{FI \cdot ID}{AI}$. Call $AI = x$, a variable; $AD = a$; $AF = b$; a and b being invariable. Then

$$\frac{FI \cdot ID}{AI} = \frac{(x - b)(a - x)}{x} = a + b - \frac{ab}{x} - x$$

This variable quantity has its maximum value when $x = \sqrt{ab}$, and its maximum value is

$$\text{maximum } \frac{FI \cdot ID}{AI} = a + b - 2\sqrt{ab} = \frac{(a - \sqrt{ab})^2}{a}$$

Hence

$$\text{maximum } E = E_0 = \frac{1}{2} w \cdot \frac{AT \cdot BD \cdot BF}{FD^2} \frac{(a - \sqrt{ab})^2}{a}$$

Draw BN perpendicular to AD. Then

$$AT \cdot BD = AD \cdot BN = AD \cdot BF \cos FBN,$$

since each of these expressions is twice the area of the triangle ABD. Hence

$$E_0 = \frac{1}{2} w \cdot \cos FBN \cdot \frac{BF^2}{FD^2} (a - \sqrt{ab})^2$$

Now

$$\frac{BF}{FD} = \frac{CI}{ID}$$

and $a - \sqrt{ab} = a - x = ID$ since, for maximum E, $x = \sqrt{ab}$; and therefore

$$E_0 = \frac{1}{2} w \cdot \cos (\alpha + \delta) \overline{CI}^2 \text{ (wall leaning forward)}$$

since

$$FBN = ICH = \alpha + \delta.$$

This is the value of E_0 if the wall leans forward, as shown. If it leans backward, α becomes negative, and

$$E_0 = \frac{1}{2} w \cdot \cos (\delta - \alpha) \overline{CI}^2 \text{ (wall leaning backward)}$$

Lay off $IK = IC$; then, since $CI \cos(\alpha + \delta) = CH$,

$$E_0 = w \text{ (area of triangle } ICK) \tag{19}$$

if AC is the true plane of rupture, or plane on which there is the greatest tendency to slide. Now bearing in mind that

$$x = AI = \sqrt{ab} = \sqrt{AD \cdot AF}$$

and

$$NBF = \delta + \alpha \text{ (if wall leans forward)}$$

or

$$= \delta - \alpha \text{ (if wall leans backward)}$$

the plane of rupture may be easily constructed as follows (Fig. 233).

From B draw BN perpendicular to AD, and draw BF making the angle $\alpha + \delta$ with BN. The angle δ may be assumed to have any value

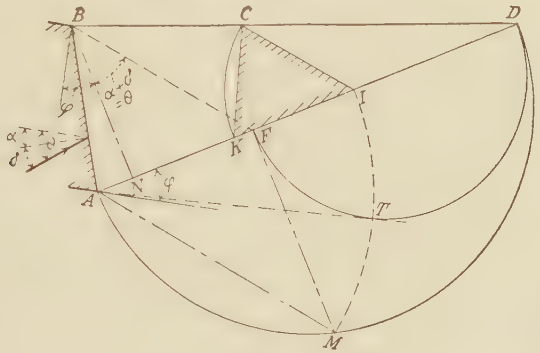


FIG. 233.

less than φ, or it may be equal to φ. Draw a semicircle with AD as a diameter. Draw FM perpendicular to AD. Lay off $AI = AM = X = \sqrt{AD \cdot AF}$. Draw IC parallel to FB, and lay off $IK = IC$; then

$$E_0 = w \text{ (area of triangle } ICK) \tag{20}$$

BF makes an angle with the vertical equal to $\varphi + \theta$ if θ is the angle that E makes below the horizontal. This is the easiest way to find F. Having found F, since $\overline{AI}^2 = AD \cdot AF$, I may be found by drawing the semicircle with FD as a diameter, drawing the tangent AT to this semicircle, and making $AI = AT$. This leads to the following rule:

From B draw BF making the angle $\varphi + \theta$ with the vertical. Draw a semicircle with FD as a diameter, the tangent AT to this semicircle, make $AI = AT$; draw IC parallel to BF, and make $IK = IC$. Then

$$E_0 = w \text{ (area of triangle } ICK).$$

Formulae may be given for this method, but they are cumbersome and unwieldy. The above construction is all that is necessary.

26. Historical.—The theory of earth pressure dates from about 1687, when the French General Vauban gave rules for the thickness of retaining walls. It is doubtful, however, whether these were founded on theory. In 1691 Bullet assumed that a prism of earth at 45° would slide, and resolved its weight into two components, one normal and one tangential to the slope, the latter being taken as the thrust on the wall. Other continental engineers used a similar method, some assuming the sliding plane prism to be the slope of repose. As late as 1858, Hoffman calculated elaborate tables based on this theory.

The next step, by Belidor (1729), was to resolve the weight of the sliding prism as before, but to deduct from the tangential component a certain amount for friction.

Coulomb (1773) was the first to assume the pressure on the surface of rupture to make the angle φ with the normal, and determine the thrust from the "wedge of maximum pressure." He considered E normal to the back of the wall.

Poncelet (1840) was the first to assume the direction of E inclined to the normal to the wall, so that his method is practically that which has been given above. This theory was elaborated by others, in analytical and graphical form (Scheffler, 1857; Culman, 1866; Rebhann, 1871).

Rankine (1856) was the first to give a theory based on the principles of internal stress. It was given independently by Winkler (1860). Mohr (1871) gave the graphical theory above explained, and was the first to call attention to the fact that according to the old theories the forces do not meet in a point.

27. Pressure of Saturated Earth.—The material sustained by a wall is sometimes saturated with water, in which case the pressure may be very different from that of dry material. There may be two cases of this kind:

1. The material may be mixed or more or less saturated with water, without there being a real hydrostatic pressure such as would exist if the water level stood above the surface of the material. In other words, the material may be holding water by its capillary action. In this case, the angle of repose will be altered and also the weight per cubic foot. Damp sand weighs more than dry sand, and has a greater angle of repose up to a certain percentage of moisture, and a smaller angle above that percentage. For soft mud the angle of repose may be zero.

This case should be treated as one of pure earth pressure, taking w as the weight of a cubic unit of the actual saturated material, and φ as its actual angle of repose, both of which values should be measured if the case is important, and if the conditions are permanent. If $\varphi = 0$, as for soft mud, it will be a case of hydrostatic pressure of a liquid having a density greater than water. If the dry material is just saturated, but with no excess of water, its weight per cubic foot will be that of the dry

material plus the weight of a volume of water just sufficient to fill the voids. If the volume of voids is 40 per cent and the dry material weighs 110 pounds, the saturated material will weigh (for fresh water) $110 + 62.5 \times 0.4 = 135$ pounds per cubic foot, and the angle of repose would probably be less than for dry material. If there were more water, the particles of earth would be farther apart, and the weight would be less than 135 pounds, but the angle of repose would be decreased; this case, however, would probably come under the next heading, in which there would be a true hydrostatic pressure.

2. There may be real hydrostatic pressure, such as would exist if the water stood above the level of the earth, or even if it did not stand above the surface but only up to a certain level, provided the condition were actually *static*, that is, *if the water were not in motion*, and if the water is not merely held by capillary action. In a case of this kind the material may not become liquid at all, but may have a considerable angle of repose, though probably less than if dry, if only on account of the lubricating action of the water. This case would occur with material like sand, gravel, or broken stone under water. A mass of crushed stone under water would certainly have a considerable angle of repose, and yet its void spaces would be completely filled and exposed to hydrostatic pressure. The same would occur in the case of a dock wall with water in front, if the material under and back of the wall were porous enough to allow the water to percolate through it and stand behind the wall at the same level as in front.

In cases of this kind the pressure on the wall would consist of two parts: (1) the hydrostatic pressure of the water, and (2) the pressure of the earthy material.

1. The water filling the voids will exert a hydrostatic pressure just as though the earth were not present. A part of the surface of the wall will be directly in contact with the water, which presses against it. Another part of the surface may be in contact with the earthy material. It is probable that a film of water will separate each earthy particle from the wall, but even if there is not, and the particle is directly in contact with the wall, there will be hydrostatic pressure on the back of this particle, or on the other particles which press against it, and this pressure is transmitted to the wall. It is thus obvious that there will be the full hydrostatic pressure on the wall below the level at which the water stands. If the water reaches the back of the wall by percolating under and around it, it is possible that, near the level of the top of the water, the water in the earth may be held by capillarity rather than hydrostatically, and the full hydrostatic pressure may not be exerted, but this merely means that the true *hydrostatic* condition is not reached, which is assumed. In the true static condition, the full hydrostatic pressure must be taken.

2. There is some difference of opinion as to whether the earth itself exerts a pressure, in addition to the hydrostatic pressure. It is obvious to the writer that it does. The pressure of the earth in any case, if it has weight which is not counterbalanced by buoyancy, arises from a sort of wedge action, by which the weight of a particle causes a pressure against adjoining particles, which is transmitted through the mass and to the wall; and this will be as true in water as in air. Under water, of course, each solid particle of earth loses a weight equal to the weight of its volume of water. Hence, if its weight dry is 110 pounds per cubic foot and there are 40 per cent of voids, then a cubic foot of earth, under water, will weigh $110 - 0.6 \times 62.5 = 72.5$ pounds (not allowing for the water which fills the voids, which is taken care of in the hydrostatic pressure). Its angle of repose under water will be less than in air, and should be determined if the problem is important enough. The pressure of the earth will be that due to this material, with a surcharge due to the weight of dry material above the water level.

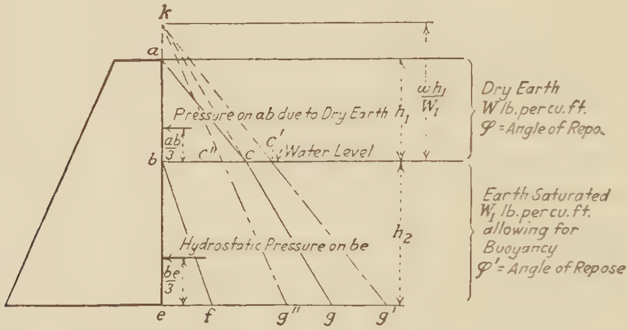


FIG. 234.

According to the above point of view, if ae (Fig. 234) is the back of a wall, with static water level at b, whether due to percolation of water from in front or from level of the ground water behind (if static and not flowing), the pressure on the back will be made up of the following parts:

1. The triangle abc, representing the earth pressure of the dry material above b, with weight w per cubic foot. For level earth surface,

$$bc = wh_1 \frac{1 - \sin \varphi}{1 + \sin \varphi}$$

2. The triangle bef, representing the hydrostatic pressure on be, with weight of 62.5 pounds per cubic foot

$$ef = 62.5h_2$$

3. The trapezoid $bcge$ (or $bc'g'e$ or $bc''g''e$), representing the earth pressure on be, with surcharge, and weight allowing for buoyancy as

above, and with angle of repose of material under water. The height h_1 represents the surcharge on the part below b. Reducing this to equivalent height of earth of unit weight w_1 and angle of repose φ', we have

$$bk = \frac{wh_1}{w_1}$$

$$bc \text{ (or } bc' \text{ or } bc'') = w_1 \frac{wh_1}{w_1} \frac{1 - \sin \varphi'}{1 + \sin \varphi'}$$

If $\varphi' = \varphi$, the pressure intensity at b will be bc, the same as on ab. If $\varphi' > \varphi$, the pressure at b, for the part below b, will be greater than bc, or it will be bc'; if $\varphi' > \varphi$, it will be less than bc, or it will be bc''. The pressure, from the earth, on be will be the trapezoid $bc'ge$ or $bc''ge$, the inclined line going through K. The total pressure on the wall will be the sum of the pressures represented by the areas abc, bef, and $bc'ge$ or $bc''ge$.

The reader may perhaps satisfy himself as to the correctness of the above by considering what the situation would be if the buoyancy were just equal to the weight of the material, so that the material below b should have no resultant weight, tending neither to sink nor to rise. In that case, the earth below b would itself exert no pressure, but the pressure bc of the surcharge would be transmitted through it, and the trapezoid $bcge$ would become a rectangle with height bc. In this case φ' is indeterminate, or may be φ.

It is claimed by some that in saturated earth there is no earth pressure to be added to the hydrostatic pressure, because in pneumatic caissons in New York it has been found that the internal pressure necessary for working is almost exactly equal to the hydrostatic pressure, without any addition for earth pressure. This, however, is inconclusive. The pressure in the working chamber of a caisson is exerted on the bottom surface which is being excavated as the caisson sinks. A pressure equal to the hydrostatic pressure is all that would be necessary to keep the water from coming up, for this hydrostatic pressure would be exerted, not merely on a vertical surface of the outside of the caisson, but equally on a horizontal surface such as the floor of the working chamber, and all that is necessary to counteract it is an equal air pressure. The vertical pressure, however, necessary to counteract and prevent the rising of the earth, if the horizontal earth pressure on the outside of the caisson is p_h, would be, as shown in Art. 11,

$p_v = p_h \frac{1 - \sin \varphi'}{1 - \sin \varphi'}$, for horizontal top surface, or considerably less than p_h, depending upon φ'.

Furthermore, if water is flowing into the working chamber and being pumped out, as is no doubt the case, the case is one of *hydraulic pressure*, and not hydrostatic pressure.

28. Hydraulic Pressure vs. Hydrostatic Pressure.—The previous discussion has assumed that the water pressure was *hydrostatic*, that is, that the water was not moving. The case is very different indeed if it is in motion.

If water percolates through the material under a dam, to find its way down the valley, the case is one of hydraulics and not of hydrostatics, and is somewhat similar to the flow in pipes. The water flows through the very fine interstices of the sand or gravel, or through the crevices in the rock, changing its direction innumerable times, and losing head by these turns as well as by friction. The loss of head in a pipe increases as the diameter decreases, and for small diameters is very large and may be a large proportion of the total head. Under these circumstances, the upward pressure that the water will exert on the base of the dam will be only a fraction—perhaps a small fraction—of the hydrostatic head. It is not unusual for engineers to assume it to vary from one-half the hydrostatic head at the heel, or upstream edge to zero at the toe, but it will actually vary with circumstances, such as the rapidity of flow, which depends upon the size of the channels through which it flows, and the number of times its direction is changed. The character of the material, its porosity, and uniformity will affect the result. The same is true of the pressure on the back of a wall below the bed, if the material behind it is saturated and the water is in motion.

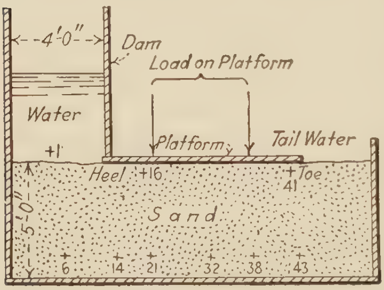


Fig. 235.

For these reasons the facts stated regarding the air pressure in caissons are inconclusive. The water is doubtless in motion.

The pressure at any point under a dam would be shown by the height to which the water would rise in an open pipe inserted at that point, or by the pressure on a gage, if one could be placed there. An attempt was made to do this in experiments described by J. B. T. Colman, in his paper on "The Action of Water under Dams."[1] A box (Fig. 235) was partly filled with sand, a watertight dam placed across it extending down only to the sand; and below the dam a platform was laid on top of the sand. A head of water up to about 5 feet was allowed to act upon the dam, and the water percolated through the sand and flowed away. The pressures were measured at various points, some just under the platform (16 and 41) and others at various (undefined) points in the sand. The pressure at point 1 showed the true head. With a natural sand, "varying from large pebbles to fine grains and containing very little clay," with a

[1] *Trans. Am. Soc. C. E.*, vol. LXXX, p. 421, December, 1916.

uniformity coefficient[1] 8, with no piling at toe or heel, the pressure at the heel was 0.7 the head, and at the toe 0.075 the head, the width of base being 8 feet 3 inches. With the larger pebbles screened out, and the uniformity coefficient 5.4, the pressure at heel was 0.76 and at the toe 0.16 the head. The greater the uniformity coefficient, the less the uniformity and the less the porosity, and consequently the greater the loss of head and the less the pressure below the heel, though the pressure at the heel should not vary much. The above results are consistent in this respect.

Sheet piling at the heel, if tight, should reduce the pressure over the entire base, and yet make it more uniform. The water must percolate down the upper face of the piling, and up the lower face, to reach the heel.

Piling at the toe, by confining the water above it, should *increase* the pressure on the base and make it more uniform. If there is sheeting under the toe, it should *not* be tight; if under the heel, it should be tight, if possible, as a slight leakage may destroy the effect of reducing the pressure. Tight sheeting is impossible of attainment in practice.

The paper referred to will repay study, and particularly the discussion thereon. There are many discrepancies in the experimental results, and the formulae should not be relied upon. In some cases a higher head gave lower pressures than a smaller head; and piling 3 feet long under the heel gave larger pressures than shorter piling; the longer piling must have leaked at the top, or there must have been other disturbing factors. It all illustrates the truth that many experimental results fail to disclose or to agree with general laws, and are applicable only under circumstances identical with those of the experiments. As shown in the last chapter of "Strength of Materials," experimental results should not be accepted blindly, and should be discarded if they disagree with reasonable *a priori* principles.

Hydrostatic or hydraulic pressure on dams will be further considered in the chapter on that subject.

29. Earth Pressure on Foundations. Foundations on Clay.—Equation (15), which gives the maximum pressure on a foundation at a given depth such that it will not force the earth outside up, has been much used. This equation will give a great variation in the pressure for a small variation in φ. Thus, the pressure when $\varphi = 45°$ is 3.77 times the pressure when $\varphi = 30°$. The value of φ is often very uncertain, and it is dependent upon cohesion. In clay there is considerable cohesion, and when wet, the material is plastic. This material illustrates the uncertainty of the subject. A large proportion of the failures involving earth pressure have been in clay. Bell states:

> Taking all available records of works subject to earth pressure which had failed, it appeared that 70 to 80 per cent of them referred to works constructed in clay.

[1] If 90 per cent is less than a certain size A, and 10 per cent is less than a certain size B, by weight, the uniformity coefficient is A/B. The more uniform the grains are in size, the greater the porosity, for small grains will not help so much to fill voids between larger grains.

Clay is, therefore, a variable and treacherous material, and works constructed in it require not only a knowledge of theory, but, more especially, judgment and experience, without which theory may lead to disaster.

30. Pressure on Braced Trenches.—When a trench is excavated, especially if it is in a city street, it is generally in material which is possessed of considerably cohesion, or which may have been compacted by traffic. It is very different from the case of earth dumped behind a retaining wall. The cohesive earth, as already shown, can resist some tension, and a bank could overhang at the top. The gradation from an absolute granular material, with no cohesion, to a solid body like rock, is gradual. The cohesive earth, therefore, to the extent of the cohesion, can resist stress like a solid material. Back of the trench wall, therefore, there can be an arch action, as indicated in Fig. 236, between the wall and some plane as AC; and it may well be that the pressure on the bracing may be greatest at the top, especially if there is a load (surcharge) on the surface BC. The cohesion may increase with the depth, so that at the bottom there may be little pressure. The character and extent of this action is beyond the power of computation, and it is a matter calling for the exercise of common sense and experience (see also Arts. 31, 32, 47).

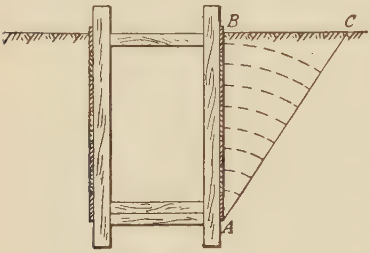


FIG. 236.

31. Uncertainty of Theory of Earth Pressure.—It is obvious that the theory of earth pressure necessarily involves many uncertainties. The theory is often criticised by practical engineers because it does not agree with their experience, and they seem to think that a theory that will so agree ought to be evolved. But this is impossible, and always will be, because of the uncertain character of the data. It is not even possible to define the term *earth* in a satisfactory way. There are variations in φ, in w, in cohesion, in fact, in every element of the problem. Moreover, the same material may change in character from day to day, owing to changes in moisture, etc.; and a given material may vary greatly from point to point, or at different depths. At depths below the reach of frost the condition may be very different from the condition near the surface. Earth in its natural undisturbed condition may be very different from that which has been excavated and backfilled.

How can accuracy be expected in dealing with a material so variable that one author finds it necessary to state—and state correctly—that the angle of repose for wet clay is from 1° to 17°, and for damp clay 18° to 45°, without even defining when it is wet and when damp? The table on page 327 shows that when the angle is 30° the pressure on a wall would be about double its value if the angle is 45°.

It is not strange that some engineers believe that since it is necessary to guess at some things, it is just as well to guess at the size of the wall to start with, or to proportion walls only by experience or empirical practical rules. It is not uncommon for engineers to assume that the width of base b of a retaining wall (Fig. 237) should be not less than some fraction, say 0.4, of the height, and to pay no attention to computing the earth pressure. The proper width, however, depends upon the slope of the earth surface back of the wall, upon whether the earth is saturated with moisture or not, and upon the shape of cross-section of the wall; and to assume a minimum width of base is illogical and unscientific. It might almost as well be urged that because the strength of the material in a steel bridge is uncertain, a truss and all its parts should be assumed empirically. If one element in the theory of a structure is uncertain, the uncertainty should be placed right there, where it belongs, and the uncertainty eliminated so far as possible by the aid of experience and common sense, so that correct theory may be applied wherever applicable. Especially, if the material is clay, is there much lack of confidence in theory. Bell concluded that "there was no available theory of earth pressure which, when applied to clay, would command the general confidence of engineers," and he was led, in consequence, to his studies on cohesion. These studies, however, while very interesting and valuable, have not yet settled the question, or afforded a basis independent of experience. If the material is clean sand or gravel, which are really granular materials with little cohesion, the theory is much more reliable than forclay, because the conditions are nearer to those assumed; but even here there will be necessary discrepancies. For earth filled in back of a wall or dam it is clearly reasonable and safe to assume that there is no cohesion, or to take as the angle of repose that of the actual material, which means assuming approximately the actual cohesion of that material.

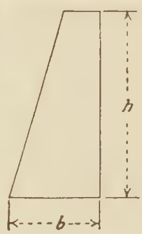


Fig. 237.

The theory is of importance, because it shows the principles governing the subject, and if the engineer is acquainted with it, realizes the assumptions made, and keeps his common sense, he may modify his designs intelligently according to circumstances. Without the theory and with only empirical rules, he will be almost sure to misapply those rules. Some engineers apply the theory to cases to which it was never intended to apply, or in which it is confessedly and perhaps unavoidably inaccurate or erroneous, and then, because the results do not agree with experience, decry the theory. They might as well decry medicine because quinine does not cure cancer. Some very capable engineers, who decry the theory because it sometimes breaks down, hold very interesting views. The reader will be convinced of this if he will read two papers by J. C.

Meem, member of A.S.C.E., named below.[1] Mr. Meem, who speaks from long experience, states that "all closely sheeted, well-braced trenches invariably show a heavier pressure at the top than at the bottom." He explains this by an arch action of the earth, assuming the material between a wall or trench and the surface of rupture to consist of a series of arches, one beneath the other, abutting against the wall and the plane of rupture, each arch acting separately, and the arch at the top, having the longest span, exerting the greatest thrust, each arch carrying only its own weight, and exerting no pressure on the arch beneath. He takes the maximum pressure at the top and the center of pressure at one-third the depth from the top. He asserts that there is no greater pressure at the bottom of the deepest trench or tunnel than at the bottom of a shallow one, and that a circular shaft may be lined with masonry having the same thickness at all depths. He takes the pressure on a wall as the weight of the prism between the wall and the plane of rupture, and acting opposite its center of gravity. Mr. Meem's experiences and facts were contradicted by others in the discussion, and they lead to some absurd results. He gave the results, however, of some interesting experiments which he had made; and his papers, with the discussion thereon, will be interesting to the student. It has been shown above that in some cases the pressure on the walls of a trench may be greatest at the top.

When it is asserted that the previous theory is valueless because it does not agree with practice, it must be remembered that that theory does not claim to take account of cohesion, and therefore cannot be expected to be correct if cohesion is present, unless corrected for that property as will be hereafter indicated. It gives the pressure assuming that the cohesion has been destroyed, as it may be to a large extent by freezing and thawing, vibration, etc., and hence takes account of contingencies that may conceivably occur. It must also be remembered that the results of theory depend entirely upon the value of the angle of repose; and, moreover, *that the angle of repose need not be, and generally will not be, the same at all depths.* Near the surface, owing to the loosening of the material by freezing and thawing, it may well be less than deeper, where frost does not penetrate, where the pressure is greater, and where the earth is more compact, and perhaps is in its natural undisturbed condition. In some cases it may be proper, according to theory, to assume that φ varies from a maximum at the bottom of the wall to a minimum at the top, and if the pressures are computed on this basis, those at the top will be increased and those at the bottom reduced, thus tending toward the results claimed by Mr. Meem and others, though not agreeing

[1] MEEM, J. C., "The Bracing of Trenches and Tunnels, with Practical Formulas for Earth Pressure," *Trans. Am. Soc. C. E.*, vol. LX, 1908; "Pressure, Resistance, and Stability of Earth," *Trans. Am. Soc. C. E.*, vol. LXX, 1910.

with them unless φ be assumed very large (nearly 90°) at the bottom and small at the top. If φ is 90° at the bottom of a trench, the pressure there, according to theory, would of course be zero. If φ is the same from top to bottom, it would be ridiculous to assume the pressure at the bottom less than at the top, unless reliance is to be placed entirely on cohesion. Regarding cohesion, too, it has already been mentioned that probably the coefficient of cohesion varies with the pressure, in some earths, and is greater the greater the depth.

32. Arch Action in Earth. Tunnel Roofs.—There is no doubt some arch action in earth, due to the presence of cohesion. Mr. Meem took a box $8\frac{1}{2}$ by $8\frac{1}{2}$ inches in horizontal dimensions, cut out the bottom except a ledge on each side $\frac{3}{4}$ inch wide, making a hole 7 by $8\frac{1}{2}$ inches, put on a loose false bottom with four bolts extending upward inside the box, with nuts and washers on top (Fig. 238), and put in dry sand, screwing the nuts down so that the washers rested on the sand. He states that

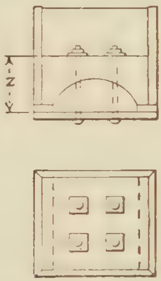


FIG. 238.

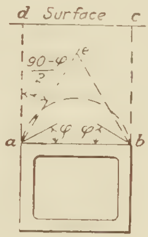


FIG. 239.

"absolute dry sand" would arch sufficiently to carry the false bottom and load of sand at a depth of $4\frac{1}{2}$ inches. Just how reliable this experiment was, or how it was made, the writer does not know, but few will dispute the statement that there is some arch action, especially in such an extremely small box as this.

The roof of a rectangular tunnel or subway (Fig. 239) should, according to theory, carry a weight equal to the weight of earth above it, or the volume *abcd*. Mr. Meem claims that it carries only the weight of the triangular volume *abe*; but according to this, the weight would be less the smaller φ is, and when $\varphi = 0$ (a liquid) it would be a minimum, which is absurd; and, really, the *greater* φ is, the more compact is the earth, the more arch action, and the less the pressure on the roof. No doubt there is arching, and the weight on the roof is that of some volume less than *abcd*, perhaps the weight below some line like one of the dotted lines (see also Art. 47).

That there could be arch action with no cohesion, or with "absolutely dry sand," if by that is meant that such earth would span an opening as an arch, the writer does not believe. Can it be conceived that a mass of

shot or of cannon balls would practically do this? It is true that an arch could be conceived which would stand, if it consisted of a series of balls whose points of tangency formed an arch which was the "linear arch" for the loads, that is, which formed a polygon, with angles on vertical lines through the load on each ball, and sides which passed through the points of tangency and which had the shape of the equilibrium polygon for the loads; but would such a condition ever exist in practice?

All the above shows that the theory of earth pressure is of great aid to the engineer if intelligently grasped, but that it must not be followed slavishly.

33. Experiments on Earth Pressure.—Many experiments on earth pressure have been made, and are found detailed in the references given in this chapter. Many of them have been on a very small scale, some with boxes of earth or sand only a few feet in dimension, or even with cylinders 6 inches in diameter! The pressures have often been measured by means of levers, determining the force which is exerted against a movable or swinging side or opening when it just begins to move, the pressure being reduced until motion begins. Evidently, when motion begins, or when it "impends," the friction on the side is brought into action. The real pressure desired is *before* movement beings or impends. The measuring apparatus has been in some cases open to criticism on this account. Recently A. T. Goldbeck, of the Bureau of Roads, U. S. Department of Agriculture, has devised a gage for measuring pressure in granular materials, which has been used in finding the pressures exerted against walls in earth.[1] The instrument is placed against the wall and connected by air pipes with a pump and manometer above ground. In the instrument a thin, flexible, metallic diaphragm makes electric contact with another piece against which it is pressed by the earth. By means of the pump, air pressure is exerted against the diaphragm until it is moved enough to break the circuit. This instrument seems quite satisfactory, and yet it is the pressure just necessary to produce the minute motion necessary to break the contact that is measured, and not necessarily the pressure existing before the air pressure was applied.

In Jamieson's experiments on pressures in grain bins a bag containing water, covered by a rubber diaphragm, was fastened to the inside of the bin, and the tube connected through the wall with the measuring gage. There was "no receding of the face by displacement of the water." This apparatus, possibly with some modifications, seems to the writer the best yet devised. These experiments will be referred to later.

Altogether, the accurate measurement of the active pressure of earth is attended with some difficulties, and on account of this, and the small scale of most experiments, the writer has not been able to place much reliance on the results.

[1] *Trans. Am. Soc. C. E.*, vol. LXXXIII, p. 1763.

34. Tables.—The following table gives values of some ratios that are useful in problems of earth pressure:

$\varphi =$	0	15°	30°	45°	60°
$\dfrac{90° - \varphi}{2} =$	45°	37½°	30°	22½°	15°
$f = \tan \varphi =$	0	0.268	0.577	1.0	1.732
$\sin \varphi =$	0	0.259	0.5	0.707	0.866
$\dfrac{1 - \sin \varphi}{1 + \sin \varphi} =$	1	0.588	0.333	0.172	0.072
$\dfrac{1 + \sin \varphi}{1 - \sin \varphi} =$	1	1.7	3.0	5.826	13.924
$\cos \varphi =$	1	0.966	0.866	0.707	0.5
$\cos^2 \varphi =$	1	0.933	0.750	0.5	0.25
$\left(\dfrac{1 - \sin \varphi}{1 + \sin \varphi}\right)^2 =$	1	0.346	0.111	0.0295	0.0052
$\left(\dfrac{1 + \sin \varphi}{1 - \sin \varphi}\right)^2 =$	1	2.89	9.0	33.94	193.8

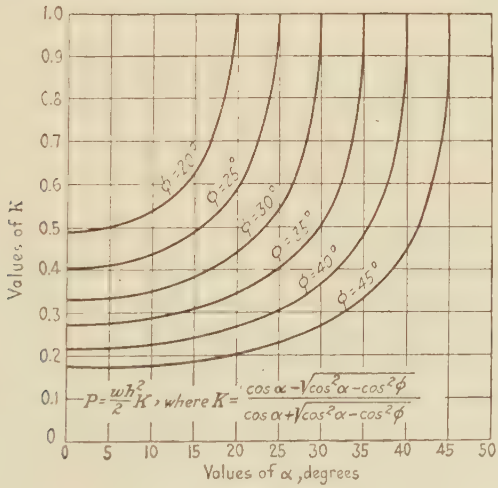


Fig. 240.

In Fig. 240, the values of k in the formula

$$E = k \cdot \frac{wh^2}{2} \tag{21}$$

are given for different values of α = slope of the surface, and φ = angle of repose.

35. The Theory of Earth Pressure for a Cohesive Material.—It remains to give a method of treating earth pressure, taking account of cohesion, supposing the coefficient of cohesion and that of pure friction to be known.

It might at first be supposed that the previous theories would apply correctly here if the angle φ were taken as the *actual* angle of greatest slope, which would of course be increased by cohesion. But it will be evident, upon consideration, that this will only give approximate results. If φ is the actual angle of repose of the cohesive earth, and φ' the angle for the same earth without cohesion, or the angle of frictional repose alone, then, as shown in Art. 8

$$\tan \varphi = \tan \varphi' + \frac{c}{p}$$

where c is the cohesion per unit area and p the normal stress intensity. The value of φ varies with p unless c varies directly as p. If c does not vary directly as p, $\tan \varphi$ is different at different depths.

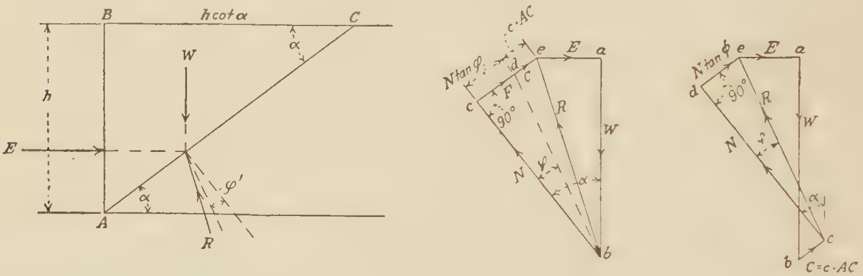


FIG. 241.

Only the case of the pressure on a vertical wall will be here considered, with the earth surface horizontal. Let Fig. 241 represent this case. The three forces, W, E, and R, are in equilibrium. The normal component of R is N, and its tangential component is made up of two parts: the friction $F = N \tan \varphi$ (where φ is not the actual angle of repose, but the angle of friction, whose tangent is f, the coefficient of pure friction) and the cohesion $C = c \cdot AC$. The maximum value of the latter is known for any given plane AC, if c (the cohesion per unit area) is known. The polygon of forces is shown as *abcdea*, which may be laid off in either of the ways shown. Using the second figure, knowing C and φ, ab may be laid off, then bc, then the directions of N and E, and as φ is known, the point e is fixed. Since W and C are known, the value of E may be found which will just keep the earth from sliding on the plane AC. This surface AC must be taken as a plane. Since

$$W = \frac{wh^2 \cot \alpha}{2}$$

and

$$C = c \cdot AC = \frac{ch}{\sin \alpha}$$

the two equations $\Sigma H = 0$ and $\Sigma V = 0$ give

$$E + \frac{ch}{\sin \alpha} \cos \alpha + N \tan \varphi \cos \alpha = N \sin \alpha$$

$$N \cos \alpha + \frac{ch}{\sin \alpha} \sin \alpha + N \tan \varphi \sin \alpha = \frac{wh^2 \cot \alpha}{2}$$

From these equations,

$$N = \frac{wh^2 \cot \alpha - 2ch}{2(\cos \alpha + \tan \varphi \sin \alpha)} \tag{22}$$

$$E = \frac{wh^2 \cot \alpha - 2ch}{2(\cos \alpha + \tan \varphi \sin \alpha)} (\sin \alpha - \tan \varphi \cos \alpha) - ch \cot \alpha \tag{23}$$

From these, it will be found by reduction that calling $\tan \varphi = f$,

$$E = \frac{wh^2 \sin 2\alpha - 2wh^2 f \cos^2 \alpha - 4ch}{2 \sin^2 \alpha + 4f \sin^2 \alpha} \tag{24}$$

The value of α which will make E a maximum must now be found. By differentiation, this will be found to be given by the equation

$$0 = \cos 2\alpha + f \sin 2\alpha$$

or

$$\cot 2\alpha = -f = -\tan \varphi = \cot (90° + \varphi)$$

Hence,

$$2\alpha = 90° + \varphi$$

$$\alpha = 45° + \frac{\varphi}{2} \tag{25}$$

Substituting this value, the value of E becomes

$$E = \frac{wh^2}{2} \cdot \frac{1 - \sin \varphi}{1 + \sin \varphi} - 2ch \frac{\cos \varphi}{1 + \sin \varphi} \tag{26}$$

The intensity of pressure at any depth h is

$$p = \frac{dE}{dh} = wh\frac{1 - \sin \varphi}{1 + \sin \varphi} - 2c\frac{\cos \varphi}{1 + \sin \varphi} \tag{27}$$

For small values of h, p will be negative; p will be zero for

$$wh(1 - \sin \varphi) = 2c \cos \varphi$$

or

$$h_1 = \frac{2c \cos \varphi}{w(1 - \sin \varphi)} = \frac{2c}{w} \cot \left\{45° - \frac{\varphi}{2}\right\}^1 \qquad (28)^1$$

[1] This equation is given by Bell, *loc. cit.*, but his method of derivation is different from the above. Equation (27) is the same as Bell's Eq. (4).

Above this depth the equation would give a tension, and as this cannot exist between the wall and the earth, it must be assumed that there is no pressure down to this depth. At any depth h, greater than h_1,

$$p = w(h - h_1)\frac{1 - \sin \varphi}{1 + \sin \varphi} \tag{29}$$

or the same as by Rankine's theory at a depth $h - h_1$.

If the earth slopes upward from the back of the wall, the thrust E may be taken as parallel to the surface, and as varying according to Rankine's formula for any depth below a depth h_1, which depth, however, will not be the same as for earth level. Just what h_1 would be in this case is uncertain. It would be safe to be guided by judgment in assuming it, or to take it somewhat less than for a level surface.

36. Cohesive Earth by the Theory of the Sliding Prism.—It has been shown that the theory of the sliding prism gives the same results as Rankine's theory if the direction of the pressure on the wall is assumed the same as in the latter theory. Cohesion may also be taken account of by the former theory.

In Art. 25 and Fig. 231 the theory of the sliding prism was explained, the thrust on the wall being taken in any desired direction so long as it made an angle with the normal less than φ. The pressure was obtained as the weight of a triangle of earth ICK in Fig. 233, and the true plane of rupture, or plane on which there is the greatest tendency to slide, AC, was found by a simple construction.

In discussing cohesion, the relation between the coefficient of cohesion and the maximum height h at which the earth would stand vertically was found in Eq. (6).

$$c = \frac{wh}{4}\left(\frac{1 - \sin \varphi'}{\cos \varphi'}\right) = \frac{wh}{4} \tan\left(45° - \frac{\varphi'}{2}\right) \tag{6a}$$

Cohesion on plane $AC = c \cdot AC$ since the prism considered is one unit thick perpendicular to the paper.

Let $h \tan\left(45° - \frac{\varphi'}{2}\right)$ be a distance $d = \frac{4c}{w}$

$$\text{Then } F = \text{cohesion on plane } AC = \frac{wd}{4}AC = w \cdot \frac{d}{2} \cdot \frac{AC}{2} \tag{30}$$

In Fig. 242 ICK is the triangle representing a prism one unit thick whose weight equals E neglecting cohesion. Let AC represent the cohesion on the plane AC, since that cohesion by Eq. (30) is a constant times AC. Resolve this cohesion parallel to E and R, by drawing AL parallel to E and CL parallel to R, making an angle φ' with the normal to AC. The component AL represents the reduction of E due to cohesion. This reduction is

$$F \cdot \frac{AL}{AC} = w \cdot \frac{d}{2} \cdot \frac{AL}{2}$$

This can be represented by a triangle and subtracted from ICK.

Lay off $KM = AL/2$; and, at right angles to KM, lay off

$$MN = d = h \tan\left(45° - \frac{\varphi'}{2}\right) = \frac{4c}{w}$$

Then the triangle KMN is $\frac{AL}{2} \cdot \frac{d}{2}$, and thus represents the reduction of E due to the cohesion; and the real E will be represented by triangle ICK minus triangle KMN, and may be represented by the triangle CKN',

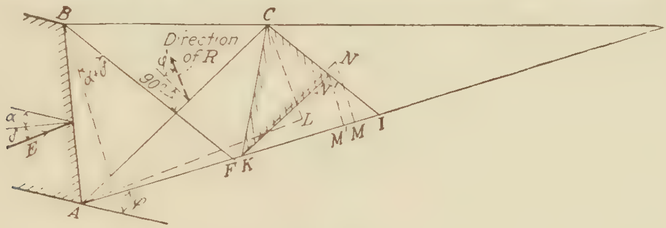


FIG. 242.

and will equal w multiplied by the area CKN'. In case the triangle KMN is larger than the triangle ICK, the cohesion would sustain the earth entirely, requiring no help from the wall. The distance $N'M'$ may be easily constructed; it is equal to $MN \cdot \frac{KM}{KI}$, since the triangles KMN and KIN' are equal.

37. Surcharge.—The treatment of a surcharge has been considered in Art. 17, but after the subsequent discussions some further remarks are appropriate.

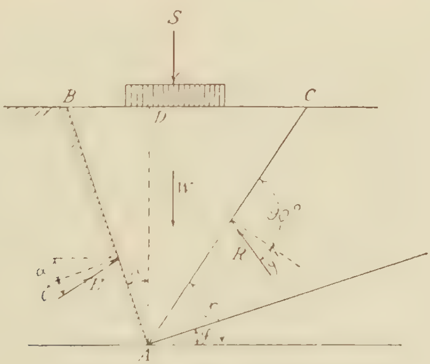


FIG. 243.

The level earth back of a retaining dock wall and in other cases is likely to be covered with merchandise, which results in a heavy load per square foot on the level earth surface. This load is not always uniform, but it should be assumed uniform in the investigation of its effect, just

as the load on a warehouse floor is assumed uniform. Sometimes, along a dock, there is a heavy crane not far from the back of the wall; but this should be distributed by a foundation, and may be considered uniform over the surface of the sliding prism, or, over DC in Fig. 243, in order to be most unfavorable to the wall. An exceptionally heavy load may be supported by its own wooden piles driven back of the wall to relieve the latter of the added thrust which this surcharge would otherwise cause.

If, however, it is desired to consider a surcharge load S (Fig. 243) acting over a comparatively small area back of the wall, this may be done by the aid of the principles which have been explained. The first and second methods which have been explained, Rankine's and the corresponding graphical method, will apply only when the surcharge is uniform or when the surcharge reduced to equivalent earth gives a plane upper surface. The method by the sliding prism must therefore be employed.

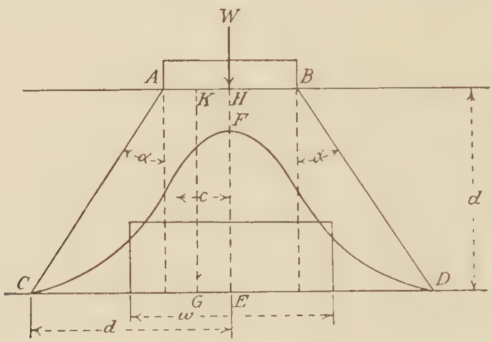


Fig. 244.

Taking any plane AC, the total weight W between this plane and AB can be found easily, and the directions of E and R may be assumed. Then, by the triangle of forces, E may be found. The plane AC_1 which gives the greatest value of E may be determined by a few trials, and the corresponding value of E. Cohesion may be allowed for by the method of Art. 36 if the angle φ' for pure friction is assumed, or determined by experiment.

38. Distribution of Pressure through a Fill.—If a load W is applied over an area AB at the horizontal surface of the ground (Fig. 244), then on any plane parallel to the surface and at a depth d below it, the load W will be distributed over a larger area than AB. It is often assumed that the distribution is uniform over a section of a cone or pyramid with angle α as shown. It is clear that the angle α_1, which determines the area over which W is actually distributed, will vary with the material, depth, and load; and also that W will not be distributed uniformly over the area CD, but will be generally greatest under

the center of the load, shading down to zero at C and D. The curve of distribution will probably be somewhat similar to the probability curve, but not asymptotic to CD. The maximum intensity will be EF, at E, and the minimum zero, at C or D. The points C and D may be difficult to fix, and the points where the pressure is very small may be taken instead. The center of pressure on CE will be to the left of GK, the center of AH.

A. T. Goldbeck, using his pressure gage referred to in Art. 33, has made some measurements of the distribution of pressure through damp sand, using depths of fill from 6 inches to 5 feet, the block AB being a circular block 8 inches or 13.5 inches in diameter.[1] The figure shows the distribution in any vertical plane through the center of W. The paper gives the distance c from the center of the load to the center of gravity of the half of the distributed load curve to one side of the center, that is, from the center to the center of gravity of half the distribution curve. The following table gives, for these experiments, the distances d and e. It is convenient to regard the load W as uniformly distributed over a width w, such that the moment, about E, of the half of the load to one side of E, is the same for the actual distribution as for the assumed uniform distribution, or

$$\frac{W}{2} \cdot c = \frac{W}{2} \cdot \frac{w}{4}; \; w = 4c$$

This is because, if W were distributed, as it actually is, over CD, and CD were part of a beam of span l whose center was E, the moment at E would be

$$M = \frac{W}{2} \cdot \frac{l}{2} - \frac{W}{2} \cdot c$$

and the uniform distribution on w will give the same moment.

Goldbeck's Experiments on Distribution

Depth of fill, inches	8-inch circular block					13.5-inch circular block				
	d, inches	Total load, pounds	Maximum pressure on CD, pounds per square inch	c, inches	$\tan \alpha$ for $w = 4c$	d, inches	Total load, pounds	Maximum pressure on CD, pounds per square inch	c, inches	$\tan \alpha$ for $w = 4c$
6	10	1,400	26.5	2.6	0.2					
							1,800	10	4.4	0.17
12	12	1,200	10.8	3.6	0.267	18	5,000	31.2	4.2	0.14
							1,800	3.6	10.5	0.59
24	24	1,400	2.5	7.0	0.42	36	4,000	7.2	8.4	0.42
							1,800	1.2	16.0	0.7
36	45	1,400	1.2±	16.8	0.82	48	5,000	3	14.0	0.59
48	..	..	..	..	..	60	5,000	1.6±	18.5	0.63

[1] *Proc. Am. Soc. Testing Materials*, vol. XVII, Part II, p. 641, 1917.

It is probably accurate enough to use this width w. The corresponding value of α is in some cases greater than 45°, and in some cases less. Except for shallow fills of, say, 18 inches or less, it is safe to take $\alpha = 45°$, according to common practice. The angle α increases with the depth of fill, and decreases with the load, as would be expected.

Other materials would unquestionably give different numerical results.

39. Earth Pressure in Confined Bins.—The previous treatment has assumed throughout that the mass of earth was unlimited, or at least of sufficient extent that the sliding wedge would not be interfered with.

If earth is confined between two walls so close together, as it might be in a U-abutment, that the two sliding wedges would overlap, this would not be true. If the distance between two parallel walls were less than twice BC in Fig. 241, the pressure on either would doubtless be less than by the previous theories. There is no theory applicable to this case. It is largely a matter of judgment.

The nearer the walls are, the more erroneous the usual theory, and when they are so close together that the plane of rupture does not meet the surface but strikes the other wall, the usual theory is of little value. Such is the case in some grain elevator bins, which are sometimes 75 to 100 feet high and 10 to 20 feet in width; and such would of course be the case in a vertical pipe of small diameter compared with its height, if filled with earth and exposed to pressure. In such cases the weight of the contents may be largely supported by friction on the sides, with pressures on sides and bottom small in comparison with the weight above. Arch action also occurs in such cases. Mr. Meem quotes from an English book a statement that a 2-inch pipe, 15 inches long, with 12 inches of sand in it and a piece of tissue paper covering the bottom, stood a blow of a heavy sledge hammer on the sand, through a wooden piston, without breaking the tissue paper; and a load of over 200 pounds could be put on top of the piston with the same result. The removal of the paper allowed the sand to drop in a mass.

If the surface of rupture strikes the earth surface, the bin is called a shallow bin; if it strikes the opposite wall, the bin is called a deep bin.

40. Theory of Pressure in Deep Vessels.[1]—The theory usually given for this case is as follows: In Fig. 245 let A be the area and U the perimeter of the vessel, both assumed constant, p the vertical and p' the horizontal pressure intensities, f the coefficient of friction on the sides; then considering a horizontal slice dy in height, the vertical forces are in equilibrium; hence

$$pA + wAdy - (p + dp)A - p'Ufdy = 0$$

$$dp = dy\left(w - \frac{p'Uf}{A}\right) \qquad (31)$$

[1] This theory is due to JANSSEN, *Zeit. Ver. deut. Ing.*, p. 1045, 1895.

If the earth is at the limiting condition of active pressure,

$$p' = p\frac{1 - \sin\varphi}{1 + \sin\varphi} = Kp \tag{12}$$

and if we call $n = KUf/A$,

$$dp = dy(w - np) \tag{32}$$

$$\frac{-ndp}{w - np} = -ndy$$

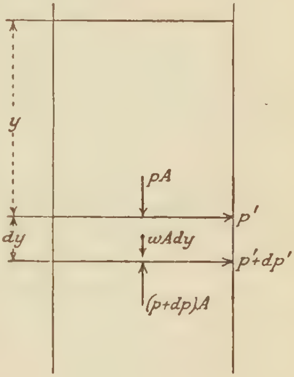


Fig. 245.

Integrating, $\log_e (w - np) = -ny + C$

But $p = 0$ when $y = 0$; hence $C = \log_e w$; hence

$$\log_e (w - np) - \log_e w = -ny$$

$$\frac{w - np}{w} = e^{-ny}$$

$$p = \frac{w}{n}(1 - e^{-ny}) \tag{33}$$

Equations (33) and (12) give values of p and p' for any value of y. Clearly

$$p < \frac{w}{n}$$

p can never equal w/n, because that would mean that $e^{ny} = \infty$, which is only true when $y = \infty$.

But p may be not far below w/n for comparatively small depths. In the table on page 343, $w/n = 500$, and p is only 10 per cent less than this when the depth is less than 30 feet, and 5 per cent less when the depth is 34 feet. It is safe and reasonable to take $p = w/n$ for such bins.

From Eq. (33),

$$\frac{dp}{dy} = \frac{w}{e^{ny}} \tag{34}$$

so that $dp/dy = 0$ only when y is infinite; but practically, the pressure will not increase appreciably with the depth when y becomes greater than a few times the width (Ketchum says $2\frac{1}{2}$ to 3; Jamieson, $3\frac{1}{2}$ to 4; Airy, 3.5).

The total weight of material in the bin is whA; the total pressure on a horizontal plane at depth y, from Eq. 33, is

$$P = \frac{wA}{n}(1 - e^{-ny}) \tag{35}$$

Hence the total weight carried by friction on the sides is

$$P' = A\left[wh - \frac{w}{n}(1 - e^{-ny})\right] \tag{36}$$

The last equation may also be found from Eq. (33) by multiplying by $fKU \cdot dy$ and integrating.

If the bin is square, with side b, $\varphi = 30°$, $f = \tan \varphi = 0.57$, and $w = 110$ pounds per cubic foot, which would be about the weight of *sand* (often taken as 100), then $K = \frac{1}{3}$, $n = 0.76/b$, $p' = p/3$, and

$$p = 145b\left(1 - e^{-\frac{0.76y}{b}}\right) \tag{37}$$

For wheat, with $w = 50$, and other values as before,

$$p = 66b\left(1 - e^{-\frac{0.76y}{b}}\right) \tag{37a}$$

Figure 246 shows p for various values of y/b, by Eq. (37). The curves become practically vertical when y/b is comparatively small. By Eq.

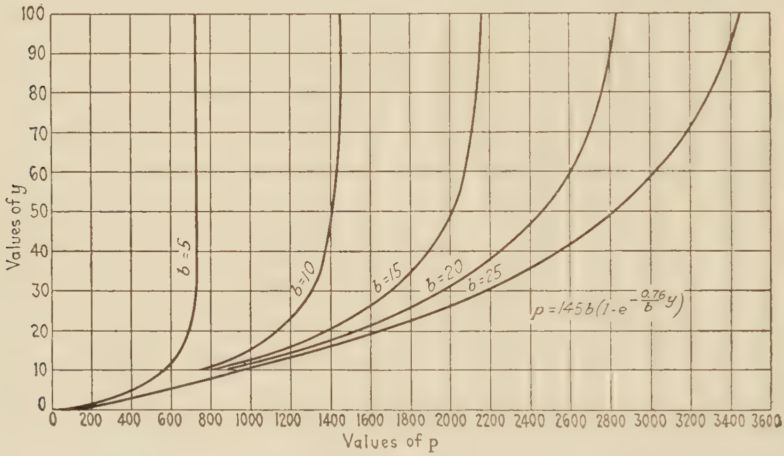


FIG. 246.

(37a), $dp/dy = 0.01$ when $y/b = 11.2$; using Janssen's constants, $K = 0.67$, $f = 0.3$ (see Art. 44), $p' = \frac{2}{3}p$, $n = 0.8\ b$, the result would be nearly the same. These results would be modified by cohesion, and the curves would become nearly vertical more quickly. Cohesion does not seem to have been determined for grain, and probably it is inappreciable, but it does exist for sand.

Using Jamieson's values, $K = 0.6$, $f = 0.4$, or, say, $Kf = 0.25$, $w = 50$, $U/A = 4/b$, $n = 1/b$,

$$p = 50b\left(1 - e^{-\frac{y}{b}}\right) \tag{37b}$$

$$p < 50b$$

where b is the width of a square bin or the diameter of a circular bin; but as we shall see, the value $K = 0.6$ is too great to correspond to *active pressure.*

The lateral pressure at any given depth is the same all around the circumference of a circular bin; but it is not uniform across the side of a square bin. It will no doubt be greatest at the center of the side, and least at the corners. Janssen computed the maximum to be 1.15 times the average and the minimum to be 0.8 the average. He considered it sufficiently close to compute the side for a uniform pressure of $p' = 0.75p$.

In the opinion of the writer, Janssen's experiments were so conducted as to increase the lateral pressure and the vertical load carried by the walls. He believes that ordinarily the ratio of 0.75 for p'/p is too great, and that Jamieson's value of 0.6 is great enough, even that being above the active pressure (see Art. 44).

Airy's Theory.—In 1897, Wilfred Airy presented a paper on "The Pressures of Grain."[1] His theory is based on the wedge of greatest

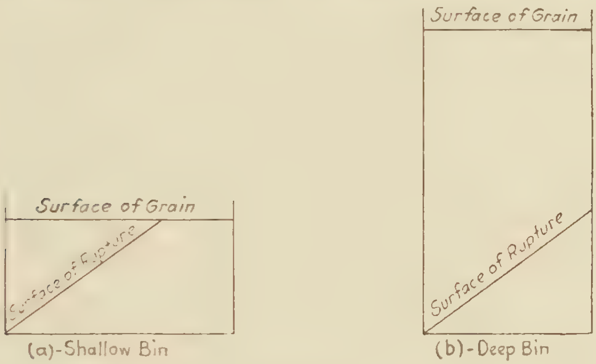


FIG. 247.

pressure, either for shallow bins (Fig. 247*a*) or deep bins (Fig. 247*b*), assuming an upward resistance on the walls equal to the normal pressure times the coefficient of friction f on the wall, and assuming that the pressure on the surface of rupture makes an angle with the normal equal to the angle of repose of grain φ. The resulting equations are too complicated for practical use, and, as given by Ketchum, appear to be erroneous anyway. The writer has not studied his original paper, as Janssen's theory appears satisfactory. Airy's results do not involve K.

41. If the cohesion of the material is to be considered, the upward force on the sides will be

$$p'Ufdy + cUdy$$

if c is the coefficient of cohesion, or the cohesion per unit area. Equation (31) then becomes

$$dp = dy\left(w - \frac{p'Uf + cU}{A}\right) \tag{31a}$$

[1] *Min. Proc. I. C. E.*, vol. CXXXI.

and Eq. (33) becomes

$$p = \left(\frac{w}{n} - \frac{cU}{nA}\right)(1 - e^{-ny}) \tag{33a}$$

which reduces to Eq. (33) when $c = 0$ (Eq. (33a) is the same as the one given by Professor Cain on page 210 of his book on "Earth Pressure, Walls, and Bins"). This equation is approximate, because, for cohesive earth, there will be no lateral pressure down to a certain depth, and the integration should not be carried to the surface; also, the cohesion will probably increase with the depth, so that p'/p will not be a constant. For deep bins the equation may be considered sufficiently accurate, though it is perhaps not quite logical to add the friction on the wall $p'Ufdy$ to the cohesion $cUdy$, as was done above, unless c is the coefficient of cohesion between earth and wall.

42. Experiments on Pressure in Deep Vessels.—Professor Ketchum, in his book "Walls, Bins, and Grain Elevators," gives an account of the experiments on this subject. There are many sources of error, and the results are discordant. Generally the pressure on the bottom, if the experiments were made on model bins, was found by having the bottom movable and resting on platform scales. In Janssen's experiments (1895) the bottom pressures were measured in this way, and from Eq. (33) the value of Kf was found; then, f being found, the value of K was obtained. Janssen gave $K = 0.67$ (which the writer believes is too large), and $f = 0.3$, for wheat in wooden bins: Kf is therefore 0.2, and if $A/U = R$, Eq. (33) becomes

$$p = \frac{wR}{0.2}\left(1 - e^{-0.2\frac{y}{R}}\right) \tag{38}$$

If d is the diameter of a circular bin, on the side of a square bin, $R = d/4$, and the formula becomes

$$p = \frac{5}{4}wd\left(1 - e^{-0.8\frac{y}{d}}\right) \tag{38a}$$

This is the formula used in Germany, and given in the handbook "Hütte."

The side pressures p' have in some cases been measured by diaphragms, either connected to platform scales by levers (Ketchum), or registering the pressure in a liquid behind the diaphragm (Jamieson, Bovey, Lufft); or the deformation of the diaphragm was observed, and afterward the load determined which would produce this deflection (Toltz, Pleissner). The last method appears to the writer unreliable.

The most reliable tests appear to be those made in 1900, by J. A. Jamieson, of Montreal, on the Canadian Pacific elevator at West St. John, N. B., having wooden bins 12 feet by 13 feet 6 inches and 67 feet 6 inches deep above the hopper bottom, with wheat weighing 49.4 pounds per cubic foot.[1] He also experimented on model bins, 6 and 12 inches square and 6 feet 6 inches deep. His paper should be studied by all who are specially interested in the subject. His measuring device is shown in Figs. 248 and 249. He found the angle of repose of the material by filling a tray

[1] JAMIESON, "Grain Pressures in Deep Bins," *Proc. Can. Soc. C. E.*, 1903.

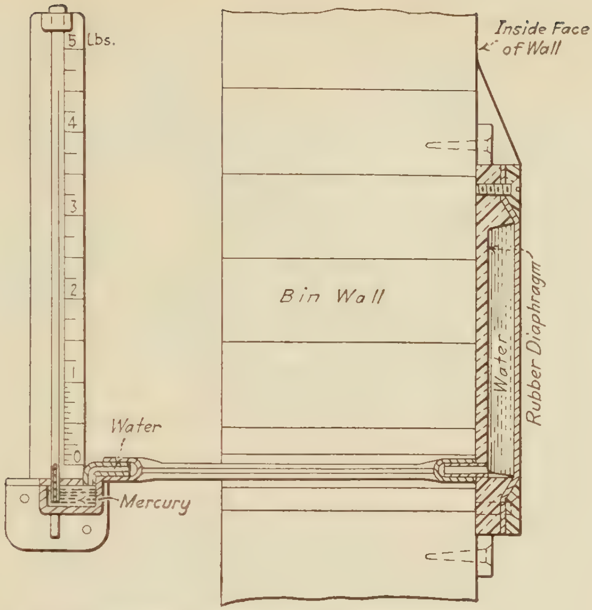


FIG. 248.—Jamieson's apparatus. (*Proc. Canad. Soc. C. E.* 1903.)

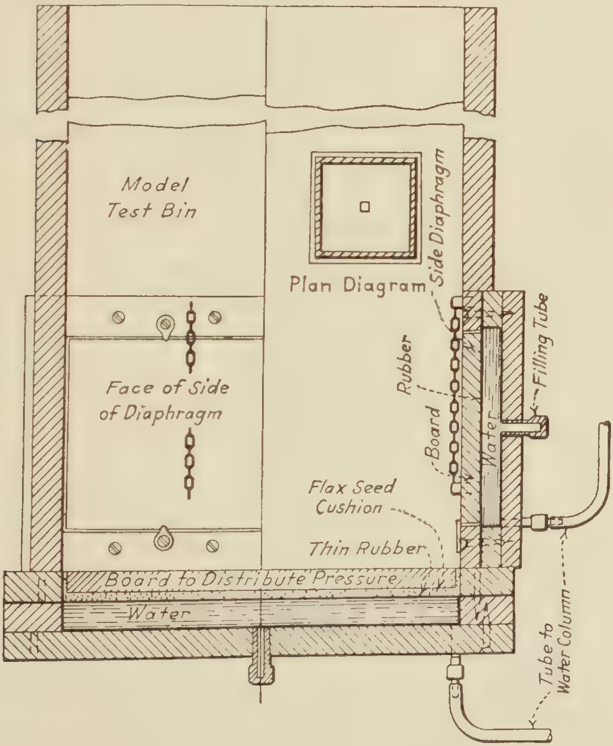


FIG. 249.—Jamieson's apparatus. (*Proc. Canad. Soc. C. E.* 1903.)

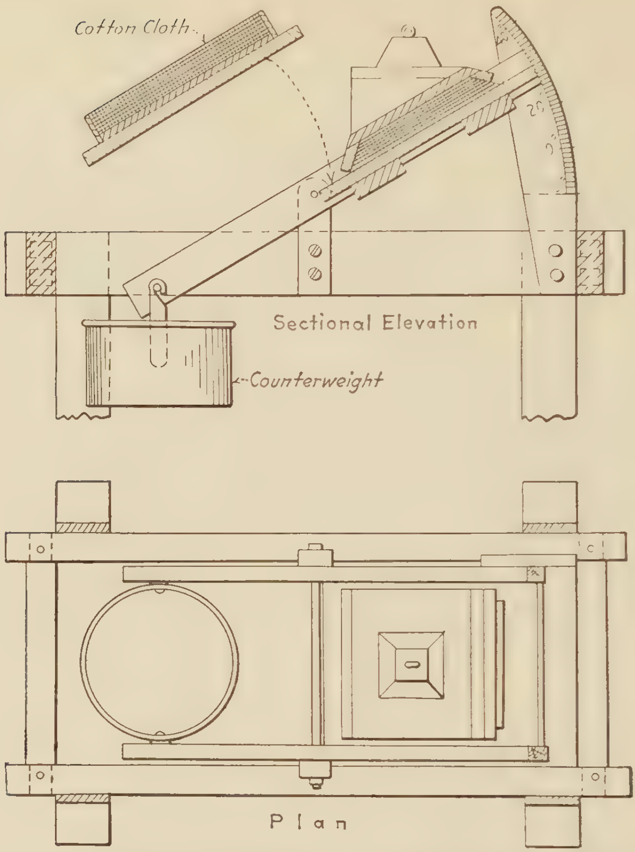


FIG. 250.—Jamieson's apparatus. (*Proc. Canad. Soc. C. E.* 1903.)

and inclining it till the material moved (Fig. 250) and the coefficient of friction by inverting the tray on a surface the same as the bin wall. For wheat, he found the following values for the coefficient of friction:

COEFFICIENT OF FRICTION FOR WHEAT, 50 POUNDS PER CUBIC FOOT, ANGLE OF REPOSE 28° (*Jamieson*)

Wheat on wheat, tan 28° = 0.532
Wheat on steel trough plate bin, 0.468
Wheat on steel flat plate, riveted and tie bars, 0.375 to 0.4
Wheat on steel cylinders, riveted, 0.365 to 0.375
Wheat on cement concrete, smooth and rough, 0.4 to 0.425
Wheat on tile or brick, smooth or rough, 0.4 to 0.425
Wheat on cribbed wooden bin, 0.42 to 0.45

Jamieson experimented with wheat, peas, corn, flaxseed, and sand.

It has sometimes been claimed that the pressure on a bin wall was the hydrostatic pressure due to a liquid having the density of the contained material. Of course this is absurd; it gives a large hoop tension, but only

a hoop tension, and neglects the load that the walls carry in direct compression. A standpipe may carry a hydrostatic pressure of water safely, but may fail if filled with grain.

The pressure on the walls when the grain is being drawn out has been much discussed. Prante[1] found the lateral pressure in motion four times the static lateral pressure when the grain was moving with a velocity of 1 millimeter per second; but he admitted that his tests were insufficient, and they are not confirmed. Pleissner obtained twice the static pressure, but Jamieson found an increase of but 4 to 7.3 per cent, and concluded that if the grain is drawn out at the center of the bottom, the increase of lateral pressure due to motion is not over 10 per cent, and considerably less when the opening is not over one-one hundred and fiftieth of the area of the bin. Bovey found an increase of 9.7 per cent, and Lufft and Ketchum no appreciable increase. If the grain is drawn out at one side, the surface tends to form a cone with the apex above the opening, and the pressure on the side opposite the opening is increased, while above the orifice it is decreased. This tends to distort the shape of the bin, and steel bins have sometimes buckled inward above the opening, due to the decreased lateral pressure there. *Grain should be drawn out from an opening in the center of the bottom*, especially in circular bins.

Ketchum states that the flow from an orifice in a deep bin is independent of the depth and approximately as the cube of the diameter of the orifice. According to experiments by Mr. Jamieson, in 1915, this is only true up to an area of about 75 square inches.

43. Weights and Coefficients of Friction of Grain.—The experiments of Airy (1897) were made to determine these constants; the results are as follows:

COEFFICIENTS OF FRICTION AND WEIGHT OF GRAIN[2]

Kind of grain	Weight per cubic foot loosely filled into measure, pounds	Coefficients of friction					
		Grain on grain		Grain on rough board, tan φ'	Grain on smooth board, tan φ'	Grain on iron, tan φ'	Grain on cement, tan φ'
		φ	tan φ				
Wheat	49	25°	0.466	0.412	0.361	0.414	0.444
Barley	39	26° 55′	0.507	0.424	0.325	0.376	0.452
Oats	28	28°	0.532	0.450	0.369	0.412	0.466
Corn	44	27° 30′	0.521	0.344	0.308	0.374	0.423
Beans	46	31° 40′	0.616	0.435	0.322	0.366	0.442
Peas	50	25° 20′	0.472	0.287	0.368	0.263	0.296
Tares	49	29°	0.554	0.424	0.359	0.364	0.394
Flaxseed	41	24° 30′	0.456	0.407	0.308	0.339	0.414

[1] *Zeit. Ver. deut. Ing.*, p. 1192, 1896.
[2] AIRY, *Min. Proc. I. C. E.*, vol. CXXXI, 1897 (from Ketchum).

44. What is the Actual Pressure Against a Wall? Ratio of Lateral to Vertical Pressure Intensity.—In the theory of earth pressure which has been given, the earth has been assumed to be just at the lower or *active* condition of equilibrium; that is, it has been assumed to be just on the point of sliding *down* on some plane, tending to tip the wall. So far as the earth is concerned, it may have any condition of equilibrium between this and the *passive* condition in which the wall is forcing the earth to slide *up* on some plane, in which case the pressure on the wall is much greater than in the active condition (for level surface, vertical wall and $\varphi = 30°$, *nine times* as great). If the wall were designed so as to be just on the point of tipping, this active pressure would be developed. But walls are built with a factor of safety of 2 or 3; that is, it would take two or three times the active pressure to tip them. Under these circumstances, the *actual pressure* may be greater than the minimum. But, on the other hand, it must be remembered that cohesion actually exists, and that before tipping can occur, the friction on the wall will be brought into action, both of which are favorable to stability. The actual pressure can never be known, and it is safe to design the wall for the active pressure with a factor of safety.

In a grain bin, if K is assumed as in Eq. (12), the grain is supposed to be in the active condition, while really K may be greater than given by that equation. Here there is no question of tipping, but the walls of the bin are exposed to hoop tension and to vertical compression, with a factor of safety of 3 to 5 or, say, 4. There is a high column of grain, and K may actually be considerably above the value in Eq. (12). The measurements of actual pressure by Jamieson gave, for wheat, as we have seen, $K = 0.6$, while φ was 28°, which, by Eq. (12), would give $K = 0.36$; and Janssen gave $K = 0.67$. If these values are correct, it simply means that the grain was not in the active condition, but between that and the passive. Other experiments give K between 0.3 and 0.6; and Pleissner found K to be a variable, depending upon the bin and the grain, and also being greater for small depths than for larger depths, all of this being reasonable. Evidently K cannot be determined merely by measuring φ; the vertical and lateral pressures in actual or model bins must be observed; then, by Eq. (12), K may be found; then f must be found separately, and all the data will be known for the use of Eq. (33), assuming K to be constant.

This is the best procedure *for deep bins*. If experiments cannot be made, Jamieson's value of 0.6 for K may be used, and his values for f (Art. 42), or Airy's values (Art. 43). *For shallow bins*, the usual construction by the sliding wedge may be used, but this will give the minimum or active pressure, and may be increased if good judgment so dictates.

45. Jamieson's "Step Process" for Grain Bins.—Jamieson computes the pressures, not by a formula obtained by integration, but by a "step

process," starting at the top and considering layers 1 foot or 5 feet thick. Thus, suppose a bin 10 feet square and 80 feet deep, filled with wheat weighing 50 pounds per cubic foot, or 5,000 pounds per foot in height, the ratio of lateral to vertical unit pressure being taken at his figure of 0.6, and the coefficient of friction on the sides being also taken at his figure of 0.41667, this value being a convenient average because 0.41667 $\times$ 0.6 = 0.25. The following table is calculated in this way. The results in the last line will be found to check closely with Eqs. (35) and (36).

VERTICAL AND HORIZONTAL PRESSURE IN BIN (*Jamieson*)

Bin 10 feet square, 80 feet deep. Wheat 50 pounds per cubic foot; weight of 1 foot of depth, 5,000 pounds. Horizontal area 100 square feet; lateral area for 1 foot of depth, 40 square feet. Coefficient of friction between grain and bin, 0.41667; lateral unit pressure = 0.6 vertical unit pressure.

1 Depth of layer, feet	2 Total weight to bottom of layer, pounds	3 Weight on top of layer, per square foot, pounds, p	4 Total weight carried by sides of layer, pounds, $40 \times 0.6p \times 0.41667 = 10p$ for 1 foot	5 Total weight carried to next layer, pounds	6 Total weight carried by sides, to bottom of layer, pounds
0–1	5,000	0	0	5,000	0
1–2	10,000	50	500	9,500	500
2–3	15,000	95	950	13,550	1,450
3–4	20,000	135.5	1,355	17,195	2,805
4–5	25,000	171.9	1,719	20,475	4,525
5–6	30,000	204.8	2,048	23,427	6,573
6–7	35,000	234.3	2,343	26,084	8,916
7–8	40,000	260.8	2,608	28,476	11,524
8–9	45,000	284.8	2,848	30,628	14,372
9–10	50,000	306.3	3,063	32,565	17,435
10–15[1]	75,000	325.6	17,851	39,704	35,296
15–20	100,000	397	20,783	43,921	56,079
20–25	125,000	439.2	22,511	46,410	78,590
25–30	150,000	464.1	23,530	47,880	102,120
30–35	175,000	478.8	24,132	48,748	126,252
35–40	200,000	487.5	24,487	49,261	150,739
40–45	225,000	492.6	24,697	49,564	175,436
45–50	250,000	495.6	24,822	49,742	200,258
50–55	275,000	497.4	24,894	49,848	225,152
55–60	300,000	498.5	24,938	49,910	250,090
60–65	325,000	499.1	24,963	49,947	275,053
65–70	350,000	499.5	24,978	49,969	300,031
70–75	375,000	499.7	24,988	49,981	325,019
75–80	400,000	499.8	24,992	49,989	350,011

[1] The results for the 5-foot layers are obtained by taking five steps of 1 foot each, as was done above.

46. The results of one of Jamieson's sets of measurements are shown in the following table.

GRAIN PRESSURE TESTS NO. 7A

Wheat. Square wooden bin. Bottom pressure tests. Smooth boards. Size of bin 12 inches by 12 inches by 6 feet 6 inches. Diaphragm on bottom, size 12 inches by 12 inches = 144 square inches. Wheat 50 pounds per cubic foot, equal to 62.2 pounds per bushel. (Jamieson in Proc. Can. Soc. C. E. 1903.)

Grain weighed into bin, pounds	Height of grain column, inches	Equivalent fluid pressure, inches of water	Pressure of grain on diaphragm, inches of water	Grain carried on bottom		Grain carried on bin side	
				Weight, pounds	Per cent of total weight of grain	Weight, pounds	Per cent of total weight of grain
25	6	4.81	3¾	19.483	77.93	5.517	22.07
50	12	9.62	5⅞	30.524	61.04	19.476	38.96
75	18	14.43	7	36.368	48.49	38.632	51.51
100	24	19.24	7⅞	40.914	40.91	59.086	59.09
125	30	24.05	8¼	42.863	34.29	82.137	65.81
150	36	28.86	8⅝	44.811	29.87	105.189	71.13
175	42	33.67	8⅞	46.110	26.35	128.890	73.65
200	48	38.48	9	46.759	23.37	153.241	76.63
225	54	43.29	9 7/16	49.032	21.79	175.968	78.21
250	60	48.10	9 9/16	49.682	19.87	200.318	80.13
275	66	52.81	9¾	50.656	18.42	224.344	81.58
300	72	57.62	9¾	50.656	16.88	249.344	83.12
325	78	62.53	9¾	50.656	15.58	274.344	84.42

Carried on bottom 50.656; on sides 274.344.

By sharply tapping sides of bin, grain settled 2⅞ inches from top, and gave maximum gage reading of 11⅞ inches of water equals total load on bottom of 61.696 pounds, or 20.56 per cent of total grain in bin.

The tapping of the sides, noted below the table, had the effect of diminishing the friction on the sides, and so caused a greater proportion of the load to be carried by the bottom.

47. Suggested Applications of the Theory of Pressure in Deep Bins to Earth Pressure.—Professor Ketchum thinks that the theory for deep bins may be applied to the pressure of a cohesive soil on braced trenches, sewers, tunnels, and temporary retaining walls. If $ACDB$ in Fig. 251 is a trench in a *cohesive earth*, and the earth would cave on the line DE, which may be calculated if the coefficient of cohesion is known, and if GH is the curve on which it would cave at a depth BG, he suggests that the pressure on BD is similar to that on the side of a bin BGH, GH being nearly vertical from H down to D, and that Janssen's formula may be applied. To the writer this does not seem admissible. The width b is

unknown, Janssen's formula does not take account of cohesion, and the value of K which Professor Ketchum uses $\left(\frac{1 - \sin \varphi}{1 + \sin \varphi}\right)$ applies only when there is no cohesion.

The uncertainty of the pressure on braced trenches has been discussed in Arts. 30 and 31, and if any theory is to be applied, it would seem as though it should be that for cohesive earth as given in Arts. 35, 36.

Professor Ketchum also applies Janssen's theory to the pressure on sewers, tunnels, and temporary retaining walls "built to sustain a cohesive filling," although that theory is not designed for a cohesive material and takes no account of the coefficient of cohesion. It will, however, apply to a cohesive material if K is determined by experiment for that

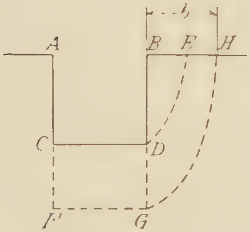


FIG. 251.

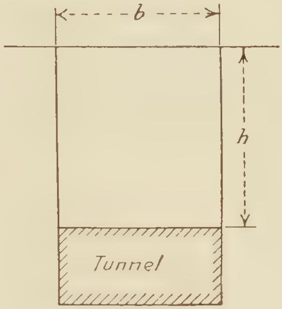


FIG. 252.

material; but K is a variable, and, in order to apply the theory as suggested, it would be necessary to determine it by experiment for the very material surrounding the trench, sewer, tunnel, or temporary material, in a deep bin, which has not been done and probably would not be done. Janssen's formulae as given by Professor Ketchum would have to be modified for a cohesive material. This modification has been shown above, in Art. 41.

Altogether, therefore, Professor Ketchum's suggestions do not appear practical to the writer.

Professor Cain, in his book "Earth Pressure, Walls, and Bins," also applies the Janssen theory to tunnels, considering a tunnel with a flat roof as the bottom of a bin of depth h (Fig. 252). He modifies Janssen's formula to allow for cohesion, as explained in Art. 41. If l is the length of the tunnel, he takes $U = 2l$, $A = bl$. This application of the theory of deep bins may be correct except that the constants are not known. Professor Cain takes $K = \frac{10}{6}\,\frac{1 - \sin \varphi}{1 + \sin \varphi}$ because Jamieson, as above explained, found $K = 0.6$ for a material having $\varphi = 28°$ and $\frac{1 - \sin 28°}{1 + \sin 28°} = 0.36$; and the ratio of 0.6 to 0.36 is $\frac{10}{6}\cdot$ He assumes

values for c, φ, b, f, and gives tables of pressures. Obviously, the results will vary with the constants assumed, but this application of the theory may be useful in estimating the very uncertain pressures on tunnels. Tunnels, however, seldom have the top horizontal, and in the case of the arch tunnel the conditions of stability of the arch may generate pressures approaching the passive pressure. The writer, therefore, in various tunnel problems that have come before him, has always preferred to endeavor to apply the usual theories with judgment.

48. Passive Pressure.—In discussing the lateral pressure in grain bins it was remarked that the pressure might exceed the active pressure, which assumes that the material is held against sliding by the smallest possible lateral force.

There are many cases in structures in which the same may be true, and in which even the passive or maximum lateral pressure should be assumed.

In an arched tunnel, if the thickness is so small that it tends to flatten vertically and bulge sidewise, the latter tendency will be resisted by an

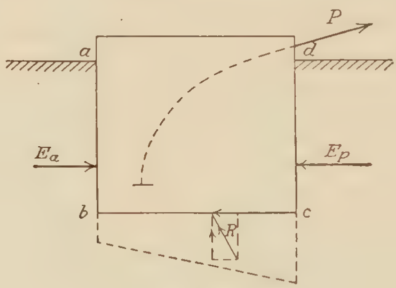


FIG. 253.

increased lateral pressure; and no *collapse* can occur until the passive pressure is exerted, although, if the earth is compressible, the movement may be dangerous before the passive resistance is reached. For this reason the writer, in studying such tunnels, after dividing the exterior surface into a sufficient number of parts which may be considered plane, finds on each part the active and the passive pressure, both in magnitude and direction, and then studies the pressures or line of resistance, remembering that the actual pressure *may* be anything between the two. This procedure is especially necessary in studying an existing structure. The so-called elastic theory is useless here unless the compressibility of the outside earth is known, which is seldom or never the case.

So, too, in the anchorage of a suspension bridge (Fig. 253) the outer forces are the pull of the cable, the weight, the earth pressure on the back *ab*, the earth pressure on the front *dc*, and the upward reaction on the foundation *bc*. The earth pressure on *ab* is the active pressure; that on

dc will be the passive pressure before the anchorage moves, and may be assumed at anything below that, depending on judgment, conditions, and compressibility of the earth. These, with the weight and the cable pull, being found, the resultant on the base *bc* may be found, and from its normal component the distribution on the base.

The principles given in this chapter will enable all such problems to be solved.

CHAPTER XXI

MASONRY: DEFINITIONS AND GENERAL PRINCIPLES

1. A masonry structure is one built of stone or concrete. When the term is used without qualification, it should be understood as meaning a structure of stone, and is to be so understood in this chapter unless otherwise stated. Concrete masonry will be called *concrete,* and will be treated in another chapter.

Stone masonry is made of stones, of various sizes and shapes, depending upon the class, either laid up without mortar, or with mortar filling the voids between the stones. Mortar is used in all important structures.

2. Kinds of Stone.—The kinds of stone are referred to in the second volume of this work, and in the larger treatises there referred to, which deal with engineering materials. The principal kinds used are granite, gneiss, sandstone, and limestone. Structures of brick also come under the head of masonry, but should be specifically termed *brick masonry.*

3. Bond.—In placing stones upon each other to form a structure, each stone should be so placed that there will be no continuous vertical joints between the stones, and that a load on any stone will be distributed upon several stones below it. There are different forms of bond, which are described below.

CLASSIFICATION OF STONE MASONRY

4. Most of the definitions of classes of masonry are based upon a paper on the "Nomenclature of Building Stones and of Stone Masonry."[1] This paper classifies masonry and describes the tools used in stone cutting. The first subject needs a new discussion, because there is much vagueness in specifications now common, and some contradictions. The tools used will be considered in the next article. Both subjects should be made clear in the mind of the student.

The classification of masonry may clearly be according to the character of the work or dressing performed upon the individual stones, which is the classification of the stone cutter; or according to the manner in which the stones are arranged in the structure, which is the classification of the mason. The two are somewhat interdependent. Clearness of definition is most important if specifications are to be without ambiguity. At present there is much ambiguity.

[1] *Trans., A.S.C.E.,* 1877.

5. Stone-cutting Tools.[1]—The stone as it comes from the quarry may be used in the work without dressing except to knock off projecting points with the hammer; or it may be dressed or cut or polished down to the greatest degree of accuracy, smoothness, or polish. Specifications and terminology should make perfectly clear just what is expected; otherwise a contractor cannot bid intelligently. Further, requirements should be reasonable, in order that the work may not cost more than is necessary to accomplish the result desired. All the tools used are of steel.

When the stone block comes from the quarry, the edges are marked on it with chalk, and the parts outside of those edges must then be cut away preparatory to dressing the surfaces to the requisite degree of smoothness. This preliminary cutting away is done with the bull set or the hand set.

The *bull set* (Fig. 254) is used by placing the square end against the edge and striking the other end with a hammer (Fig. 255). One man

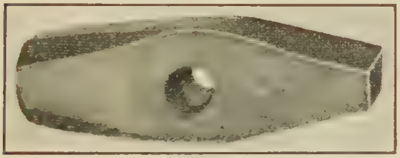


FIG. 254.—Bull set.

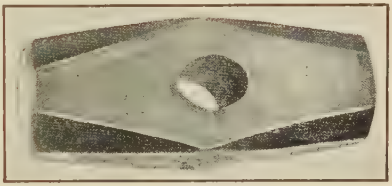


FIG. 255.—Hammer.

holds the set, and another strikes it with the hammer. The *hand set* is a smaller tool for the same purpose (Fig. 256), which one man can operate. It is often called a pitching tool, and consists of octagonal steel, ¾ inch to 1¼ inches, spread on the cutting edge to a rectangle about ⅛ inch by from 1⅜ to 2 inches.

For roughly evening a surface, the *point* and the *chisel* are used.

The *point* (Fig. 257) is made of octagon or ¼-octagon steel rods, from ¼ to 1 inch in diameter and about 8 to 12 inches long, with one end brought to a point. It is used until its length is reduced to about 5 inches. It is employed for dressing off the irregular surface of a stone, either for a permanent finish or preparatory to the use of the axe. According to the hardness of the stone, either the hand hammer (Fig. 255) or the mallet (Fig. 258) is used with it. The mallet is used with the softer rocks.

The *chisel* (Fig. 259) is of round, octagon, or ¼-octagon steel, ¼ to ¾ inch in diameter, and about 10 inches long, with one end brought to a cutting edge ¼ inch to 2 inches wide. It is also used for cutting drafts or margins on the face of stones.

[1] See catalogues of the following companies: Harrison Supply Company, 5 and 7 Dorchester Ave. Extension, Boston, Mass.; Thomas H. Dallett Company, 165 West Clearfield St., Philadelphia, Pa.; The Trow and Holden Company, Barre, Vt.; Livingstone Manufacturing Company, Rockland, Me.; Vulcan Tool Manufacturing Company, Quincy, Mass.

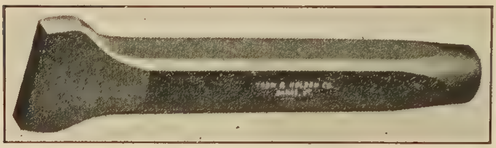


Fig. 256.—Hand set.

Fig. 257.—Point.

Fig. 258.—Mallet.

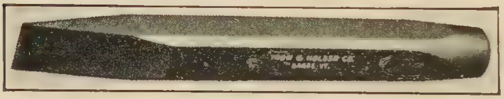


Fig. 259.—Chisel.

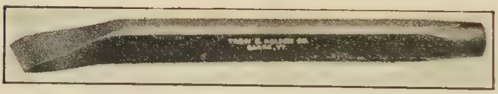


Fig. 260.—Chipper.

Fig. 261.—Chisel.

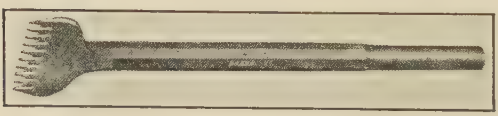


Fig. 262.—Tooth chisel.

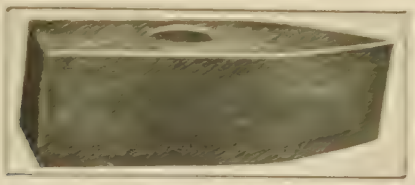


Fig. 263.—Face hammer.

Fig. 264.—Ax or peen hammer.

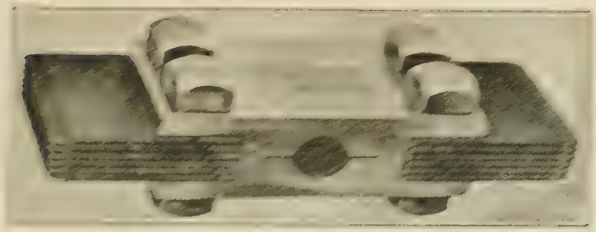


Fig. 265.—Tooth hammer.

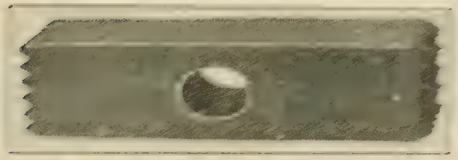


Fig. 266.—Point bush hammer.

Fig. 267.—[illegible]

Fig. 268.—Bush chisel.

The *chipper* (Fig. 260) is also used for making a more accurate edge, or making a draft along the edge.

There are other forms of chisel, such as the *marble-cutting chisel* (Fig. 261) and the *tooth chisel* (Fig. 262) in which the cutting edge is divided into teeth, and is used on marbles and sandstones. For architectural carving there are still other forms of chisels which are shown in the catalogues referred to.

For evening the surface, various forms of hammers are used, by which blows are struck nearly at right angles to the surface, either by hand, or almost universally at present by means of tools attached to a pneumatic hammer which strikes a rapid succession of blows.

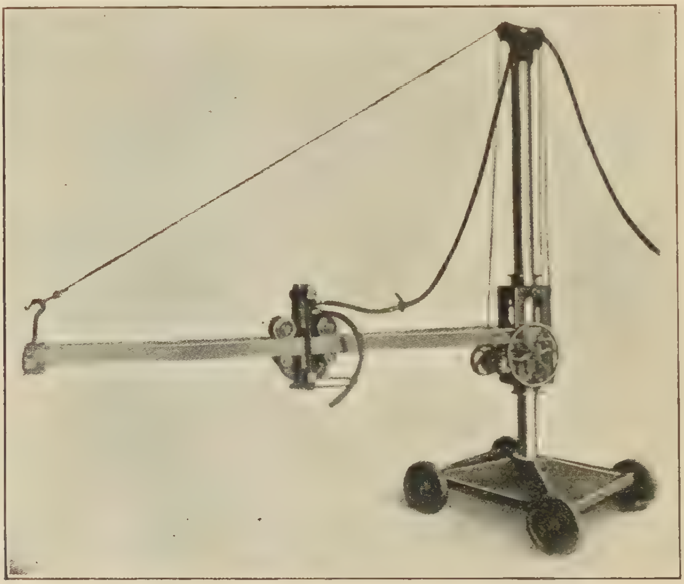


Fig. 269.—Pneumatic surfacer.

The *face hammer* (Fig. 263), and the double face hammer, which has both ends flat, are used for roughly evening the surface as well as for knocking off projections. The sharpened edge of the former is also used for splitting. Granite can be split in this way, just as wood is split. The *cavil* has one blunt and one pointed end, and is used in the quarry. The striking hammer (Fig. 255) has a slightly curved face, with no sharp edges. The *pick*, with both ends pointed, is sometimes used on limestones and sandstones, but not on granites, for rough dressing. The *axe* or *peen hammer* (Fig. 264), about 10 inches long and with edges about 4 inches wide, is also used, for dressing the surface and for splitting. The *tooth axe* is like the axe, except that its cutting edges are divided into teeth, the number of which vary with the kind of work required.

The *bush hammer* (Fig. 265) is formed of a number of blades. It is now used in pneumatic surfacing more than by hand. It is sometimes called the *patent hammer*. The *point bush hammer* is like the double face hammer except that the striking surfaces are made of a number of points (Fig. 266). The *Scotia* is a thin bush hammer (Fig. 267), and the *bush chisel* (Fig. 268) may be used by hand or with the pneumatic surfacer.

The pneumatic surfacer consists of a pneumatic hammer attached to a crane, either movable or fixed (Fig. 269). The arm of the crane extends

Fig. 270.—Tools used with surfacer.

over the surface to be dressed, and the necessary tool is attached to the pneumatic machine. The tools used (some of which are shown in Fig. 270) are modifications of those already described.

For splitting a stone along a desired plane, a slight groove is cut around the stone with the *tracer* (Fig. 271), and a properly directed blow with the hammer will split the stone along the lines so marked.

For splitting a stone in the quarry, the *plug* and *feathers* are used. A row of holes is made with the drill, along the line to be split, and, in each of these, two feathers are inserted and the plug driven in between them.

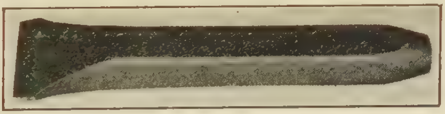


Fig. 271.—Tracer.

Saws are also used in cutting granite, limestone, sandstone, and marble. The reciprocating saw has a thin flat blade of soft steel, and the cut is made by abrasives applied with water between the blade and the stone. The abrasives used are sand or carborundum for the softer stones, and chilled steel pellets, known as shot, or crushed steel, for granite. Circular saws use the same abrasives in some instances, and in others have diamonds or carborundum plates inserted for the cutting edge, or the whole saw is of carborundum. Some saws have teeth and some have not. In either case it is the abrasive or shot which does the cutting, being caught between the teeth. Some saws are rotary saws and some are reciprocating saws working horizontally.

When a surface has been dressed with the bush hammer, it is sometimes designated as four-cut or six-cut work, depending upon the number of blades in the hammer.

Various other tools are used for carving.

6. Stonework may be classified as follows, but there is no sharp distinction between the classes, except between class I and the others. Class II gradually becomes class III, depending upon the fineness of the dressing.

Class I.—Unsquared stones or *rubble*. In this class, the stones are used as they come from the quarry, without other preparation than the removal, with the hammer, of very acute angles and excessive projections from the general figure. This is known as rubble. When rubble masonry is specified, it is universally and properly understood to mean that no stone cutting will be required, except to knock off projections with the hammer. Rubble may be dry, that is, without mortar, or it may be laid in mortar.

Class II.—Stones roughly squared and dressed on beds and joints. "The distinction between this class and the third lies in the degree of closeness of the joints which is demanded"; that is, in the fineness of dressing of the joints. Where the distance between the general planes of the surfaces of adjoining stones is ½ inch or more, the masonry properly belongs in this class.[1] Three subdivisions of this class may be made, depending on the character of the face of the stone.

a. Quarry-faced stones are those whose faces are left untouched as they come from the quarry.

b. Pitch-faced stones are those on which the arris (edge) is clearly defined by a line beyond which the rock is cut away by the pitching chisel (bull set or hand set), so as to give edges that are approximately true (straight).

c. Drafted stones are those on which the face is surrounded by a chisel draft (approximately plane surface), the space inside the drafts being left rough. Ordinarily, however, this is done only on stones in which the cutting of the joints is so fine as to bring them in class III.

In specifying stones of class II, the specifications should state the *depth* (perpendicular to the face) of the bed and end joints, and how far the surface of the face may project beyond the plane of the edge. This projection may vary from one to six inches. It should also be stated whether or not the face stones are to be drafted.

Class III.—Stones accurately squared and with joints finely dressed. As a rule all the edges of stones of this class are drafted, and between the drafts the joints are smoothly dressed, though the face is often left rough, since there is no structural necessity for dressing it. For appearance, as in buildings, the face may be dressed in various ways. It is

[1] *Trans. Am. Soc. C. E.*, p. 300, 1877.

rough pointed when projections of more than an inch are removed by the point. This operation is also done on the joints prior to finer surfacing. For a smoother final finish the face may be *fine pointed*, the projections being less than an inch. The bush hammer may be used for finer dressing, producing peen-hammered, four-cut, six-cut and eight-cut work. A smoother surface may be obtained if the face is cut by a carborundum saw, and it may be rubbed or polished with grit or some preparation, to any desired degree of smoothness, without drafts. If finer dressing than eight-cut is desired, specify rubbed. Sometimes the face is made as a *diamond panel*, with four triangular planes sloping to an apex at the center.

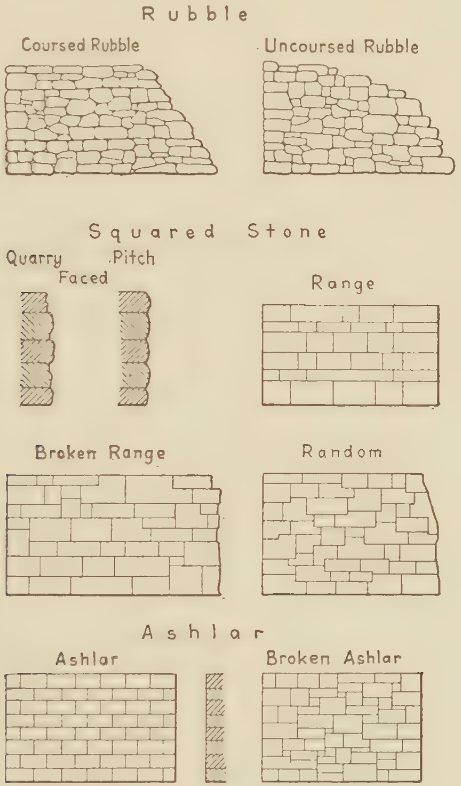


FIG. 272.—Classes of masonry.

7. Classes of Masonry.—*a. Rubble masonry*, as already stated, is composed of stones on which no dressing has been done. It may be *uncoursed*, that is, laid without any attempt at making regular courses (layers), or *coursed*, that is, leveled off at intervals to a horizontal surface. The stone may be required to be roughly shaped with the hammer, but if any dressing is required, it is not rubble.

b. Squared Stone Masonry.—This is either *quarry faced* or *pitch faced.* If laid in regular courses of about the same rise throughout, it is *range work.* If laid in courses that are not continuous, it is *broken range work.* If not laid in courses at all, it is *random work.* Unless the specifications call for range work, this class of masonry is assumed to be random work. Corner stones and the sides of openings are usually hammer dressed.

c. Ashlar Masonry or Cut-stone Masonry.—This is always laid in courses, generally continuous; if not continuous, it is *broken ashlar.*

These three classes of masonry correspond accurately with the three classes of stonework of Art. 6.

Often the face of a masonry is of ashlar, while the *backing* behind it is of rubble or concrete.

Figure 272 shows the classes of masonry described above.

8. In view of the fact that the last two classes have no strict line of demarkation, masonry specifications should be particularly definite, explicit, and unambiguous, in order that the contractor may know precisely what he is to furnish. He cannot otherwise estimate closely. It is no benefit to the engineer for the contractor to lose money. Both are in business for the profit or wage that it brings them, and are entitled to it. If the contractor furnishes the stone from his own quarry, he knows the condition in which the blocks will come from the quarry, and the expense of cutting them to the required specifications. If the work is remote, and the contractor is obliged to open a quarry near the work, he will not know just how the blocks will come from the quarry. The engineer should assist him as much as possible, by giving him all possible information. He should never mislead him, as, for instance, by calling for rubble masonry and then specifying the joints in such a way that stone cutting will be required. If rubble masonry is called for, the contractor is justified in assuming that the engineer knows and guarantees that the stones will come from the quarry in such shape that the further specifications as to the joints may be fulfilled with no further work than knocking off projections with the hammer, and that any dressing will be paid for as an extra.

The mortar joint is to prevent the entrance of water and to distribute the pressure from one stone over the stone beneath or adjoining. It should be thick enough so that no projecting point of one stone shall touch the other, or else there will be a concentration of pressure at this point. But a hollow in a stone surface does no harm, if not too large or too near the surface; it merely increases the mortar joint. In strong stones, where the strength of the stone is much greater than that of the mortar, even a hollow close to the face may do no harm. If too near the face of soft stones, however, it might tend to crack a piece off. Many inspectors are unreasonable on this point. An inexperienced, ignorant, or

obstinate engineer may reject many stones which could be used without injury to the work; but there must be a limit.

9. Bond.—In order that there may be no continuous joints through the structure, the stones are *bonded*, that is, arranged so that a joint between two stones has a stone above and below it, and on each side. The arrangement to accomplish this is the *bond*.

A stone may be a *stretcher* or a *header*. Its length is always, in squared stone masonry, greater than its width or thickness. If laid with its length lying longitudinally in the wall, it is a stretcher; if the length is perpendicular to the face, it is a header. A header projects back of the stretcher on which it rests, into the backing. A header in one face, as a *quoin* or corner stone, is a stretcher in the adjacent face. In order to secure proper bond, there must be a sufficient number of headers showing in the face of the structure.

In *English bond*, used especially in brickwork, each alternate course consists entirely of headers and stretchers. In *Flemish bond*, each course has on the face alternate headers and stretchers. In order that brickwork may lie properly, the length of a brick should be twice the width, plus the thickness of a vertical joint. In common bond there is one course of headers to several courses of stretchers, not more than five for a strong bond.

In stone masonry it is generally required that a certain percentage of the area of the face shall be headers. The percentage should in general be not less than 20.

10. Stereotomy or Stone Cutting.—This is a branch or application of descriptive geometry, showing how, having designed a masonry structure, the shape of each separate stone and of each of its surfaces, may be determined and drawn, so that the stone cutter may be furnished with drawings showing each dimension. This is often difficult. Of course, in an ordinary wall, pier, or abutment, having specified the limiting dimensions of headers and stretchers, nothing more is necessary. The contractor furnishes and lays the stone without further directions. But in structures like groined or cloistered arches, or skew stone arches, the shape of the various stones, or of some of them, must be drawn for the use of the stone cutter. The study of this subject[1] is a valuable training for the student.

11. Unless the structure is one in which the dimensions of each stone are determined, specifications should state the limiting dimensions of headers and stretchers, and the proportion of headers. The thickness of joints, allowable depressions, and character of face should also be stated.

[1] The outstanding work known to the writer is the French work: LEJEUNE, "Traité pratique de la Coupe des Pierres," Baudry, Paris.

In English, the following works are recommended: SIEBERT and BIGGIN, "Modern Stone Cutting and Masonry," John Wiley & Sons, Inc.; and FRENCH and IVES, "Stereotomy," John Wiley & Sons, Inc.

There is no necessity or advantage in dressing the bed joints (top and bottom) of headers for a greater distance from the face than the width of the stretchers. Behind the stretchers they are in the backing, and may as well be left rough. They may also be allowed to taper, horizontally and vertically, back of the stretchers. The *builds*, *i.e.*, vertical joints, or those perpendicular to the bed joints, need not be dressed back of the stretchers; indeed, since there is in general no pressure transmitted through the builds, it is not necessary to dress them even for the full width of the stretchers. If dressed for a reasonable distance from the face, the builds may be left rough and the headers allowed to taper horizontally back of this, provided they do not flare outward but leave a sufficient joint to the adjoining stretcher.

There is a tendency at the present time to make specifications unduly rigid, and to exact requirements which increase the cost with no advantage. Just as there is a tendency to indulge in complicated mathematical theories based upon assumptions which are generally untrue, and then, forgetting the assumptions, to imagine that accuracy is obtained because the work is complicated and difficult, so in drawing specifications there is a tendency to exact requirements which are unnecessarily rigid and to imagine that, because they are rigid, better work is obtained. The engineer should aim to have a valid reason for everything which he specifies, and should study to make specifications, as well as computations, as simple and practical as possible. This is a situation in which a well-balanced mind is shown.

12. In a bridge structure with which the writer was connected, involving steel spans, piers, and abutments, the masonry specifications were as follows:

The stone used in the masonry shall be a superior grade of granite, of quality approved by the engineer. It must be sound, free from flaws or defects, and of fairly uniform quality and color. Each bidder will submit with his bid a sample of the stone which he will use if the contract is awarded to him.

The piers and abutments shall consist of regular coursed ashlar laid with horizontal beds and vertical joints on the face.

The dimensions and arrangement of the stones are to be substantially as shown on the plans, but may be varied with the approval of the engineer.

Courses shall be from twenty to twenty-four (20 to 24) inches in thickness, decreasing from the bottom to the top of the work. No course shall be thicker than the one below it. All corners and batter lines shall be run with an inch and one-half ($1\frac{1}{2}$) chisel draft.

Courses shall be continuous through and around the piers. The masonry shall consist generally of one header and two stretchers alternating; at least one-fifth of each face shall be headers. The headers and stretchers shall be so distributed throughout each course as to bond the work in the best manner. Each stone must bond with the stone of the underlying course; no bond of less than twelve inches will be allowed.

Stretchers must not be more than six and one-half ($6\frac{1}{2}$) feet or less than three (3) feet in length. Headers must not be less than three and one-half ($3\frac{1}{2}$) feet in length,

and must be through headers in all courses six and one-half (6½) feet or less in width. All stones must have at least as much bed as rise.

In abutments, through headers must extend at all points to the surface of the back of the wall as indicated on the plans; but no masonry outside of this surface will be estimated or paid for. Headers which are not through headers must have at least eighteen inches between their back surfaces and the back surface of the wall as indicated on plans.

The beds of the lower courses, resting upon the concrete, shall be dressed to lay not over one-inch joints.

All other beds (*i.e.*, top and bottom surfaces) are to be dressed to lay one-half-inch joints, but this will not exclude stones whose joint surfaces may have cavities not more than six inches across or one inch deep, provided that the total area of such cavities in any stone shall not exceed one-third the area of the joint surface in which they exist and that no such cavities are nearer than three inches to the edge of the joint. Upon that portion of a stone which is to be directly covered with concrete, the surface may be quarry split.

The vertical joints are to be dressed for one-half-inch joints for one foot back from the face and dressed or split for from one- to three-inch joints for the balance of the depth.

Faces of stone are to be quarry faced, pitched to line and to the batter required, and to be out of wind and full to line; to have no projections or more than four inches and no hollow faces, and must show no drill or dog holes.

Backs of stones are to be quarry split.

All stones are to be laid solid in cement mortar and vertical joints filled with the same.

All spaces between the stones in the piers, and in the abutments the space between the back of the stone masonry and the back surface of walls, as indicated on the plans, except the joints, are to be filled with concrete, laid to the satisfaction of the engineer, and finished flush with each course.

All face joints are to be finished in a satisfactory manner as the work progresses, or raked out and neatly pointed as may be required.

Clamps are to be used in the piers, where shown on the plans, set four inches into each stone in neat cement mortar.

The face of retaining walls and wing walls shall be what is known as broken-range masonry, with horizontal beds and vertical joints. It shall be laid in fairly regular courses and shall be well bonded, with at least one header to every three stretchers, uniformly distributed. No stone shall be less than eight (8) inches thick, nor measure less than twelve (12) inches in its smallest horizontal dimension, nor less than three (3) square feet in horizontal area, nor have less width than thickness. Headers must be at least three feet long. The thickest stones shall be placed toward the bottom. Joints shall be broken at least eight (8) inches. Beds and builds shall be dressed to lay not over ¾ inch joints, with provision for cavities as in pier and abutment masonry, except that cavities in the top surface of a stone may be two inches deep. Vertical joints are to be dressed to ½ inch joints for eight inches back from the face. Face of stones to be split or quarry faced, pitched to line and batter, with no projections over five inches, and no hollow faces. All stones are to be laid solid in cement mortar, vertical joints filled with the same, and face joints pointed. Backing is to be of concrete as provided in abutment masonry.

The retaining walls on the north approach are to have a coping. Retaining walls on the south approach are to have no coping, but the top course must be fairly uniform in depth and have a fairly even top surface.

Coping stones on piers are to have dimensions indicated on the plans; they are to be dressed for one-half-inch bed joints and vertical joints, and peen hammered full to

line on top. Stones under truss bearings are to be finished on top to grade with four-cut work. Faces are to be quarry faced, pitched to line, and of same quality of finish as specified for pier masonry.

The end stones of each coping course are to be doweled to the course below with two one and one-fourth ($1\frac{1}{4}$)-inch round iron dowels, holes for dowels to extend through coping stones and five inches into course below, and to be filled with neat cement mortar above and around dowels.

End stones are to have one and one-half ($1\frac{1}{2}$) inch chisel draft on each side of corners.

The coping courses are to be carefully laid in Portland cement mortar and the joints pointed.

The coping for wing walls and retaining walls on the north approach must be of stone averaging at least six feet in length laid true to line and out of wind. The tops of coping included within the sidewalk must be fine pointed to a lateral slope of one-half inch to one foot.

The face stones of bridge seats are to have tops rough pointed, full to line and to have a one and one-half ($1\frac{1}{2}$)-inch chisel draft cut its entire length after the course is laid.

Stones under truss bearings are to be finished on top to grade with four-cut work.

The backing of the bridge-seat course is to be of concrete.

The parapet courses are to be of the form and dimensions shown. In the lower course the stretchers are to be two (2) feet deep and the headers the full depth of the wall.

The vertical joints and beds are to be dressed for one-half ($\frac{1}{2}$)-inch joints.

The face of the parapet and its top under the roadway is to be rough pointed; the top of the parapet at the sidewalks and the exposed face at the curb to be peen hammered.

Provision shall be made for drainage of the water in the banks behind the walls, by weep holes three (3) inches wide and at least one (1) foot high, one such hole for not more than five square yards of front of wall.

The tops of back walls must be dressed true to form of roadway surface.

Back walls must be set to elevation of surface of roadway on the basis of using four (4)-inch paving block. Flooring with two (2)-inch wearing surface may be blocked up at the end panels to meet this elevation.

Suppose, as in an actual specification for a rectangular tapering pier, the specification reads:

Header stone shall have a width on the face not less than one and one-quarter times the vertical dimension of the course, and they shall be at least three-quarters of this width for their full depth. The header stones of any course shall average full width for their full depth. Headers shall have a length of 24 inches. Stretchers shall as a rule have a width on the face equal to not less than two times their rise. They shall have a depth of 12 inches.

The dimensions used here in the first sentence are: "width on the face," "vertical dimension," and "depth." "Depth" must, therefore, mean the dimension perpendicular to the face, and the header stones must be at least three-quarters of the width on the face for their full depth perpendicular to the face, and yet the next sentence says, "The header stones of any course shall average full width for their full depth," which contradicts the first sentence, unless it is intended or expected

that back of the stretchers the headers could be very irregular, having a width at some points of three-quarters the width on the face and at other points, in order to average the full width, having a width of about one and one-quarter times the width on the face. The third sentence introduces a new dimension, "length." What is this dimension? Obviously, it must mean the dimension perpendicular to the face, or what is above called "depth." The next sentence introduces still another dimension of "rise," which, of course, is the same as "vertical dimension." The last sentence defines the depth of stretchers, which obviously means the dimension perpendicular to the face, but it does not say whether this dimension is measured from the pitch line or from the required projection of the face in front of the pitch line. The paragraph is inexact.

If a specification states that granite masonry shall be paid for "per cubic yard of granite in place," the question will arise whether the joints are included. One specification for a rectangular pier specifies that the facing shall be paid for "per cubic yard of granite in place, the volume to be determined by the face dimensions multiplied by the specified depth of the stones." This is indefinite, because, in the first place, the depth of the stones is not specified, but only *minimum* dimensions. The contractor should not be paid for minimum dimensions only, unless it is *required* that these shall be the actual dimensions. But, in the second place, if the face dimensions are multiplied by the depth, the quoins or corner stones, will be counted twice; they require more dressing, and cost more than if only one surface showed on the face, and it is fair to count them twice; but it is clear that the specification contains two conflicting requirements. What is the contractor to do if the engineer makes two conflicting statements in his specifications? Clearly he should protect himself.

The quarryman's basis is "rectangular measure," that is, the smallest rectangular block from which the required stone can be cut. To the cost of getting out this block he adds the cost of cutting. If measured in the work, joints must be added; if at the quarry or on cars, they are not. It might be specified that the quarryman should be paid for the number of cubic yards in place, rectangular measure, but this is not done. The quarryman is expected to modify his price so as to yield him a profit when paid for the actual quantity in place.

13. Masonry Specifications of the (A.R.E.A.).—These read as follows, remarks in parenthesis having been added by the writer:

SPECIFICATIONS FOR STONE MASONRY[1]

General

Standard Specifications:

1. The requirements for cement and concrete shall be those adopted by the American Railway Engineering Association.

[1] A.R.E.A. *Manual*, 1921.

Engineer Defined:

2. Where the term "Engineer" is used in these specifications, it refers to the Engineer actually in charge of the work.

General Requirements

Stone:

3. Stone shall be of the kinds designated and shall be hard and durable, of approved quality and shape, free from seams, or other imperfections. Unseasoned stone shall not be used where liable to injury by frost.

Dressing:

4. Dressing shall be the best of the kind specified.

5. Beds and joints or builds shall be square with each other, and dressed true and out of wind (*i.e.*, not warped surfaces). Hollow beds shall not be permitted.

6. Stone shall be dressed for laying on the natural bed (*i.e.*, parallel to the stratification). In all cases the bed (*i.e.*, the width or length) shall not be less than the rise.

7. Marginal drafts shall be neat and accurate.

8. Pitching shall be done to true lines and exact batter. (*Batter* is the slope of the surface, along the line of greatest slope, and is generally expressed in inches horizontally per foot vertically.)

Mortar:

9. Mortar shall be mixed in a suitable box, or in a machine mixer, preferably of the batch type, and shall be kept free from foreign matter. The size of the batch and the proportions and the consistency shall be as directed by the Engineer. When mixed by hand, the sand and cement shall be mixed dry, the requisite amount of water then added, and the mixing continued until the cement is uniformly distributed and the mass is uniform in color and homogeneous.

Laying:

10. The arrangement of courses and bond shall be as indicated on the drawings, or as directed by the Engineer. Stone shall be laid to exact lines and levels, to give the required bond and thickness of mortar in beds and joints.

11. Stone shall be cleansed and dampened before laying.

12. Stone shall be well bonded, laid on its natural bed, and solidly settled into place in a full bed of mortar.

13. Stone shall not be dropped or slid over the wall, but shall be placed without jarring stone already laid.

14. Heavy hammering shall not be allowed on the wall after a course is laid.

15. Stone becoming loose after the mortar is set shall be relaid with fresh mortar.

16. Stone shall not be laid in freezing weather, unless directed by the Engineer. If laid, it shall be freed from ice, snow or frost by warming. The sand and water used in the mortar shall be heated.

17. With precaution, a brine may be substituted for the heating of the mortar. The brine shall consist of one pound of salt to eighteen gallons of water, when the temperature is 32° F.; for every degree of temperature below 32° F., one ounce of salt shall be added.

Pointing:

18. Before the mortar has set in beds and joints, it shall be removed to a depth of not less than one inch. Pointing shall not be done until the wall is complete and mortar set; nor when frost is in the stone.

19. Mortar for pointing shall consist of equal parts of sand, sieved to meet the requirements, and Portland cement. In pointing, the joints shall be wet, and filled with mortar, pounded in with a "set-in" or calking tool and finished with a beading tool the width of a joint, used with a straight-edge. (More or less water will always be

absorbed by the joints near the face, and freezing and thawing may cause this mortar to disintegrate. When this occurs, the disintegrated mortar is raked out and the joint filled with rich mortar firmly pressed in with a trowel or pointing tool which leaves the joint generally with a concave surface. A pointed joint may be struck flush, inclined, or concave, as in Fig. 273.)

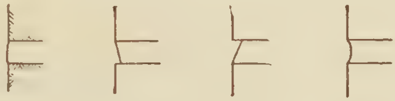


FIG. 273.

Bridge and Retaining Wall Masonry, Ashlar Stone

20. The stone shall be large and well proportioned. Courses shall not be less than 14 inches or more than 30 inches thick, thickness of courses to diminish regularly from bottom to top.

Dressing:

21. Beds and joints or builds of face stone shall be fine pointed, so that the mortar layer should not be more than ½ inch thick when the stone is laid.

22. Joints in face stone shall be full to the square for a depth (*i.e.*, perpendicular to the surface) equal to at least one-half the height of the course, but in no case less than 12 inches.

Face or Surface:

23. Exposed surfaces of the face stone shall be rock faced, with edges pitched to the true lines and exact batter. The face shall not project more than 3 inches beyond the pitch line.

24. Chisel drafts 1½ inches wide shall be cut at exterior corners.

25. Holes for stone hooks shall not be permitted to show in exposed surfaces. Stone shall be handled with clamps, keys, lewis, or dowels.

Stretchers:

26. Stretchers shall not be less than 4 feet long with at least one and a quarter times as much bed (*i.e.*, width perpendicular to face) as thickness of course.

Headers:

27. Headers shall not be less than 4 feet long; shall occupy one-fifth of face of wall; shall not be less than 18 inches wide in face; and, where the course is more than 18 inches high, width of face shall not be less than height of course.

28. Headers shall hold in heart of wall the same size shown in face, so arranged that a header in a superior course shall not be laid over a joint, and a joint shall not occur over a header; the same disposition shall occur in back of wall.

29. Headers in face and back of wall shall interlock when thickness of wall will admit.

30. Where the wall is 3 feet thick or less, the face stone shall pass entirely through. Backing shall not be permitted (in this case).

Backing:

31*a*.[1] Backing shall be large, well-shaped stone, roughly bedded and jointed; bed joints shall not exceed 1 inch. At least one-half of the backing stone shall be of same size and character as the face stone and with parallel ends. The vertical joints in back of wall shall not exceed 2 inches. The interior vertical joints shall not exceed 6 inches. Voids shall be thoroughly filled with { *concrete.* / *spalls, fully bedded in cement mortar* } [spalls are chips broken from a large block].

[1] Paragraphs 31*a* and 31*b* are so arranged that either may be eliminated according to requirements. Optional clauses printed in italics.

31b.[1] Backing shall be of { *concrete.* / *headers and stretchers, as specified in paragraphs* 26 *and* 27, *and heart of wall filled with concrete.*

32. Where the wall will not admit of such arrangement, stone not less than four (4) feet long shall be placed transversely in heart of wall to bond the opposite sides.

33. Where stone is backed with two courses, neither course shall be less than eight (8) inches thick.

34. Bond of stone in face, back and heart of wall shall not be less than 12 inches. Backing shall be laid to break joints with the face stone and with one another.

Coping (*i.e.*, the top course, which generally projects at the face beyond the course below, to shed water):

35. Coping stone shall be full size throughout, of dimensions indicated on the drawings.

36. Beds, joints, and top shall be fine pointed.

37. Location of joints shall be determined by the position of the bedplates, as indicated on the drawings.

Locks:

38. Where required, coping stone, stone in wings of abutments, and stone on piers shall be secured together with iron cramps or dowels, to the position indicated on the drawings. (Dowels are straight iron bars inserted in holes in adjoining stones. Cramps are iron bars with the ends turned at right angles with the length, to be inserted in holes in adjoining stones.)

Bridge and Retaining Wall Masonry, Rubble Stone

39. The stone shall be roughly squared and laid in irregular courses. Beds shall be parallel, roughly dressed, and the stone laid horizontal to the wall. Face joints shall not be more than 1 inch thick. Bottom stone shall be large, selected, flat stone.

40. The wall shall be compactly laid, having at least one-fifth the surface of back and face headers arranged to interlock, having all voids in the heart of the wall thoroughly filled with { *concrete.* / *suitable stones and spalls, fully bedded in cement mortar.*

Arch Masonry, Ashlar Stone [See Chapter 24]

41. Voussoirs shall be full size throughout and dressed true to templet, and shall have bond not less than thickness of stone.

Dressing:

42. Joints of voussoirs and intrados shall be fine pointed. Mortar joints shall not exceed ⅜ inch.

Face or Surface:

43. Exposed surface of the ring stone shall be { *smooth* / *rock faced, with a marginal draft.*

44. Number of courses and depth of voussoirs shall be indicated on the drawings.

45. Voussoirs shall be placed in the order indicated on the drawings.

Backing:

46. Backing shall consist of { *concrete.* / *large stone, shaped to fit the arch bonded to the spandrel and laid in full bed of mortar.*

47. Where waterproofing is required, a thin coat of mortar or grout shall be applied evenly for a finishing coat, upon which shall be placed a covering of approved waterproofing material.

48. Centers shall not be struck until directed by the Engineer (see Chap. 26).

Bench Walls, Piers, Spandrels, Etc.:

49. Bench walls, piers, spandrels, parapets, wing walls and copings shall be built under the specifications for Bridge and Retaining Wall Masonry, Ashlar Stone.

Arch Masonry, Rubble Stone

Dressing:

50. Voussoirs shall be full size throughout, and shall have bond not less than thickness of voussoirs.

51. Beds shall be roughly dressed to bring them to radial planes.

52. Mortar joints shall not exceed 1 inch.

Face or Surface:

53. Exposed surfaces of ring stone shall be rock faced, and edges pitched to true lines.

54. Voussoirs shall be placed in the order indicated on the drawings.

Backing:

55. Backing shall consist of { *concrete.* / *large stone, shaped to fit the arch, bonded to the spandrel, and laid in full bed of mortar.*

56. Where waterproofing is required, a thin coat of mortar or grout shall be applied evenly for a finishing coat, upon which shall be placed a covering of approved waterproofing material. (For information on Waterproofing Masonry, see page 292 (of *Manual*).)

57. Centers shall not be struck until directed by the Engineer.

Bench Walls, Piers, Spandrels, Etc.:

58. Bench walls, piers, spandrels, parapets, wing walls and copings shall be built under the specifications for Bridge and Retaining Wall Masonry, Rubble Stone.

Culvert Masonry

59. Culvert masonry shall be laid in cement mortar. Character of stone and quality of work shall be the same as specified for Bridge and Retaining Wall Masonry, Rubble Stone.

Side Walls:

60. One-half the top stone of the side walls shall extend entirely across the wall.

Cover Stones:

61. Covering stone shall be sound and strong, at least 12 inches thick, or as indicated on the drawings. They shall be roughly dressed to make close joints with each other, and lap their entire width at least 12 inches over the side walls. They shall be doubled under high embankments, as indicated on the drawings.

End Walls, Coping:

62. End walls shall be covered with suitable coping, as indicated on the drawings.

Dry Masonry

63. Dry masonry shall include dry retaining walls and slope walls.

Retaining Walls:

64. Retaining walls and dry masonry shall include all walls in which rubble stone laid without mortar is used for retaining embankments or for similar purposes.

Dressing:

65. Flat stone at least twice as wide as thick shall be used. Beds and joints shall be roughly dressed square to each other and to face of stone.

66. Joints shall not exceed ¾ inch.

Disposition of Stone:

67. Stone of different sizes shall be evenly distributed over entire face of wall, generally keeping the larger stone in lower part of wall.

68. The work shall be well bonded, and shall present a reasonably true and smooth surface, free from holes or projections.

Slope Walls:

69. Slope walls shall be built of such thickness and slope as directed by the Engineer. Stone used in this construction must reach entirely through the wall. Stone shall be placed at right angles to the slopes. The wall shall be built simultaneously with the embankment which it is to protect.

14. Principles Governing the Design of Stone Masonry.—The characteristic of stone masonry is that is it not supposed to be able to resist a tensile stress. Although there is mortar between the stones, and although that mortar has a certain tensile strength, that strength is not relied upon, because it may be destroyed by freezing and thawing; and because it is not only that tensile strength, but the adhesion of the mortar to the stones, which would be called into play by a tension, and that adhesion cannot be depended upon.

It follows that on any plane surface or joint which is imagined to cut the structure in two, the resultant outer force on one side of that section must be a compression, and must cut the joint within the kernel of the section. In masonry, the sections are always or generally rectangles, and the forces act in a plane parallel to one side; hence the resultant must act within the middle third.

There are therefore three principles or conditions governing the design of masonry structures:

1. The resultant on any (rectangular) joint, acting in a plane parallel to one side, must cut the joint within the middle third.

2. The angle between the resultant and the normal to the plane on which it acts must not exceed the angle of repose of stone on mortar.

3. The maximum intensity of pressure must not exceed the allowable stress on the material. (This is generally taken as the allowable stress on the stone, ignoring the mortar joint, because this, being a thin layer, would have a large strength, or, practically, could not fail.)

15. The angle of repose will depend on the smoothness of the joint. Bed joints, however, since they are filled with mortar, are never cut smooth, and indeed it is a disadvantage to cut them any smoother than will prevent direct bearing of any point of one stone on another. The angle of repose will therefore be never less than 30°.

16. A concrete structure, if not reinforced, is considered subject to the first and third of the above conditions, because tension of the concrete is not relied upon. There are no bed joints in such a structure, but planes are taken passing through it in the same direction that the joints would have in stone masonry. The second condition is replaced by the requirement that the shear must not exceed what is allowable.

A reinforced concrete structure, since steel rods are placed where tension tends to exist, is not subject to the first condition. The resultant on a section may pass outside the middle third, or outside the section itself. The structure is an elastic one and theoretically should be so considered; while a structure of stone masonry is only an elastic structure subject to the restriction of the first condition, and in view of the existence of actual joints, is considered as a series of blocks or courses placed one on the other and subject only to compression.

17. Maximum Compressive Intensity.—If the width of a joint is b, the thickness unity (Fig. 274) and if the resultant R, whose component

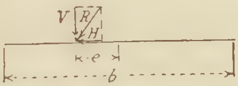


FIG. 274.

perpendicular to the joint is V, cuts the joint at a distance e from the center b, the moment on the joint is Ve, and the maximum and minimum stresses are given by the formula for combined direct stress and bending:

$$p = \frac{V}{A} \pm \frac{Mv}{I}$$

or

$$\text{maximum } p = \frac{V}{b} + \frac{Ve\frac{b}{2}}{\frac{1}{12}b^3} = \frac{V}{b} + \frac{6Ve}{b^2} \tag{1}$$

$$\text{minimum } p = \frac{V}{b} - \frac{6Ve}{b^2} \tag{2}$$

If $e = b/6$, that is, if V acts at the middle third point,

$$\text{maximum } p = \frac{2V}{b} \tag{1a}$$

$$\text{minimum } p = 0 \tag{2a}$$

or the maximum is twice the average.

18. Design of a Masonry Structure.—The design of a masonry structure requires merely that the dimensions be tentatively assumed at first, and then, from the known outer forces, that the resultant on joints at proper distances apart be found. If these fulfil the requirements of Art. 14, the section is safe. If the margin is too great, a new and smaller section may be assumed; if the conditions are not fulfilled, the section must be changed until they are fulfilled. If the second condition is the only one which is not fulfilled, the actual joints may be changed in direction so that it will be fulfilled. In this way a correct and economical design may be arrived at. The sections investigated must be taken at distances apart determined by judgment, so as to insure that at no inter-

mediate section will the conditions fail to be fulfilled. Sometimes, as in a retaining wall, it may be necessary to consider only one section, namely, the base, where the wall rests on the foundation.

The allowable compression will depend upon the kind of stone.[1] The coefficient of friction may generally be taken at 0.5 with safety.

19. Workmanship.—In laying stone masonry, workmanship is important as well as material. Professor Baker gives the following rules[2] which should be adhered to.

General Rules

The following general principles apply to all classes of stone masonry:

1. The largest stones should be used in the foundation to give the greatest strength and lessen the danger of unequal settlement.

2. A stone should be laid upon its broadest face, since then there is better opportunity to fill the spaces between the stones.

3. For the sake of appearance, the larger stones should be placed in the lower courses, the thickness of the courses decreasing gradually toward the top of the wall.

4. Stratified stones should be laid upon their natural bed, *i.e.*, with the strata perpendicular to the pressure, since they are then stronger and more durable.

5. The masonry should be built in courses perpendicular to the pressure it is to bear.

6. To bind the wall together laterally, a stone in any course should break joints with or overlap the stone in the course below; that is, the joints parallel to the pressure in two adjoining courses should not be too nearly in the same line. This is briefly comprehended by saying that the wall should have sufficient lateral bond.

7. To bind the wall together transversely, there should be a considerable number of headers extending from the front to the back of thin walls or from the outside to the interior of thick walls; that is, the wall should have sufficient transverse bond.

8. The surface of all porous stones should be moistened before being bedded, to prevent the stone from absorbing the moisture from the mortar and thereby causing it to become a friable mass.

9. The spaces between the back ends of adjoining stones should be as small as possible, and these spaces and the joints between the stones should be filled with mortar.

10. If it is necessary to move a stone after it has been placed upon the mortar bed, it should be lifted clear and be reset, as attempting to slide it is likely to loosen stones already laid and destroy the adhesion, and thereby injure the strength of the wall.

11. An unseasoned stone should not be laid in the wall, if there is any likelihood of its being frozen before it has seasoned.

In laying brick masonry, the following rules are given by C. C. Williams.[3]

1. Lay the brick in a full bed of mortar.
2. Shove the brick into place and apply sufficient pressure to bed firmly.
3. Keep all leads and corners plumb.
4. Slosh the joints full of mortar.
5. Use good mortar.

[1] See vol. II of this series for data.

[2] "Masonry Construction," New York, John Wiley & Sons p. 282, 1909.

[3] "Design of Masonry Structures, and Foundations" p. 44, McGraw-Hill Book Company, Inc., 1922.

6. Put in plenty of headers.

7. Wet all brick before laying, especially if dry and porous (that it may not absorb the water from the mortar).

8. Strike all exposed joints neatly (*i.e.* finish the joints neatly flush with the brickwork).

20. References on Masonry.—For nomenclature and classification of masonry, reference is made to the paper, above referred to,[1] though the description of stone-cutting tools is a little out of date owing to the general introduction of pneumatic surfacers.[2]

[1] *Trans. Am. Soc. C. E.*, 1877.

[2] Good treatises on masonry are the following: BAKER, IRA O., "A Treatise on Masonry Construction," John Wiley & Son, Inc., tenth ed., 1909 (a standard and excellent work); WILLIAMS, CLEMENT C., "The Design of Masonry Structures and Foundations," McGraw-Hill Book Company, Inc., 1922 (a recent and excellent work); HOOL and KINNE, "Reinforced Concrete and Masonry Structures," McGraw-Hill Book Company, Inc., 1924 (the latest work, consisting of a series of chapters by different authors and dealing mainly with concrete). One or both of the first two of these works should be in the library of the engineer who has to do with stone masonry.

CHAPTER XXII

RETAINING WALLS

1. A retaining wall is a wall of masonry or concrete which sustains (or retains) behind it a mass of earth, which may be either in its natural or unloaded condition, or may be loaded or surcharged with some applied load, as in the case of a dock wall behind which are loads of merchandise, cranes, or railroad tracks, or as in the case of a railroad which has been elevated and is carried on an earth embankment between retaining walls in order that too much ground may not be occupied by the slopes.

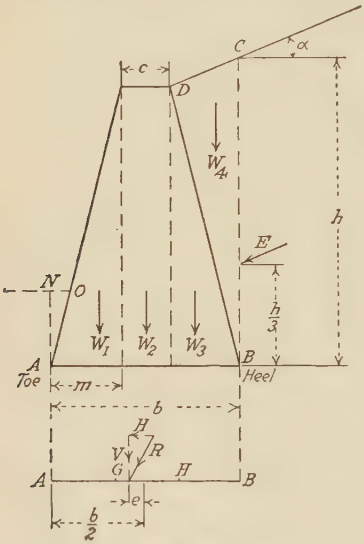


Fig. 275.

2. The first problem in the design of a retaining wall is to find the earth pressure, which is the outer force applied to it. This may be done by the principles explained in Chap. XX. In the present chapter the earth pressure E will be assumed to be known in direction, point of application, and magnitude, so that all that remains to be done is to proportion the wall.

3. General Principles of Design of Masonry or Plain Concrete Walls.—Bearing in mind the three principles which govern the design of masonry structures, including plain or unreinforced concrete, as stated in the previous chapter, the design simply requires that those three principles be fulfilled. Considering the base of the wall, it is necessary that (1) the resultant force on the base must lie inside the middle third; (2) the resultant must not make with the normal to the base a greater angle than the angle of repose of masonry on earth; (3) the maximum pressure on the base must not exceed the allowable pressure intensity on the foundation.

To determine these points, it is only necessary to find the resultant on the base.

The shape and dimensions of the wall must first be assumed. Let the shape be that in Fig. 275. Then the earth pressure E and the weight of the wall are fully known. In the opinion of the writer, it is most

convenient to consider the material to the left of a vertical BC through the heel or inside edge B of the wall. In this case the only earth pressure that need be considered is that on a vertical plane BC, which is assumed to act parallel to the surface, and at a height above B equal to $\frac{1}{3}BC$. The pressure on the back of the wall BD will be this pressure E on BC combined with the weight of the prism of earth BDC, but it is entirely unnecessary to find this pressure on BD. The pressure on BC is given by the formula

$$E = \frac{1}{2}wh^2 \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\varphi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\varphi}} \tag{1}$$

where φ is the angle of repose of earth. For horizontal earth surface, this becomes

$$E = \frac{1}{2}wh^2 \frac{1 - \sin\varphi}{1 + \sin\varphi} = \frac{1}{2}wh^2K \tag{2}$$

w is the weight of a cubic unit of earth.

4. If the pressure were a hydrostatic pressure due to a material weighing w pounds per cubic foot, the pressure would be $\frac{1}{2}wh^2$, so that the actual pressure is this hydrostatic pressure multiplied by the factor $\frac{1 - \sin\varphi}{1 + \sin\varphi} = K$. It is well to remember the following values of this factor K:

For $\alpha = 30°$, $K = 0.333 = \frac{1}{3}$
For $\alpha = 45°$, $K = 0.176 =$ nearly $\frac{1}{6}$
For $\alpha = 60°$, $K = 0.072 =$ nearly $\frac{1}{14}$.

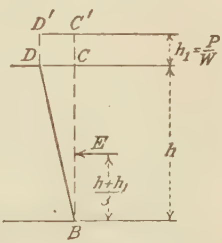


Fig. 276.

5. If the earth slopes, as in Fig. 275, it is said to be *surcharged*. Only if it is horizontal, with no load upon it, is it not surcharged. If it is horizontal but with a uniform load of p per foot upon it, the load p is equivalent to a height of earth h, equal to p/w (Fig. 276) and the pressure E on BC' is

$$E = \tfrac{1}{2}w(h + h_1)^2K \tag{3}$$

acting at a distance $\frac{1}{3}(h + h_1)$ above B. The condition will be most unfavorable if the surcharge is considered to act only up to C and not over CD, because the load C to D would add to the stability.

If the earth surface slopes at the angle α, it may also have a superimposed load, which should be reduced to an equivalent height of earth h_1.

6. The section of the wall may be divided into triangles and rectangles, as indicated, and the weight of each section for 1 foot of length may be found. E may be resolved into vertical and horizontal components. The total vertical force on AB will be $V = W_1 + W_2 + W_3 + W_4 +$

the vertical component of E; the horizontal force H on AB will be the horizontal component of E. The ratio of the latter to the former must not exceed the coefficient of friction on the base.

To find the resultant R on the base it is only necessary to take moments about A of the forces W_1, W_2, W_3, W_4, and E. Dividing the total moment about A by the total vertical force on AB gives the distance from A to the point where the resultant strikes the base, if the base AB is horizontal. This distance should exceed $AB/3$ and be less than $2AB/3$, in order to lie inside the middle third.

If this resultant lies within the middle third, the maximum and minimum intensity of pressure will be

$$\left.\begin{aligned}\text{maximum } p &= \frac{V}{b} + \frac{6Ve}{b^2}\\ \text{minimum } p &= \frac{V}{b} - \frac{6Ve}{b^2}\end{aligned}\right\} \tag{4}$$

if e is the distance from the center of AB to the point where the resultant strikes AB.

If the resultant passes through the outer middle third point G, $e = b/6$, and

$$\left.\begin{aligned}\text{maximum } p &= \frac{2V}{b}\\ \text{minimum } p &= 0\end{aligned}\right\} \tag{5}$$

In other words, the pressure at B is zero, and that at A is twice the average on AB.

If the resultant passes outside the middle third, at a distance d from A, the total vertical pressure V is distributed over a distance equal to $3d$, or over less than the base, and the maximum which occurs at A is

$$\text{maximum } p = \frac{2V}{3d} \tag{6}$$

In this case there is a tendency for the wall to rise from the foundation over the portion of the base on which there is no pressure.

The base of a wall is usually horizontal; if it is inclined, the moment about A divided by the component of R *perpendicular to the base* will give the distance from A to the point where R cuts the base.

If there is earth to the left of the wall, as indicated by the dotted line, the pressure on AN should be found; and this pressure, together with the weight of the prism of earth ANO, must be taken account of in taking moments about A. In any case. it is only necessary to find the earth pressure on a vertical surface.

These simple principles are all that are necessary in order to judge of the safety of the wall. No other formulae are necessary.

7. In what has preceded, the dimensions of the wall have been assumed as given. If the wall is to be designed, these dimensions are the very things to be determined. This may be done by assuming dimensions, determining the pressures, and then changing them till desired dimensions are found which will give proper pressures. In Arts. 10 to 12 some data are given which will serve as a guide in making the first tentative design.

8. Factor of Safety.—This term is sometimes used in regard to retaining walls to signify the factor by which E must be multiplied in order that R may pass through the toe A; or, it may be referred to the outer middle third point G. There seems to the writer no special importance to be attached to this factor of safety. It may easily be computed in any given case by prolonging E on the back of the wall to meet the resultant weight of the wall, and finding the value of E necessary for R to pass

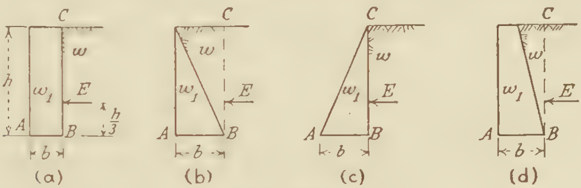


FIG. 277.

through A, and dividing this by the actual E. If the wall is triangular with a vertical back (Fig. 277c) and the resultant actually goes through the middle third

$$E\frac{h}{3} = \frac{w_1 h b^2}{6}$$

In order that the resultant should go through A, the earth pressure should be E_1 and

$$E_1\frac{h}{3} = \frac{w_1 h b^2}{3}$$

or the factor of safety is two.

If the wall is rectangular (Fig. 277a),

$$E\frac{h}{3} = \frac{w_1 b^2 h}{6}$$

$$E_1\frac{h}{3} = \frac{w_1 b^2 h}{2}$$

or the factor of safety is 3.

9. Influence of Character of the Foundation Soil.—If the resultant R on the base does not cut the base at its center, the normal pressure will be distributed unevenly over the base, being greatest at the edge which is nearer the resultant and least at the other edge.

If the foundation is incompressible, as rock, or hard clay or hardpan which is not likely to become compressible by the action of moisture, it may be unnecessary for the pressure to be uniformly distributed. It may be allowable for the resultant to be even outside the middle third, if its position does not cause more than the allowable pressure on the material at the nearer edge. A position outside the middle third, however, would tend to make the foundation joint open at the inner edge since no material is absolutely incompressible, and so may allow the entrance of water, if water is present in the soil. Generally the resultant should not be allowed in any case to pass outside the middle third, in order to insure some pressure at every point of the base.

If the wall rests on compressible material, the case is very different. Here, if the pressure is uniform and the material uniform, and if the resultant cuts the center of the base, the pressure will be uniform, and the compression will be the same at every point, so that the wall will merely settle vertically if the base is horizontal. But if the resultant cuts the base not at the center, there will be a greater pressure at one edge than at the other, and the wall *will surely tip*. If the resultant cuts the base inside the center, *i.e.*, toward the earth, the wall will tend to tip inward; this will merely increase the earth pressure, which we know from Chap. XX may vary from the active to the much greater passive pressure; and while this may compress the earth and cause a slight tipping, it can never become dangerous or cause failure. But if the resultant cuts the base outside the center, the wall will tend to tip, and will actually tip, outward. Even this, if not too great, may not be dangerous, as in the case of a railroad fill between retaining walls; but it is not desirable. If a retaining wall supports the brick wall of a building, any tipping outward is very objectionable and even a slight tipping may be dangerous and may cause collapse; in such a case, any movement of the top of the wall outward makes the weight of the brick wall act excentrically on the base, and so causes increased tipping.

On compressible soil, therefore, the resultant should in general cut the base *at or inside the center*. If a retaining wall carries a brick wall on top, as in a wall for underpinning, the resultant should under no circumstances on a compressible soil cut the base outside the center.

10. Shapes of Section for Retaining Walls.—Three typical shapes of section are indicated in Fig. 277, the rectangle, the triangle with vertical face, and the triangle with inclined face. The top must always have a certain width, as in section (d) but for the present we may consider the other three sections only. In all three, the earth pressure on the plane BC will be the same. In (a) and (b) the resultant on the base *must* always lie to the left of the center of the base, because the weight of the wall itself together with that of the earth above its back as in (b) acts at or to the left of the center. In section (c) the width b may be such as to make

the resultant go through the center; in all, it may be made to go through the outer middle third point.

Calling w_1 and w the weight per cubic foot of wall and of earth, let us find the value of b which will make R go through the outer middle third point.

$$\left.\begin{aligned} &\text{Section } (a)\colon \frac{Eh}{3} = \frac{w_1hb^2}{6};\ b = \sqrt{\frac{2E}{w_1}} \\ &\text{Section } (b)\colon \frac{Eh}{3} = \frac{whb^2}{6};\ b = \sqrt{\frac{2E}{w}} \\ &\text{Section } (c)\colon \frac{Eh}{3} = \frac{w_1hb^2}{6};\ b = \sqrt{\frac{2E}{w_1}} \end{aligned}\right\} \tag{7}$$

The necessary width is the same in sections (a) and (c) and larger in section (b). Section (c) is the most economical of material, because w is less than w_1. If w is assumed to be $\frac{2}{3}w_1$, and it will generally be as large as this, or larger, the width b in section (b) will be 1.22 that in sections (a) or (c).

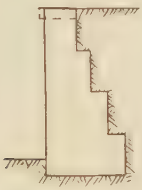


FIG. 278.

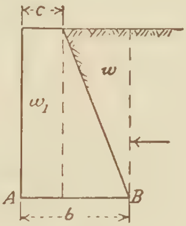


FIG. 279.

In stone masonry, it is expensive to cut the face to a slope. A vertical face is simpler, less expensive, and looks better. The section is therefore generally made with the back face rough and in irregular steps, as in Fig. 278. There should be an overhanging coping or upper course, as indicated by the dotted lines to prevent water from the ground running over and down the front of the wall.

Equations (7) may be used as a guide in making a first tentative design even for an actual section in which there is a top width, as in Fig. 277d. Formulae may be deduced for such cases, however, if a top width is assumed, as in Fig. 279. The equation of moments about the outer middle third point will be

$$E\frac{h}{3} = w_1ch\left(\frac{c}{2} - \frac{b}{3}\right) + \frac{w_1(b-c)h}{2}\left[\frac{2b}{3} - \frac{2(b-c)}{3}\right] + \frac{w(b-c)h}{2}\left[\frac{2}{3}b - \frac{b-c}{3}\right] \tag{8}$$

$$b = \sqrt{\frac{2E - c^2(w_1 - w)}{w}} \tag{9}$$

A similar equation may be found for a section like Fig. 280 in which both front and back faces are inclined. The equation will be

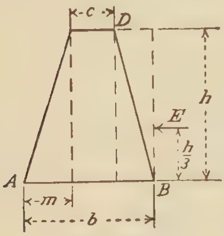


FIG. 280.

$$E\frac{h}{3} = \frac{wh(b-m-c)}{2}\left[\frac{2}{3}b - \frac{b-m-c}{3}\right] + \frac{w_1h(b-m-c)}{2}\left[\frac{2}{3}b - \frac{2}{3}(b-m-c)\right] + w_1ch\left[\frac{2}{3}b - \frac{c}{2} - (b-m-c)\right] + \frac{w_1mh}{2}\left(\frac{2}{3}m - \frac{b}{3}\right)$$

from which

$$b = -\frac{w_1m}{2w} + \sqrt{\frac{2E}{w} + (c+m)^2 - \frac{w_1}{w}(c^2+2cm) + \frac{w_1^2m^2}{4w^2}} \qquad (10)$$

If $m = 0$, Eq. (10) reduces to Eq. (9), as it should.

These equations tell us nothing about the maximum pressure on the foundation. They merely have reference to the resultant passing through the outer middle third of the base. In each case the maximum pressure intensity, at the outer edge, is twice the average. Hence:

$$\left.\begin{array}{l} \text{Section } 3(a)\text{: maximum } p = 2w_1h_1 \\ \text{Section } 3(b)\text{: maximum } p = (w+w_1)h_1 \\ \text{Section } 3(c)\text{: maximum } p = w_1h_1 \\ \text{Section Fig. 279: maximum } p = (w+w_1)h + \frac{hc}{b}(w_1 - w) \\ \text{Section Fig. 280: maximum } p = (w+w_1)h + \frac{hc}{b}(w_1 - w) - \frac{mh}{b}w \end{array}\right\} \qquad (11)$$

Hence not only is (c) the most economical of material, but the maximum pressure on the base is less than in any of the other sections. Therefore for economy of material the section should approach as near to c as practicable.

Almost all stone retaining walls, unless of rubble, are laid with horizontal joints, a coping on top, and stepped at the back.

11. Necessary Width of Base.—Equations (7), (9), or (10) may be used to find the necessary width of base if the resultant is to go through the outer middle third, by substituting the value of E in Eq. (2). The width will depend upon the value of K, *i.e.*, upon the angle of repose assumed. From Eqs. (7) and (2):

$$\text{For section } (b)\text{: } b = h\sqrt{K} \qquad (12)$$

$$\text{For section } (c)\text{: } b = h\sqrt{\frac{wK}{w_1}} \qquad (12a)$$

From Eq. (9), if $c/h = r$, and $w/w_1 = K_1$

$$b = h\sqrt{K + r^2 - \frac{r^2}{K_1}} \tag{13}$$

If $r = 0$, Eq. (13) becomes Eq. (12), as it should.

For ordinary earth fill, it is very common to assume the angle of repose as 30°; or as the angle corresponding to a slope of 1½ horizontal to 1 vertical, which is a little less than 34°, giving a value of K for horizontal earth surface of 0.286. If we assume $K = 0.3$, $K_1 = 0.667$, we have from Eqs. (12) and (12*a*):

For section 3(*b*): $b = 0.55h$

For section 3(*c*): $b = 0.45h$

and from Eq. (13),

$$h = h\sqrt{0.3 - 0.5r^2}$$

or, if $r = \frac{1}{5}$, $b = 0.53h$

This is really a flatter angle of repose than it is necessary to assume; for an earth fill behind a wall, if properly put in, will become compacted in time. If we assume $K = 0.2$, $K_1 = 0.667$, and $r = 0.2$, we have

From Eq. (12): $b = 0.45h$

From Eq. (12*a*): $b = 0.365h$

From Eq. (13): $b = 0.424h$

The effect of a sloping earth surface, as in Fig. 275, will not be great, but the effect of a superimposed surcharge, as, for instance, railway tracks behind the wall, may be very considerable, particularly if impact is considered.

12. Arbitrary Rules for Width of Base.—Though the width of base will obviously depend upon conditions, yet, on account of the uncertainty of the earth pressure, arbitrary rules have often been suggested.

Baker's Rule.—The English engineer, Sir Benjamin Baker (with Sir John Fowler the designer of the Forth bridge), stated[1] that a wall to sustain earth with a level surface, having a batter or slope of 1 or 2 inches per foot on the front, when the backing and foundation are favorable, would be safe if

$$b = \frac{h}{4}$$

and that for no ordinary conditions of surcharge or backing if the foundation is solid, need the width be greater than

$$b = \frac{h}{2}$$

In his own practice, in ground of average conditions, he made

$$b = \frac{h}{3}$$

[1] *Proc. Inst. C. E.*, vol. LXV, p. 184.

The retaining walls of the London underground railways were designed on the basis, without any instance of settlement, sliding, or overturning.

Trautwine gives the following:

For wall of cut stone, or first-class rubble in mortar. . . . . $b = 0.35h$
For good common mortar rubble, or brick. . . . . $b = 0.4h$
For well scabbled dry rubble. . . . . $b = 0.5h$

13. For the sections thus far considered, it is only necessary to consider the bottom of the wall in computing pressures. On any other horizontal plane through the wall, not only is the allowable pressure on the masonry greater than that on the foundation, and the allowable angle of the pressure on the joint with the normal greater than on the foundation, but if the resultant strikes the base at the outer middle third point, the resultant on a joint higher up will be inside the middle third. The earth pressure above any horizontal joint (Fig. 275), will vary as the square of the distance below the surface, and its moment about the outer edge will vary as the cube of that distance. At a depth of $CB/2$, the moment of E will be one-eighth of the moment for the base. On the other hand, in Fig. 277(a) the moment of the weight will vary as the depth, and in Fig. 277(c) as the cube of the depth. The moment of the weight will not decrease as we ascend in the wall faster than the overturning moment of E. Hence if the lower joint is safe, any joint above it will be safe.

14. In Arts. 3 and 10, the resultant earth pressure E has been assumed to act at one-third the height above the base. In the chapter on Earth Pressure it has been explained that in some cases it is found to lie higher. Some writers recommend taking it at $0.4h$ above the base. The writer uses one-third. This is one of the uncertainties of the subject.

15. Other Forms of Section.—In order that as great a weight of earth as possible may act to increase the stability, or may have a right-handed moment about the toe, the shape of the section is often made as in Fig. 281. In this case all the earth to the left of BC acts to increase the stability. It is necessary, however, to investigate the section HG. This wall is really only a wall $HGDK$, which is like those previously discussed, set upon a base $ABB'A'$. Such a shape may be desirable if the allowable pressure on the foundation is small. The section FG must also be investigated, and in stone masonry the resultant force on FG should act within the middle third of FG. The projection below AB, shown dotted, may be desirable as a precaution against sliding.

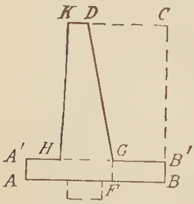


Fig. 281.

16. Distribution of Surcharge.—If the earth behind a retaining wall carries railroad tracks, the surcharge per square foot p equals the total load on the track per lineal foot divided by the distance from center to center of tracks if this is less than 14 feet; if not, divided by 14 feet. If

there is a building wall, the surcharge is the weight on the wall per foot divided by the width of the foundation. The equivalent height of surcharge is the surcharge per square foot divided by the weight of a cubic foot of earth.

There may be question, however, as to the surface over which this surcharge acts, depending upon the position of the tracks or the building wall. If the tracks or wall lie entirely outside the "wedge of rupture," that is, to the right of BN (Fig. 282), they clearly do not affect the earth

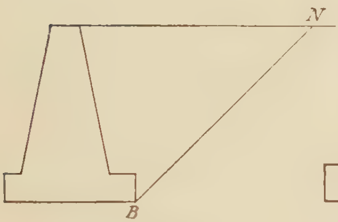


Fig. 282.

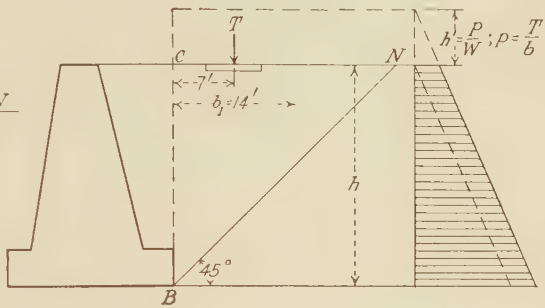


Fig. 283.

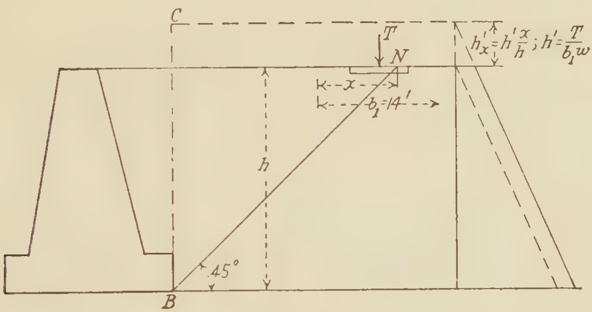


Fig. 284.

pressure. But the wedge of rupture is somewhat uncertain, and it is a question what to do if the surcharge acts partly to the left of BN. The rules of the A.R.E.A. are as follows:

If the 14 feet or b_1 reaches up to or to the left of C, that is, if the center of the nearest track is 7 feet or less to the right of C, the surcharge is assumed to extend up to C, but not to the left of C (Fig. 283).

If the 14 feet or b_1 extends to the left of the line BN drawn at 45°, extending x to the left of N, the surcharge is still assumed to cover the ground to the right of C (Fig. 284) but the value of the surcharge is reduced to $h'_x = h'\frac{x}{h}$. A somewhat similar procedure is followed if a wall overlaps the line BN.

If the 14 feet lies entirely to the right of BN, no surcharge is assumed.

17. Reinforced Concrete Retaining Walls.—Thus far only walls of stone masonry or unreinforced concrete have been considered. Walls of reinforced concrete may be made much thinner, becuse it is not necessary to confine the point of application of the resultant on a horizontal section *through the concrete* to the middle third. It may cut the section outside the middle third, or even outside the section, since the section can resist tension on the inside.

On the base, however, the principles previously explained still apply. The vertical wall may therefore be thin, but the base must be wide, perhaps even wider than for a masonry wall. This is because, in computing the pressure on the base by taking moments about the outside or toe[1] of the base, there are two moments of opposite sign, the "overturning moment" of the earth pressure, and the "moment of stability" of the weights. The overturning moment will be the same for a reinforced concrete wall as for a masonry wall; but the moment of stability may be

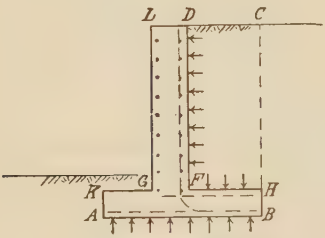


Fig. 285.

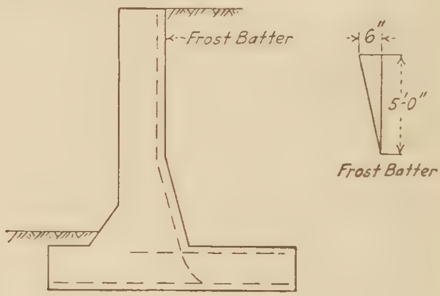


Fig. 286.

less for a reinforced concrete wall, depending upon the shape, and the total weight or vertical resultant on the base may also be less, because the area of section is less.

The section of a reinforced concrete wall may be composed essentially of a thin vertical wall, and a wide and flat base, as in Fig. 285. The position of the vertical wall upon the base may be different according to circumstances.

The vertical wall is a vertical beam or cantilever fixed at the bottom and exposed to a horizontal load of earth pressure, increasing in intensity from zero at the top, if there is no superimposed load, to a maximum at F. The moment will be a maximum at F, and the inside face must be reinforced by vertical rods.

There are also, in general, horizontal rods near the front and back faces, to reduce the tendency to crack. The horizontal shear must be provided for in the way suitable for reinforced concrete beams; usually the area of concrete will be sufficient for this.

[1] In Fig. 275 the point A is called the *toe*, and B the *heel* of the section.

The base is exposed to upward loads, distributed according to the position of the resultant, and is therefore exposed to bending moment and shear, as in any beam. The entire design is nothing more than that of a concrete beam. There will be tension at the bottom of the part KG, and at the top or bottom of FH, depending upon the pressures, and shear in both of these parts. Frequently there are horizontal rods at the top and bottom, as indicated.

Since an angle in a beam is bad, particularly a large angle, it is better to make the section as in Fig. 286, though the arrangement of rods may be varied. The front and back may be inclined from the top down. There may also be counterforts behind the vertical stem, as described in Art. 22. The earth, when freezing, may adhere to the back of the wall and tend to lift it, so that a so-called "frost batter" is often used as in Fig. 286, extending down about 5 feet, or as far as frost penetrates, and about 6 inches offset at the top.

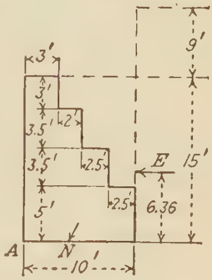


Fig. 287.

Further discussion of retaining walls of reinforced concrete will be found in the chapter on Reinforced Concrete in the next volume.

18. Examples.

1. Consider the wall shown in Fig. 287, and suppose that back of the wall there are railroad tracks. Let w_1 = 150 pounds per cubic foot, w = 110, and the weight on the heaviest engine driving axle 75,000 pounds. The engine load may be considered distributed over the distance between tracks, or 13 feet, and over a longitudinal distance of 6 feet, giving, say, 1,000 pounds per square foot. Impact will not be added. This load corresponds to a height of surcharge of about 9 feet. Let the angle of repose be about 35°, or take $K = 0.28$.

Then $E = 110 \times \frac{24 \times 24}{2} \times 0.28 - 110 \times \frac{9 \times 9}{2} \times 0.28 = 8{,}870 - 1{,}247 = 7{,}623$ pounds

Overturning moment = $8{,}870 \times 8 - 1{,}247 \times 18 = 48{,}510$ foot-pounds.

Moment of the weight about A =

$$
\begin{array}{lll}
3 \times 15 \times 150 \times 1.5 & = 6{,}750 \times 1.5 & = 10{,}125 \\
+2 \times 12 \times 150 \times 4 & = 3{,}600 \times 4.0 & = 14{,}400 \\
+2.5 \times 8.5 \times 150 \times 6.25 & = 3{,}187.5 \times 6.25 & = 19{,}922 \\
+2.5 \times 5 \times 150 \times 8.75 & = 1{,}875 \times 8.75 & = 16{,}406 \\
+2 \times 3 \times 110 \times 4 & = 66.0 \times 4.0 & = 2{,}640 \\
+2.5 \times 6.5 \times 110 \times 6.25 & = 1{,}787.5 \times 6.25 & = 11{,}172 \\
+2.5 \times 10 \times 110 \times 8.75 & = \underline{2{,}750} \times 8.75 & = \underline{24{,}062} \\
 & 20{,}610 & 98{,}727
\end{array}
$$

$$\text{Distance } AN = \frac{98{,}727 - 48{,}510}{20{,}610} = 2.44 \text{ feet}$$

This shows that N is outside the middle third, and

$$\text{maximum } p = \frac{2 \times 20{,}610}{3 \times 2.44} = 5{,}630 \text{ pounds per square foot}$$
$$= 2.8 \text{ tons per square foot}$$

The section is sufficient if this pressure is within the bearing power of the foundation, but the wall will tip outward slightly if on compressible soil, and the base must be wider if the resultant is to lie within the middle third.

Without the surcharge,

$$E = 110 \times \frac{15 \times 15}{2} \times 0.28 = 3{,}465 \text{ pounds}$$

Overturning moment $= 3{,}465 \times 5 = 17{,}325$ foot-pounds.
Moment of stability $= 98{,}727$ foot-pounds.

$$AN = \frac{98{,}727 - 17{,}325}{20{,}610} = \frac{81{,}402}{20{,}610} = 3.95 \text{ feet}$$

or nearer the center than it was with surcharge, and well within the middle third.

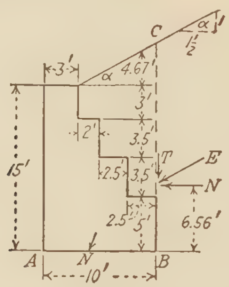


Fig. 288.

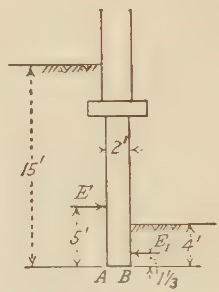


Fig. 289.

2. Consider the section shown in Fig. 288, which is the same as Fig. 287 except that the earth slopes $1\frac{1}{2}$ to 1. Then $\cos \alpha = 1.5/\sqrt{3.25} = 0.833$. Assume $\varphi = 45°$, or $\cos \varphi = 1/\sqrt{2}$. The factor K in Eq. (1) is

$$\frac{0.833 - \sqrt{0.692 - 0.5}}{0.833 + \sqrt{0.692 - 0.5}} = \frac{0.395}{1.271} = 0.31$$

$$E = 110 \times \frac{\overline{19.67}^2}{2} \times 0.31 = 6{,}597 \text{ pounds}$$

This acts parallel with the surface 6.56 feet above the base. Its components normal and tangential to BC are

$$N \text{ (normal)} = 6{,}597 \cdot \frac{15}{18} = 5{,}498 \text{ pounds}$$

$$T \text{ (tangential)} = 6{,}597 \cdot \frac{10}{18} = 3{,}665 \text{ pounds}$$

The overturning moment is

$$5,498 \times 6.56 - 3,665 \times 10 = -583 \text{ foot-pounds}$$

The minus sign shows that the line of E produced meets the base to the right of A. The moment of stability may be found from the last part of the previous problem to be

$$98,727 + 110 \times \frac{1}{2} \times 7 \times 4.67 \times (\frac{2}{3} \cdot 7 + 3) = 112,501 \text{ foot-pounds}$$

$$AN = \frac{112,501 + 583}{20,610 + 110 \times \frac{1}{2} \times 7 \times 4.67 + 3,665} = \frac{113,084}{26,072} = 4.34 \text{ feet.}$$

3. A brick wall of a building was under-pinned by piers 2 feet thick (Fig. 289) flaring longitudinally so as to form a continuous foundation at the bottom. The earth was excavated, let us say, to a depth 4 feet above the base on one side of the wall, while on the other side it was about 15 feet above the base. The weight, per foot of wall, on the foundation was about 18,360 pounds. The earth was sand and gravel, let us say, for about 7.5 feet down, the remainder being clay or clay and sand.

The hydrostatic pressure on the left, taking the earth as weighing 100 pounds per cubic foot, was $\frac{1}{2} \times 100 \times 15 \times 15 = 11,250$ pounds and on the right, $\frac{1}{2} \times 100 \times 4 \times 4 = 800$ pounds. If an angle of repose for the entire mass of a little less than 60° is assumed, or $K = 0.1$, $E =$ 1,125 pounds and $E_1 = 80$ pounds.

The overturning moments are

On the left, $1,125 \times 5 = 5,625$ foot-pounds $= 67,500$ inch-pounds

On the right, $80 \times 1\frac{1}{3} = 106\frac{2}{3}$ foot-pounds $= 1,280$ inch-pounds

having a resultant of 66,220 inch-pounds

The moment of stability was

$$18,360 \times 12 = 220,320 \text{ inch-pounds}$$

The distance of the resultant on the base from A was

$$\frac{220,320 - 66,220}{18,360} = 8.39 \text{ inches}$$

or inside the middle third.

The maximum and minimum pressures on the base were

$$\text{maximum } p = \frac{18,360}{2} + \frac{6 \times 18,360 \times 3.61}{4 \times 12} = 8.7 \text{ tons per square foot}$$

$$\text{minimum } p = \frac{18,360}{2} - \frac{6 \times 18,360 \times 3.61}{4 \times 12} = 0.46 \text{ ton per square foot}$$

As the lower part of the foundation was in clay, which may *for a time* stand vertically without support, the computation may be made assuming

pressure from the left only on the upper half, with $K = 0.2$. The pressure was

$$E = 100 \times \frac{1}{2} \times 7.5 \times 7.5 \times 0.2 = 562.5 \text{ pounds}$$

acting at a height of 10 feet above the base.

The overturning moment = 5,625 foot-pounds

The moment of stability was = 18,360 foot-pounds

The distance of the resultant from the edge was

$$\frac{18{,}360 - 5{,}625}{18{,}360} = 0.69 \text{ foot}$$

or just inside the middle third, and 0.31 foot from the center.

$$\text{maximum } p = \frac{18{,}360}{2} + \frac{6 \times 18{,}360 \times 0.31}{4} = 8.86 \text{ tons per square foot}$$

$$\text{minimum } p = \frac{18{,}360}{2} - \frac{6 \times 18{,}360 \times 0.31}{4} = 0.32 \text{ ton per square foot}$$

The foundation was compressible, and the safe load was assumed to be 4 to 5 tons per square feet. Anyway, the pier tipped, and the wall and building collapsed, causing the loss of 43 lives.

19. Direction of Pressure on Wall.—In the preceding treatment, the direction of the pressure on the back of the wall is not found. That pressure is the resultant of the pressure on BC (Fig. 275) and the weight of the prism of earth BDC. When the back of the wall inclines to the left, or away from the earth, the pressure on the wall always has a downward vertical component.

Many engineers think that the pressure on the back of the wall should be found, and that it should make an angle with the normal to the back equal to the angle of repose of earth on masonry (or of earth on earth if a film of earth is supposed to adhere to the wall). This opinion is held because, obviously, the wall cannot actually tip without the earth sliding downward along its back. To make this assumption adds to the stability of the wall, because the earth pressure is taken as acting in the most favorable direction possible. If this method is followed, the pressure must be found in the manner explained in Art. 19 of Chap. XX.

But the wall does not actually tip, so that the basis of the assumption above given does not hold. In this case it is said that the earth will settle or compact itself back of the wall, and so tend to slide down on the back, even if the wall does not tip. Possibly this may be true in some cases and at some time; but it seems to the writer more reasonable to suppose that in time, and after the effects of moisture have been felt, the earth will attain a condition of equilibrium and will not be on the point of sliding in any direction.

Of course, if experiments are made by which the pressure is found when the wall is just tipping, or just on the point of tipping, the pressure will not correspond to the Rankine theory. In other words, *a wall may not tip in the experiments even when that theory indicates that it should. The theory is too safe; it gives too unfavorable pressures so far as tipping, or any motion, is concerned.* But it may not be wrong so far as crushing strength is concerned, or allowable pressure on the foundation. This is but another illustration of the truth that Nature always helps the engineer. A wall, like any other structure, will not fail if it is possible for it to stand. Before failing, it will utilize all statically possible elements of safety.

20. Reasonable Limits of Application of the Rankine Theory. Walls Inclining Backward.—Is the Rankine theory applicable in all cases, even under its own assumptions? That theory gives the pressure on *any plane* in a granular mass of earth of unlimited extent. Will it give correctly the pressure on any wall which replaces the earth on one side of that plane? Suppose the wall to incline backward, as in Fig. 290.

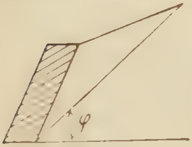


Fig. 290.

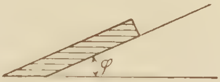


Fig. 291.

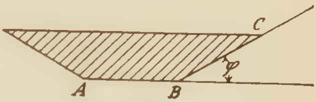


Fig. 292.

The pressure on the wall, according to the Rankine theory, may then be inclined upward. It is stated by some competent writers[1] that "there can be no active upward thrust on a wall." The writer, however, has never been able to understand or to agree to this. The criterion whether there will be pressure on a wall is whether the earth would slide if the wall were not there. If the earth would slide provided the wall were not there, then if the wall is there, the earth will exert a pressure against it. This is certain. If the earth slopes at its actual angle of repose, it is not tending to slide and would not slide; and if a wall is laid on top of it, as in Fig. 291, there will be pressure on the wall, but it will only be because the wall is lying on the earth, and the earth is reacting against it, just as any foundation does. If the wall has a shape (Fig. 292) such that its weight falls inside its base, or even at the center of its base AB, which may be supposed on an incompressible foundation, so that the wall has no tendency to tip or to move vertically, and if the surface BC is just in contact with the earth, there will be no pressure on BC, because the earth has no tendency to slide. In Fig. 291, there is no *active* pressure of the earth on the wall, but merely a *resisting* pressure; the wall is active. But if the slope BC is steeper than the actual angle of repose (Fig. 293),

[1] Ketchum, Williams.

then even if the wall were balanced so that by itself it would not move, the earth will press against BC because it would slide if the wall were not there, and the pressure, according to the Rankine theory, may be inclined upward. Will this be the real pressure? If the wall were not there, the tendency to slide would be resisted by the mass of earth to the left of BC below the dotted line. In the earth alone, if the wall were not there, there would be a pressure on BC, the earth to the left would be pressing on the earth to the right, and the latter would be resisting, or *vice versa;* both would be active pressures because the earth on either side would slide without the other. The earth to the left, of course, being above BC, may be said to be lying upon BC, and to that extent it may be proper to call the pressure exerted on BC by the earth to the right in part a resisting pressure, and in part an active pressure due to its own tendency to slide. When the earth to the left of BC is removed and replaced by a wall, the part to the right may be kept from sliding by any one of a variety

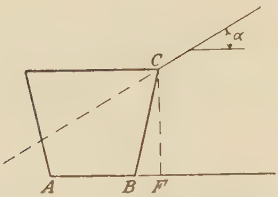


FIG. 293.

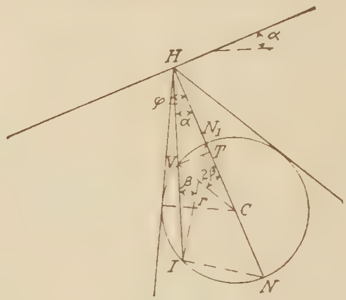


FIG. 294.

of pressures on BC: the question is, which one of these is the one really exerted by the wall. The difference between the wall to the left of BC and the earth to the left of BC is that the former, by itself, may (depending upon its section) exert no pressure on BC, that is, may require no resistance from the earth to the right, as the earth to the left does. If the wall requires the same support that the earth does, the pressure on BC will be the same as in the mass of earth alone, and the pressure on BC may have an upward component. The matter is uncertain, depending upon the shape of the wall, and its tendency to tip to the right, if standing by itself. But the assumption that there can be no upward pressure on the wall appears to the writer arbitrary, and no more inherently correct than to assume it to be according to Rankine's theory. However, it is one of the assumptions often made.

Mohr suggested as the limit of the application of his circular diagram, which is the same as Rankine's theory, that the direction of the maximum principal stress at B (Fig. 293) should lie in the earth and not in the wall. Drawing the diagram (Fig. 294), the maximum principal stress is on a plane parallel to IN, and is in the direction IN_1, so that this criterion

means that the wall must not incline farther to the right than IN_1, or must not make an angle to the right of the vertical exceeding $HIN_1 = \beta$. It can be shown that

$$\sin 2\beta \cdot \sin \varphi + \cos 2\beta \cdot \sin \varphi \cdot \tan \alpha = \tan \alpha \qquad (14)$$

for $\alpha = 0$; $\qquad \beta = 0$
for $\alpha = \varphi$; $\qquad \beta = 45° - \varphi/2$

This is shown as follows (Fig. 294):

$$HP = \frac{r}{\sin \varphi} - r \cos 2\beta;\ VP = HP \tan \alpha = r \sin 2\beta$$

$$\therefore \tan \alpha = \frac{\sin 2\beta \cdot \sin \varphi}{1 - \cos 2\beta \cdot \sin \varphi}$$

from which Eq. (14) follows.

This criterion seems to the writer the most reasonable one.

21. Actual Pressure on a Wall.—The actual pressure on a wall can never be known, because the real angle of repose and coefficient of cohesion cannot be known, and they may vary from the top to the bottom. All the designer can do is to assume reasonable values and then see that the wall is properly constructed and the filling properly placed, if placed at all. If a wall fails and a law suit results, attorneys will surely endeavor to show that the *actual pressure* should be known before it can be stated that it caused the collapse; and they may claim that there was no pressure at all. A failure due to earth pressure may be slow, because the earth may be able to stand for a time at a vertical slope, and the pressure may be generated gradually, as weathering and moisture conditions determine.

In the case illustrated in the third example of Art. 18, the earth had been excavated on one side of the piers, down to within a couple of feet of the bottom of the piers. There had been heavy rains, and the clay was wet and slimy. The building on the left had been damaged by fire and the roof leaked, so that water got into the basement. The earth to the left of the piers was therefore wet. Yet the load on the foundations, from vertical load alone, was about 4 tons per square foot, or above what most engineers would consider proper; and *any earth pressure* would certainly still further increase the pressure at the toe; and would cause some tipping of the piers and endanger the 50-foot brick wall above it. The wall collapsed, all the piers tipping outward.

22. Buttresses and Counterforts.—A *buttress* is properly a projection on the front of a wall; a *counterfort* (sometimes also called a buttress) is a projection on the rear. These make the horizontal section of the wall as in Fig. 295. If the foundation plan is as in Fig. 295(a) the method of computation for a stone wall would be to consider a length of the wall between centers of buttresses a and d, find the center of gravity of this

section *abcd* and the total earth pressure behind it, and take moments about the axis SS through this center of gravity. Then the usual formula for flexure

$$p = \frac{N}{A} \pm \frac{Mv}{I}$$

would give the intensity of vertical pressure at any point distant v from that axis, and so the distribution of pressure on the base would be found.

Retaining walls have been built with deep counterforts and arches between them, so-called *relieving arches*, the object being to relieve the wall itself of earth pressure, so that the wall may be thin, and the section would be somewhat as in Fig. 296, the earth sloping down on top of the relieving arches. Such walls, however, are now rarely built.

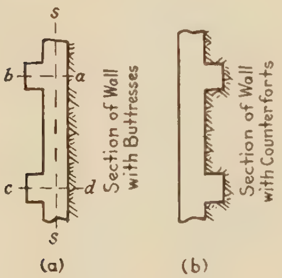


Fig. 295.

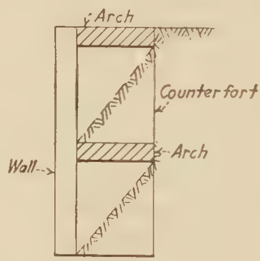


Fig. 296.

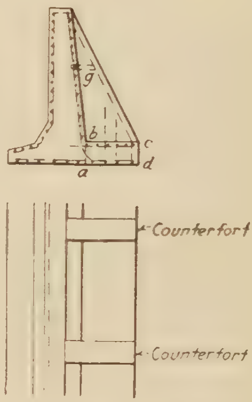


Fig. 297.

Wall of reinforced concrete with vertical stem and bottom slab are often built with counterforts at intervals (Fig. 297). In such cases the vertical stem may be considered as a horizontal slab having a span equal to the distance between counterforts, whose only function is to carry the load to the counterforts, as shown in the plan. The main reinforcement in this vertical stem or slab consists then of horizontal rods near the front face, closer together toward the bottom because the earth pressure increases toward the bottom. There must also be horizontal rods near the back face in front of the counterforts, to carry the negative moment there, together with adequate shear reinforcement.

The bottom slab back of the vertical stem is a slab supported on three sides, namely, the vertical stem, and two adjoining counterforts. Its true condition is therefore uncertain and statically indeterminate. But if the vertical stem is considered as merely a horizontal beam or slab distributing the earth pressure to the counterforts, it is not considered to exert any vertical pressure on the horizontal slab beneath, unless the earth pressure has a vertical component. It is thus advisable to consider the slab *abcd* in Fig. 297 as a horizontal slab of span equal to the distance between counterforts, loaded by a downward load of the earth above it

and by an upward load of the reaction of the foundation. Rods will be needed top and bottom.

The horizontal slab in front of the vertical stem will be a projecting cantilever loaded with an upward load of the reaction of the foundation.

There will also be needed vertical and horizontal rods, as indicated in Fig. 297.

The counterfort is exposed to the earth pressure behind it, the load on the bottom from the slab *abcd*, and the *outward* load on the front from the vertical stem.

The counterfort must thus be anchored to the vertical stem, either by looped rods as indicated at g (Fig. 297) or by horizontal rods running through stem and counterfort, *with nuts at the ends*, or by some other means. These horizontal rods may be run through horizontal angle irons in the vertical stem, with nuts, as shown in Fig. 298. The counterfort must also be anchored to the horizontal slab beneath it, at least over all the portion of the section *bc* where there is tension.

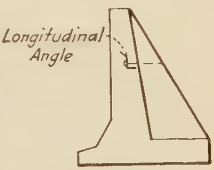


Fig. 298.

A reinforced concrete wall with counterforts will be further discussed and worked out in the chapter on Concrete, in the next volume.

23. The construction of retaining walls is well discussed in Baker's "Masonry Construction," in Ketchum's "Walls, Bins, and Grain Elevators," and in Cain's "Earth Pressure, Walls, and Bins."

If the earth is *deposited* back of the wall, it is better to have it in layers sloping downward away from the wall. The foundation should be below frost. The earth back of the wall should be well drained, generally by depositing along the bottom of the wall a longitudinal drain of broken stone or porous gravel, with openings or "weep holes" through the wall at intervals of 15 to 25 feet. Clay should not be used as a filling.

The possibility of sliding of the wall on the foundation must be carefully investigated. The coefficient of friction between masonry and dry earth may be from 0.5 to 0.66, and on dry sand or gravel it may be even greater; on dry clay it may be 0.5 or over, but on wet clay it may fall as low as 0.2 to 0.33. Clay is a treacherous foundation, and should be avoided if possible, or if it cannot be avoided, the possibility of its becoming wet should be reduced as much as possible. Sliding resistance may be increased by inclining the foundation, or by having an offset as in Fig. 281.

It is unnecessary to state that it has not been the object, in this chapter, to give a complete discussion of reinforced concrete retaining walls, but merely to indicate a few fundamental principles. Reinforced concrete structures all reduce, in principle, to finding the direct force, bending moment, and shear on a section, and providing for them by the principles governing columns and beams of reinforced concrete.

CHAPTER XXIII

PIERS AND ABUTMENTS

1. Each end of a single span of a bridge rests on an *abutment*, which supports it and also holds back the earth fill behind it.

If a bridge consists of several spans, adjacent ends of two adjoining spans rest upon a *pier*.

PIERS

2. Loads.—The loads or outer forces on a pier are:

a. Its own weight.

b. The weight from each adjoining span, live and dead.

c. The pressure of the water if the pier is in a stream.

d. The pressure from ice and the impact from ice and other floating objects.

e. Wind pressure on the pier and on the bridge spans.

f. Tractive force or that due to expansion from temperature changes.

The first two items require no discussion.

The hydrostatic pressure on the pier being balanced, it is only the pressure due to the current that is to be considered under *c*. If v is the velocity and A the area of the upstream end of the pier exposed to the current, this pressure will be

$$cA\frac{wv^2}{2g}$$

where c is a constant depending upon the shape of the upstream end. This constant varies with the shape of the upstream and downstream ends of the pier. If the ends are plane and at right angles to the current, the value of c is from about 1.1 to 1.4, depending upon the relation between the length of the pier and the cross-section; the smaller value of c being where the length divided by the square root of the submerged cross-section is 3, according to Du Buat, and the larger value when this ratio is very small, because in this case there is a greater reduction of pressure on the downstream end.

If the upstream end is not square, the pressure will bear the following ratio to the pressure on a square end, according to Poncelet:

Triangle with angle at the point 90°	0.728
Triangle with angle at the point 60°	0.52
Pointed, with sides circular arcs	0.39
Half cylinder, circular	0.57
Half cylinder, elliptical	0.43

The point of application of the impact of the water may be taken at about half the depth.

The upstream end is frequently made triangular, with an inclined edge on which the ice may break, as in Figs. 299 and 300. It is also in

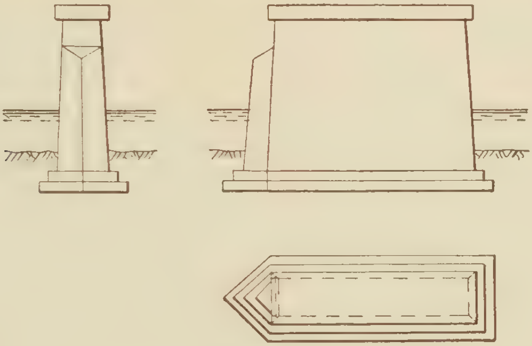


FIG. 299.

many cases made with two intersecting circular arcs, with the edge vertical is inclined (Figs. 301, 302, 303). Above high water it is often semicircular. The downstream end is generally square or semicircular, less often pointed. A paper by Floyd A. Nagler on "Obstruction of Bridge

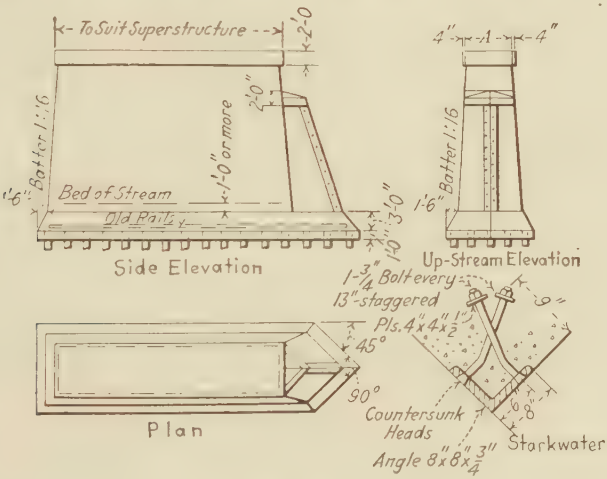


FIG. 300.

Piers to the Flow of Water"[1] describes experiments made in a flume about 2 feet wide with model piers 6 inches wide and of various forms, from which the writer of the paper concludes that the best form of the upstream end is half-round or elliptical, and of the downstream end double curved,

[1] *Trans. Am. Soc. C. E.*, p. 334, 1918.

like a fish tail. Such a form for the downstream end, however, is not practical. The force exerted by floating ice and other objects, or due to the freezing of ice, is uncertain. Professor Baker found the crushing strength of ice to vary from 370 to 760 pounds per square inch.

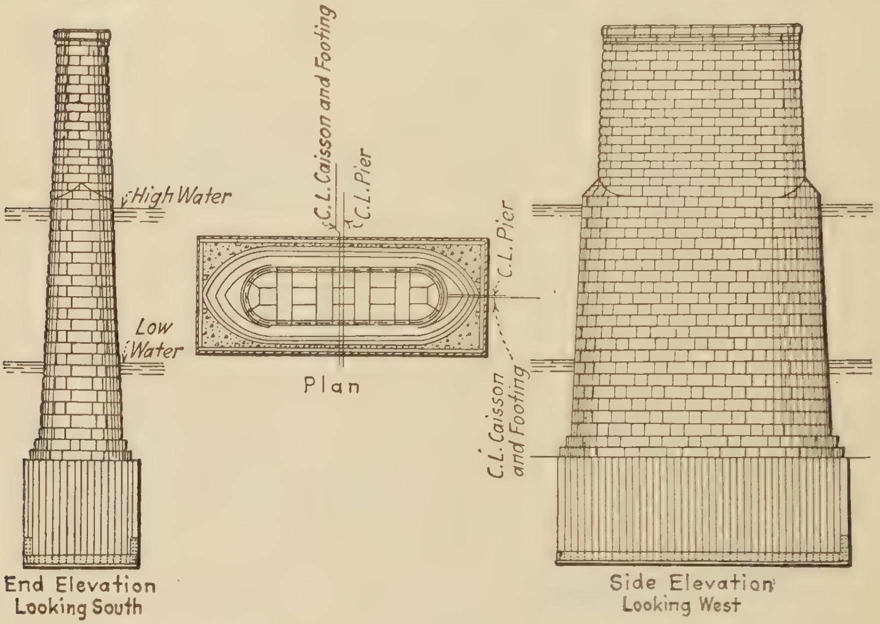


Fig. 301.

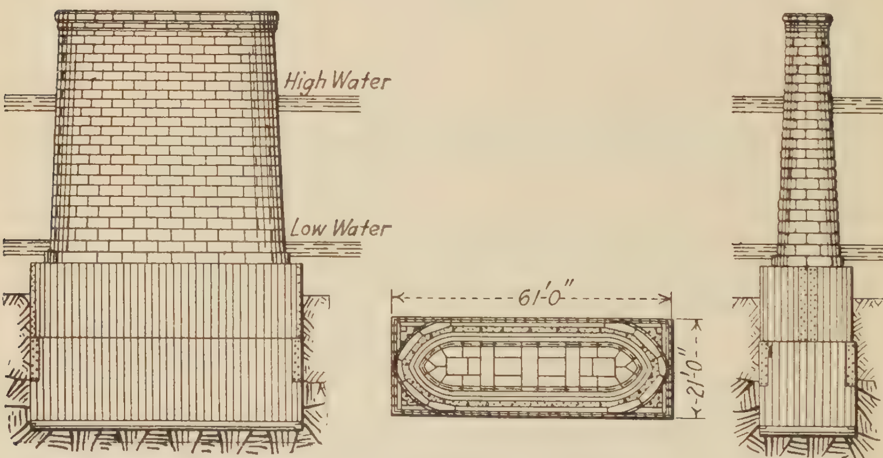


Fig. 302.

The wind pressure transverse to the bridge will be that on the upstream end of the pier plus that on one-half the adjoining spans. The latter will be either that on the unloaded spans at 50 pounds per square foot of

exposed surface of two trusses, or that on loaded spans at 30 pounds per square foot. For the most unfavorable condition, the train should be light. The pressure on the truss will act at about half its height; that on the train at a height of about 7 feet above the rail on freight trains, and about 9 feet on passenger trains.

There is generally no danger to the pier from forces acting transverse to the bridge.

Fig. 303.

In considering the stability of the pier lengthwise of the bridge, the wind pressure on the long side of the pier should be considered, the brakes and tractive force of the train, and the force due to expansion and contraction. All these may act in either direction. Generally a pier carries one roller end and one fixed end of a span, though it may carry two fixed ends or two roller ends.

The tractive force is the weight on driving wheels multiplied by a coefficient of friction of 0.2 or 0.25; the braking force is the weight on all

the wheels of the train which have brakes, multiplied by the same coefficient. Either force may act in either direction. The tractive or braking force on a span should be considered as all resisted at the fixed end, though some may be resisted at the roller end if the rollers are not frictionless. It acts at the center of gravity of the train, or above the top of the rail.

The force due to expansion and contraction is due to the fact that there is friction at sliding ends and even at roller ends, since the latter often get rusted or clogged so that they do not roll freely. The force is the reaction at the movable end multiplied by the coefficient of friction of, say, 0.25.

3. Shape of Piers.—Piers are generally given a batter on all sides of 1 or ½ inch per foot. The downstream end may be square, but is better rounded. The upstream end may be square or rounded above high water; below high water it should be pointed. The general shape is shown in Fig. 299.

Piers are sometimes made entirely of concrete. In that case there should be an angle iron on the upstream pointed edge, anchored back into the concrete (Fig. 300). Concete corners are likely to be damaged by floating objects, and it is best to make piers with a stone facing and concrete or rubble backing.

Figures 301, 302, and 303 show views of piers.

For further details of piers, see Baker's "Masonry Construction." The most important and generally the most difficult part of a pier is the foundation, and this subject is not treated in this book.

ABUTMENTS

4. The function of an abutment is to support the end of the bridge span and to hold back the earth. Hence it is a sort of modified retaining wall, or a modified pier.

5. Forms of Abutments.—The forms are shown in Fig. 304. In Fig. 304 (*a*) of which (*e*) is an elevation, the portion BC supports the end of the bridge span, which rests on the *bridge seat;* while AB and CD are *wing walls* or *wings*, which in this case are in line with the abutment, and are carried out to the foot of the slope of the approach embankment. The wings may be sloped, as in AB, or stepped as in CD. If built of stone, they are generally stepped; if of concrete they are sloped, and sometimes curved as in $A'B'$. Sometimes they are not carried to the foot of the slope but end at E, and the earth is allowed to flow around the end to a slope FG. The thickness of the wings may be reduced as the height increases, keeping either the face straight, or the back straight as at the right in Fig. 304 (*a*). Instead of being straight, the wings may be turned at an angle or *splayed*, as in Fig. 304 (*b*), and here again may be sloped or stepped, and carried to the foot of the earth slope or not. When the wings are turned at 90°, the

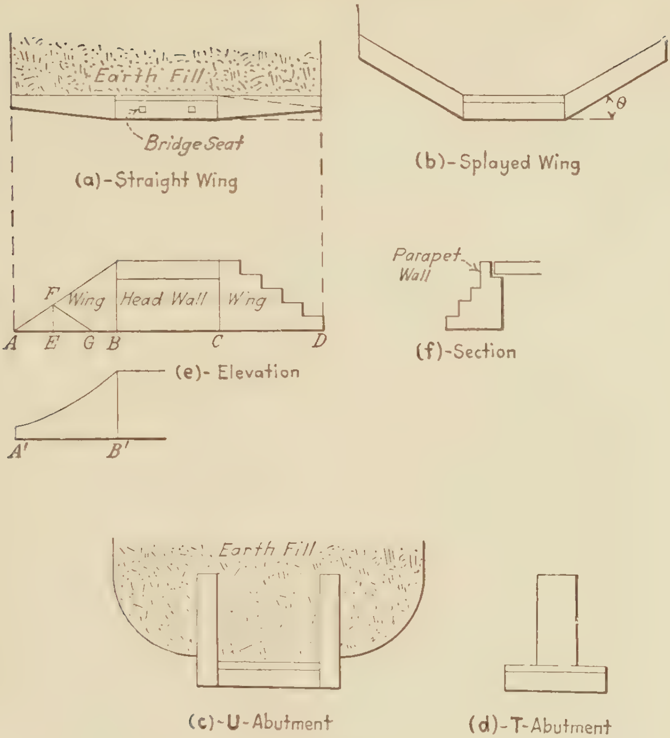


FIG. 304.

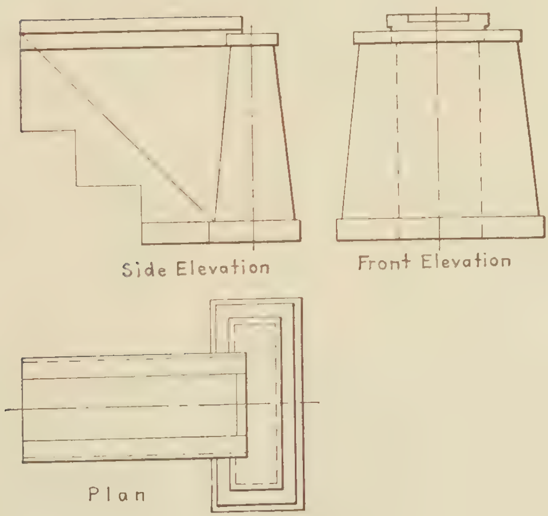


FIG. 305.

abutment is a U-abutment, as in (*c*); here the earth slope curves around against the wings in conical form.

In the T-abutment there are no wings, but a tail wall at the center of the head wall. The tail wall must be long enough to reach to the top of the earth slope and wide enough to carry the tracks on the roadway. In some cases the ties are laid directly on the tail wall, which is but 8 or 9

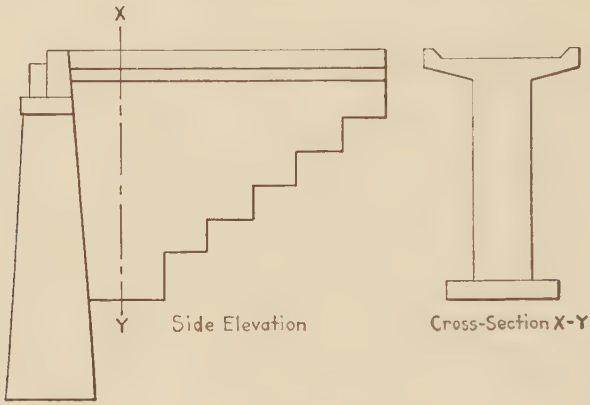


Fig. 306.

feet wide, but it is better to make the tail wall wider and to lay the ties in ballast, as in Fig. 305. If of concrete, the tail wall may be narrow and cantilevered out at the top, as in Fig. 306. The tail wall may be stepped, as shown. The wings of the U-abutment may also be stepped if the ground rises back of the head wall. Instead of the tail wall, there may be a pier at the top of the slope, with a short span between it and the head

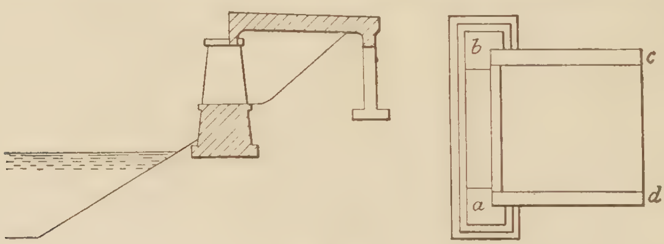


Fig. 307.

wall; or two single square piers with a girder between. Indeed, the entire abutment may be made of four columns, *a*, *b*, *c*, *d* (Fig. 307) with a concrete slab and girders.

Back of the bridge seat is a thin *parapet wall* or *back wall*. The back of the abutment, both head wall and wings, may be vertical, sloped, or stepped. The weight on the bridge seat is considered to be uniformly distributed over the length of the head wall *BC*.

6. Loads.—The loads on an abutment are: its own weight, the load from the bridge span, wind pressure, tractive or expansion forces, and the earth pressure. These have already been sufficiently discussed.

7. It is obvious that the action of the abutment proper or head wall is very different from that of the wings. The head wall is subjected to the earth pressure behind it, like any retaining wall, and to its own weight and the forces acting through the bedplates of the bridge. As the bed-plate is usually set near the face of the wall, the vertical load from the bridge, unless the face has a considerable batter, which is not common, acts near the toe of the wall and so tends to cause the head wall to tip inward. The base must therefore be considerably wider than for a simple retaining wall of the same height. It may be difficult to keep the resultant within the middle third of the base, and especially difficult to keep it back of the center of the base, unless the face is given a considerable batter, or the foundation extended in front. Often the face must be vertical, as in a bridge over a street. *There is, accordingly, a considerable tendency for an abutment to tip forward*, if on a compressible foundation. The wings, on the other hand, are simple retaining walls to hold back the earth, and their tendency to tip forward may be much less than that of the head wall. Such a tendency of the head wall is resisted, in part at least, by the longitudinal resistance of the bridge span, but this resistance is zero if one end of that span is on frictionless rollers, and the tractive or braking forces may increase the tendency of the head wall to tip forward. The action of the head wall and of the wings is thus radically different; and *there is much to be said in favor of making a continuous vertical joint separating the two*, so that they may act independently, though this is not commonly done. It is required by the Pennsylvania Railroad. Instances are not uncommon, however, where cracks have occurred at these points.

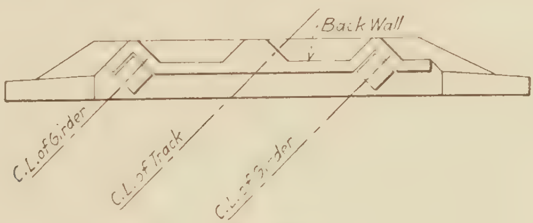


Fig. 308.

8. Whether wings should be straight or splayed, and, if splayed, at what angle, depends upon circumstances. If the bridge crosses a street, the abutment is generally on the street line, in order to make the span as small as possible, and the wings are straight. If the bridge crosses a stream, the wings should be splayed, in order to facilitate the passage of the water, but not at too great an angle, for if at a large angle there

will be an eddy at the angle where the upstream wing joins the head wall, which may wash the foundation at this point. Probably the angle θ should not exceed 30° in this case. If the bridge is not over a stream, the angle may be greater, and the wing made short, letting the earth fill extend around its end.

The arrangement of piers and abutments depends to some extent on hydraulic conditions, such as the flood discharge, climate, character of floating bodies, etc., which will not be discussed here.

9. Abutments for skew arches often have the back wall in a series of offsets, as in Fig. 308.

10. Crib Walls.—It has long been common to make retaining walls along railroads by using old ties, in the form of cribs, with stringers longitudinally, and, at intervals between these, headers reaching back into the earth or to a rear line of stringers. The earth flows down into the crib, but the slope is held and sliding is prevented.

Recently, similar crib retaining walls have been built of concrete. The units are cast, and are reinforced by rods. Such walls are built by the R. C. Products Company, Inc., Engineers Building, Cleveland, Ohio, and by the Massey Concrete Products Corporation, Peoples Gas Building, Chicago, and 50 Church St., New York. The units of the R. C. Co. are I-shaped, the stringers having the web horizontal and the headers having the web vertical. The headers have enlargements at the ends, which fit into and interlock with the stringers, without dowels or other connections. The Massey Co. makes both stringers and headers rectangular, the headers having T-shaped enlargements at the ends, which project beyond the stringers and hold them in place. As seen from the front, such a wall consists of lines of concrete stringers with spaces between, and in these spaces the ends of the headers appear at intervals. The earth may be seen in the spaces. Fig. 308*a* shows one of the walls of the Massey Co., along the Pennsylvania Railroad at Sandusky, Ohio.

The units being pre-cast, they can be made under the most favorable conditions, and properly cured, so that they avoid some of the uncertainties inherent in concrete which is poured at the site. Such walls can be easily placed, and may be taken down and the units used again if desired; and they may be built in any season.

Walls of this kind may be built curved or with angles. The thickness may be increased with the height. The top thickness of a high wall may be the length of a header, which interlocks with stringers front and back. At an appropriate depth below the top, the thickness may be increased to the length of two headers, interlocking with three lines of stringers; and at greater depths the thickness may be further increased.

Such walls have been used in many places, and have much to recommend them.

11. Many rivers in the middle west have beds and banks of material which is easily eroded, and the streams are subject to severe floods. Bridges are often made too short, for economy, and the abutments are frequently undermined or the material behind them washed away in times of flood. Many abutments in this region are built by driving steel I-beams and supporting the earth behind them by planking or otherwise. It is found that in a compressible material such beams, particularly if the flanges are parallel and not tapered, acquire a very large resistance or supporting power when well driven. The great bearing power of

FIG. 308*a*.—Crib wall at Sandusky, Ohio. (*Courtesy, Massey Products Corporation.*)

such steel piles appears to have been discovered, about 1900, by Robert Z. Drake of Omaha and John A. Crook, now of Denver. It appears to be due to the compression in driving of the material between the parallel flanges, especially when driven by a drop hammer. When used as a bridge abutment, these steel piles are solidly driven and braced, and connected at the top by a horizontal girder designed to support the end of the bridge; and, if there is filling behind, it is held back by planking or concrete construction.

12. References.—Further details regarding piers and abutments may be found, among other sources, in the following works.[1]

[1] BAKER, IRA O., "A Treatise on Masonry Construction," tenth ed., John Wiley & Sons, Inc., 1909; WILLIAMS, C. C., "The Design of Masonry Structures and Foundations," McGraw-Hill Book Company, Inc., 1922.

CHAPTER XXIV

THE STONE ARCH[1]

On fait une voûte d'après les voûtes faites: c'est affaire d'expérience.

L'Ingénieur chargé de projeter, de construire une voûte, trouvera dans cet ouvrage ce qui a été fait, ce qu'il faut faire, ce qu'il ne faut pas faire.

Séjourné, "Avant-Propos to Grandes Voûtes," 6 vols

1. Definitions.—Any structure which, when under vertical loads, exerts outward horizontal pressures on its supports, may be called an arch. Arches may be of wood, steel, or masonry. If of masonry, they may be monolithic (that is, of concrete), or made up of separate wedge-shaped stones called voussoirs, with plane mortar joints between (that is, the stone arch).

Steel and metal solid arches are elastic structures, in the sense that they are curved elastic ribs without discontinuity of cross-section. Steel trussed arches are merely statically undeterminate frames, unless they have three hinges or are otherwise made determined. In a statically indeterminate arch it is the outer forces that are indeterminate by statics, and the reactions must be found by elastic conditions, such as the condition that the span does not change if the abutments are immovable.

The solid unreinforced concrete arch without hinges is also a curved elastic rib, with continuous section unless a crack occurs in the concrete, in which case its section and axis are discontinuous. The same is true of the stone arch unless a joint opens. As long as these are continuous they may be treated by the same methods as solid metal ribs. The stone arch, however, is not homogeneous, but consists of stones with mortar joints between.

The reinforced concrete arch, as long as no cracks form, is a solid continuous rib. If, however, the reinforcing rods are depended on to carry tension which would otherwise exist in the concrete because the resultant on a section cuts the section outside of the kernel, the assumption is made that there is no tension in the concrete and that the section consists of a rectangle of concrete in compression and steel rods in tension. Under this assumption the section of the arch has discontinuities (see Art. 18).

[1] Since the stone arch is an elastic arch, differing only in degree from a monolithic concrete arch, it is impossible in this chapter to distinguish sharply between the two. Before the student of structural engineering begins the study of elastic arches, it is desirable that he should study carefully this chapter on the stone arch, notwithstanding the fact that stone voussoir arches are now seldom built.

It is assumed that in a stone arch a joint can take no tension. It follows, since the joints are rectangular, that if the resultant pressure on any joint falls outside the middle third, the joint will open and the pressure will be distributed over only a part of the joint. It has therefore been the custom to treat the stone arch, not as a curved elastic rib, but as a series of voussoirs with joints between, which may or may not open. This treatment will be explained in the present chapter. The reader

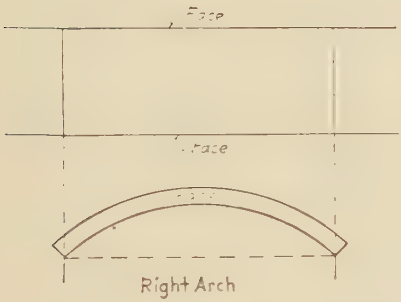


Fig. 309.

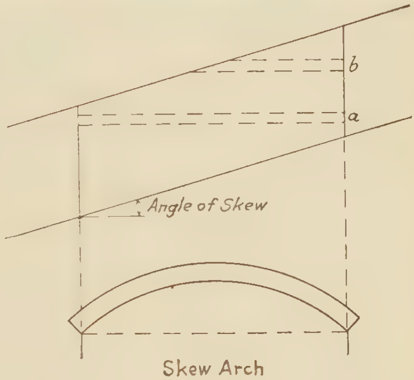


Fig. 310.

must be prepared, however, to find it inexact, in view of necessary uncertainties. The theory of the curved elastic rib will be fully treated in the last volume of this work.

2. Parts of the Stone Arch.—The barrel-shaped ring or rib which supports the loads is the *arch ring.* If the axis of this ring is at right angles to the roadway, so that the arch as seen in plan is a rectangle, the

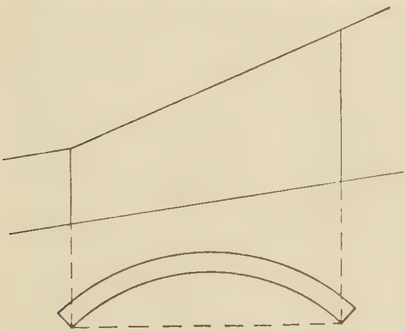


Fig. 311.

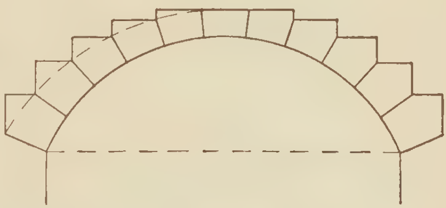


Fig. 312.

arch is a *square arch* or *right arch* (Fig. 309); if the axis makes an angle with the roadway, so that the ring in plan is not a rectangle, the arch is a *skew arch* (Fig. 310); the two faces may even be not parallel (Fig. 311). The end surfaces of the ring are the *faces.* The under surface of the ring is the *soffit,* and the intersection of the soffit with a vertical plane parallel with the face is the *intrados;* sometimes the soffit is called the intrados

(A.R.E.A.). The intersection of the outer surface with a plane parallel with the face is the *extrados*, but sometimes the outer surface itself is called the extrados. The soffit is a single curved surface generated by a straight line remaining parallel to a given line and revolved about an axis or axes which are parallel to the same line. If it revolves about one center, the arch is single centered or the soffit is part of a circular cylinder; if it revolves about three centers in succession, it is three centered. Some arches are five centered; others are elliptical, or of some similar shape.

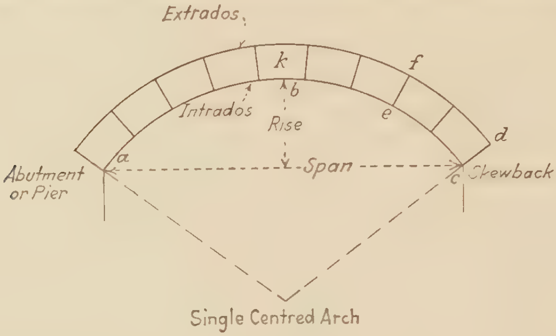


FIG. 313.

The extrados or outer surface may be generated in the same way, and may or may not be parallel with the soffit or concentric with it; if concentric with the soffit, the arch thickness is the same everywhere; but generally the thickness increases toward the abutments. The outer surface may not be a curved surface at all, but may be a succession of steps, as in Fig. 312; in this case the arch ring should be taken as bounded by an extrados like the curved dotted line.

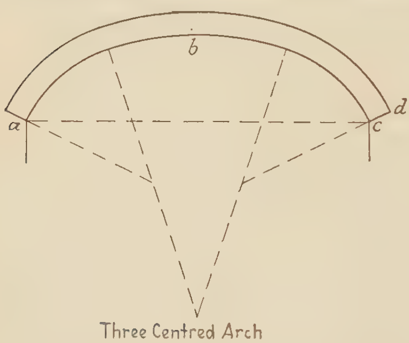


FIG. 314.

The inclined or horizontal surface where the arch begins, *cd* in Figs. 313, 314, is called the *skewback;* the inner edge of the skewback, or the axial line through *a* or *c* is the *springing*, or *springing line*. The highest point or line of the intrados *b* is the *crown*. There is no joint at the crown, although in the analysis of the arch it is always assumed that there is one there. The arch stones are called *voussoirs*, and the one at the crown is the *keystone k*. The joints as seen on the face are generally radial, like *ef*, and such joints are continuous from one face to the other, and are called *coursing joints*. A course of stones extending from face to face is a *string*

FIG. 315.

course; the interior stones, not showing in the face, are sometimes called *sheeting.* There are also joints parallel to the faces, but these are not continuous so as to separate the arch into rings, but the joints in one string course are offset with the joints in adjacent string courses, giving a bond, as in Fig. 315 which distributes a load from face to face. These discontinuous joints perpendicular to the coursing joints are called *heading joints* or *ring joints.*

The *span* is the horizontal distance between springing lines, and the *rise* is the height of the crown above the line joining the springings. Generally the springings are at the same level, but not always. The *haunches* of an arch are the portions of the ring between the upper part near the crown and the lower part near the springings. The *spandrel*

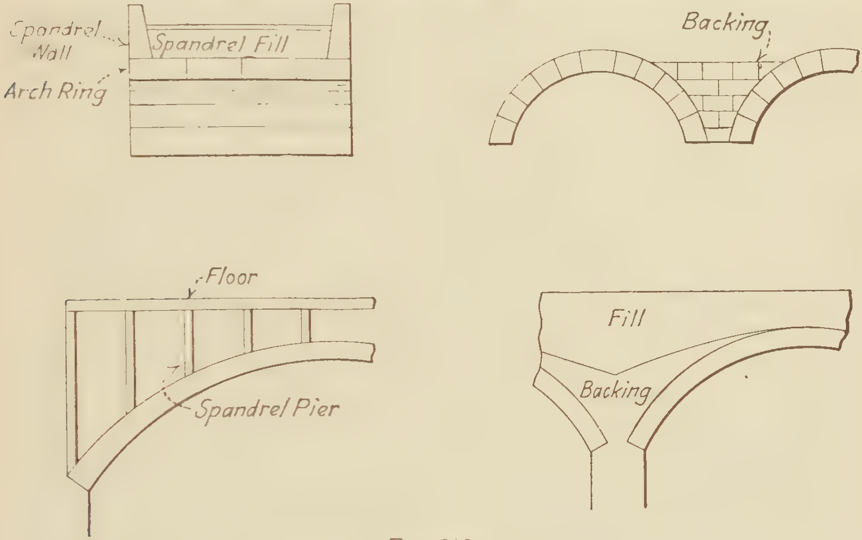


Fig. 316.

is the indefinite space between the extrados and the roadway; the spandrel must be filled in, by filling confined between walls at each face, called *spandrel walls* or *face* walls, or by solid masonry or concrete *backing,* or by *spandrel piers* supporting the roadway and resting on the arch ring, as in Fig. 316.

In studying the arch, a strip one unit thick parallel to the face is considered. This is because a load is generally distributed over the roadway, or is distributed over the arch by the filling, or is distributed by the bond between string courses. In a skew arch, the ring should logically be taken at right angles to the abutments, as at a in Fig. 310, but in that case a strip like b, springing from the skewback on the right would end at the face on the left with nothing to push against there. The strips must therefore be assumed parallel to the face. In some cases the skew stone arch has been made of a series of square arches offset,

with continuous heading joints, as in Fig. 317. The separate rings may be tied together by clamps at the crown, or otherwise, but the construction is not good. The real skew stone arch was a complicated structure, requiring each stone and each face to be carefully studied and drawn. These arches are seldom or never built now. Concrete is generally used for skew bridges, so that stone cutting is unnecessary, but the stress analysis is uncertain (see Art. 24).

3. The Stone Arch without Hinges Is Statically Undetermined.—If the loads are given, there are two reactions to be found, each of which requires three quantities for its determination, namely, a vertical component, a horizontal component, and a point of application. Each reaction acts at a springing joint, whose position is known, but it may act at any point of that joint unless fixed by a hinge at a definite point, which is sometimes done but will not here be assumed. There are thus six unknown quantities and three statical conditions, so that three more conditions are necessary. In the elastic theory, if applicable, these are that the span and the inclination at each springing are unchanged. In the present treatment the conditions required are assumed or inferred, as will be explained presently.

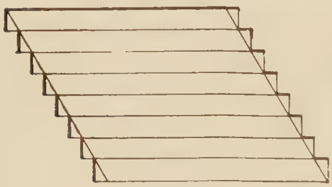


Fig. 317.

4. Center of Pressure and Line of Resistance.—If the resultant of all the outer forces on either side of a joint (loads and reactions) is found, the point where this resultant cuts the joint is its *center of pressure*, and the line joining the centers of pressure on all joints is the *line of resistance*. The actual joints are not known till the arch is built, but in designing, or even in studying an actual arch, it is assumed that the joints may be anywhere, and they are assumed at convenient distances apart, and radial, or in the directions in which the actual joints would be placed. As the joints may be anywhere, the line of resistance is a curve; but as the joints are assumed it is a broken line. It is assumed that there is a joint at the crown, though there is none in an actual arch. This is perfectly proper.

5. Loads.—The loads are the weight of the arch, and any forces applied to the extrados. If there is filling above the extrados, the latter loads are really the earth pressures on the extrados. In practice, it is near enough to assume the load on the extrados of any voussoir to be that vertically above it. It is not only simpler to assume the loads on the extrados to be vertical, but it is safer to do so unless horizontal inward loads would tend to increase instability. Generally the reverse is the case, for the arch tends to spread outward, and an inward reaction from the spandrels would resist this tendency. If concentrated loads act on the roadway, such loads should be distributed through the fill, as

explained in Art. 38 of Chap. XX. Concentrated loads being distributed in this way, the load acting on any voussoir is vertical, and may be fully determined when the roadway loads are known. It is not necessary, however, to consider the loads from an earth fill to be vertical. The earth pressure on each voussoir may be found, and the horizontal component as well as the vertical component used; but as the mass of earth is not of indefinite extent, as is assumed in the theory of earth pressure, this determination of the pressure is by no means exact. The writer has always used vertical loads, but each engineer may do as he pleases; only, if he finds the pressures by the usual theories, he must not delude himself by thinking that he is necessarily more exact.

6. Conditions of Safety.—The conditions which must be fulfilled, in order that the arch may be safe, and that no joint may tend to open, are:

1. The (true) line of resistance must lie within the middle third of the arch ring in order that no joint may open; or it must lie within the arch ring, otherwise the arch would collapse.

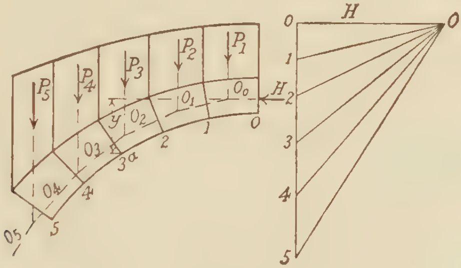


FIG. 318.

2. The (true) pressure on any joint must not make with the normal to that joint an angle exceeding the angle of repose of stone on stone or stone on mortar.

3. The (true) maximum intensity of pressure at one edge of any joint must not exceed the allowable stress.

In order to be sure that these conditions are fulfilled, the *true line of resistance* must be known, or, at least, something must be known about it. This is impossible unless the true reactions are known, and these cannot be found by statics, without making assumptions.

7. Construction of the Line of Resistance.—The line of resistance may be drawn if the thrust on any joint is fully known. Thus, in Fig. 318, if the thrust at the crown is H, and is fully known, then if P_1, P_2, etc. are the loads on successive assumed voussoirs, the resultant of H and P_1 is the thrust on joint 1, the resultant of this and P_2 is the thrust on joint 2, and so on down to the springing. It is only necessary to lay off the loads and H as shown, and with O as a pole draw the resultant equilibrium polygon. The centers of pressure are the intersections of the proper strings of this

polygon with the corresponding joints; thus the center of pressure on joint 2 is the intersection of *string* $O2$ (not ray $O2$) with joint 2. At joint 5, string $O5$ must be produced backward to find the center of pressure on joint 5.

The crown joint is the one on which the thrust is assumed because it is obvious that, *for a symmetrical arch and symmetrical load, the thrust at the crown must be horizontal.* This is not an assumption, but a necessary truth. The point of application of the thrust, however, is not known. In Fig. 318 a symmetrical load was assumed.

For each possible crown thrust there is a line of resistance, which may be easily drawn if the crown thrust is given. Only one of this infinitude of lines of resistance is the true or actual one.

In order to study the matter, it is necessary to perceive the relations between different lines of resistance; and in order to do this, certain properties of a line of resistance must be observed.

8. Properties of a Line of Resistance.—Assuming an arch and loading symmetrical about a vertical line through the crown, the crown thrust is horizontal and clearly the following properties exist:

1. The line of resistance is a curve concave downward, and itself symmetrical about the same line.

2. The greater the crown thrust, the flatter the line of resistance.

3. The crown thrust may be so assumed as to make the line of resistance pass through any given point: thus, to make it pass through a of joint 3 (Fig. 318), let ΣM be the sum of the moments about a of the loads P_1, P_2, P_3, between joint 3 and the crown, and y the vertical distance from a to the line of action of H. Then

$$\Sigma M = Hy \tag{1}$$

Hence if either H or y is assumed, the other may be found; the line of resistance will then pass through the point symmetrical with a on the other side of the center. If the loads have horizontal components, the moments of these components must be included in ΣM.

4. The line of resistance may be made to pass through any two points; for if a and a' are the two points, and ΣM and $\Sigma M'$ are respectively the sum of the moments of loads between the crown and the joint on which a lies and between the crown and the joint on which a' lies,

$$\Sigma M = Hy, \text{ and } \Sigma M' = Hy' \tag{2}$$

from which H and y may be found; a and a' may be on the same or on opposite sides of the center, and in each case the line of resistance will pass through symmetrical points on the other side.

5. If the loading or the arch is not symmetrical, there will be three unknown quantities regarding the crown thrust, H_c, V_c, and the distance y_c of the point of application above the intrados. In this case, the line

of resistance may be made to pass through any three points, a, a', and a''; for (Fig. 319)

$$\left.\begin{aligned} \Sigma M &= V_c x + H_c(y_c + y) \\ \Sigma M' &= V_c x' + H_c(y_c + y') \\ \Sigma M'' &= V_c x'' + H_c(y_c + y'') \end{aligned}\right\} \tag{3}$$

from which the three unknown quantities may be found.

6. If the crown thrust is given, the center of pressure on any joint may be found analytically, with ease; for the symmetrical arch and load (Fig. 320), if P is the resultant load between the joint and the crown, the moment, about a, of the resultant force on the joint is $Hy - Px$, and this,

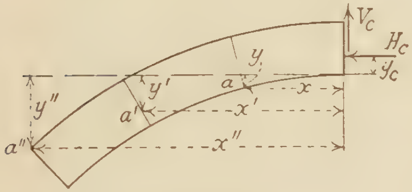


Fig. 319.

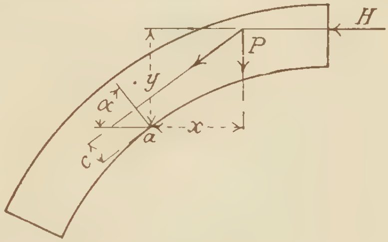


Fig. 320.

divided by the normal component of that resultant, will give the distance c of the center of pressure from a, measured along the joint, or

$$c = \frac{Hy - Px}{H \sin \alpha + P \cos \alpha} \tag{4}$$

If the arch is unsymmetrical (Fig. 321),

$$c = \frac{Hy + Vx_1 - Px}{H \sin \alpha - V \cos \alpha + P \cos \alpha} \tag{5}$$

7. Under vertical loads, the horizontal component of the thrust on every joint is of course the same.

9. Relations between Two Lines of Resistance.—If arch and load are symmetrical about a vertical through the crown, the thrust at the crown, as already stated, must be horizontal. If arch or load is not symmetrical, there is some joint on which the thrust is horizontal, generally not at the crown. In Fig. 322 ab may represent in general the joint on which the thrust is horizontal. Then the following principles are obvious:

1. Two lines of resistance, having the same H, but applied at different points on ab, do not intersect; for if they did, y and y' being the distances of the point of intersection below H, Hy would equal Hy', since either equals the moment of the same loads; hence $y = y'$. By moving up the point of application of H, leaving its magnitude unchanged, the entire line of resistance is obviously raised.

2. If the point of application of H is raised, but its magnitude diminished to H', the two lines may meet. If they do, for the point of intersection,

$$Hy = H'y'$$

If through the point of intersection a horizontal line be drawn to meet one of the lines on the other side of the center, the other line will go through this same point. Two lines of resistance cannot intersect at two points not on the same level.

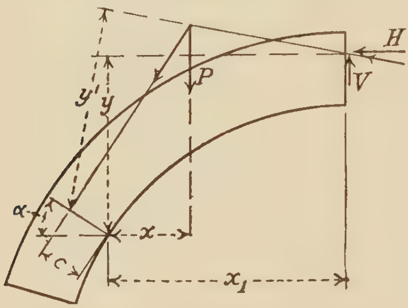


Fig. 321.

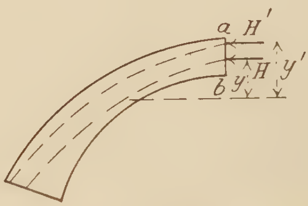


Fig. 322.

10. Maximum and Minimum Lines of Resistance.—The maximum line of resistance is that one, lying within the arch ring, which has the largest horizontal thrust; the minimum line is the one, lying within the arch ring, which has the smallest horizontal thrust.

Suppose a line of resistance drawn and lying entirely within the arch ring. By the previous article, if the center of pressure at the crown (or, in general, at the joint where the thrust is horizontal) is lowered, the entire line of resistance is lowered, and it may go below the intrados somewhere; if it does, then, keeping the same point of application at the crown, the thrust may be increased, and this will raise the entire line except at the crown, making it flatter. By this process it is seen that the maximum line of resistance may be found, and that its characteristic is this:

The maximum line of resistance is the flattest one possible within the arch ring; it touches the extrados of the complete arch at two points near the springing, one on each side of the center, and it touches the intrados at one point (or possibly two) near or at the crown.

By *flattest* is meant the line most extended horizontally and most contracted vertically. The reason why the maximum line may touch the intrados at two points is that the shape of the arch and the loads may be such that it may be above the intrados at the crown and tangent to it a little to each side of the crown, though this would rarely happen. Where the line *touches* either intrados or extrados, it is meant that it is

tangent to it except that at the springing it may not be tangent but may cut the springing joint at the extrados.

In a similar manner it is obvious that the minimum line of resistance is the steepest one possible within the arch ring, i.e., the one most contracted horizontally and extended vertically; it touches the intrados at two points near the springing, one on each side of the center, and it touches the extrados at one point (or possibly two) near or at the crown.

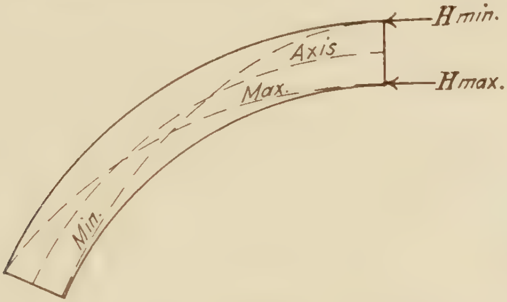


Fig. 323.

In a semiarch, the minimum and maximum lines may be as indicated in Fig. 323.

11. Modes of Failure of Arches.—If the actual line of resistance is either the minimum or maximum in the middle third, the arch is just on the point of opening at some joints. If the actual line is either the minimum or maximum in the entire ring, the arch is just on the point of failure or collapse. This is not from lack of *strength*, primarily, but from lack of *stability*, although if the line approaches too near the edge

Fig. 324.

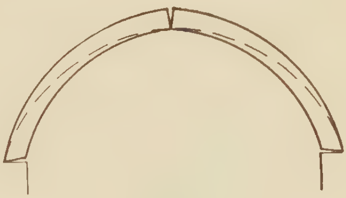


Fig. 325.

of a joint the allowable pressure may be exceeded; for if the actual thrust is less than that of the so-called minimum line or greater than that of the maximum line, no line of resistance will be possible within the arch ring (or middle third), and it will collapse (or have tension). The mode of failure is different in the two cases, and Figs. 324 and 325 indicate them. If the thrust is less than the minimum, the joint at or near the crown tends to open at the intrados, and the two joints near (or at) the springing tend to open at the extrados; and if the thrust is greater than the maximum, the opposite is the case.

But it is clear, *since any line between the minimum and the maximum is possible* in the arch, *it will actually fail only if the minimum and the maximum coincide*, that is, *if only one line is possible, and if that line fulfils the conditions for both maximum and minimum.* If only one line of resistance can be drawn within the arch ring (or the middle third), that line must have the characteristics of the so-called maximum and the so-called minimum lines. It may not be clear, however, that if only one line is possible,

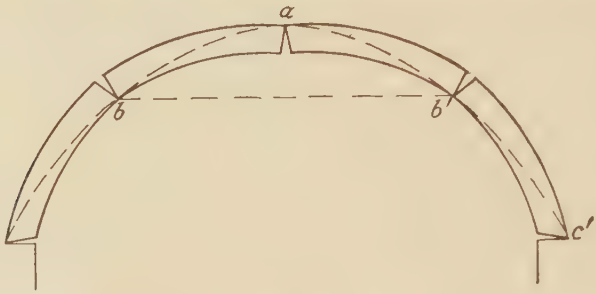


Fig. 326.

the arch will stand (stability only being considered, as in all this discussion). Will an arch stand if it is possible to draw only one line of resistance within it? Clearly it will, because any slightest tendency to open at any joint would throw the center of pressure at that joint toward the other edge; and so, by a conceivably gradual shifting of the line of resistance, it would take the position of a possible line, if one is possible. *The arch will stand up if it can.*

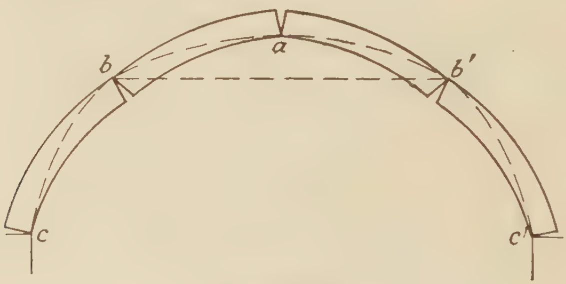


Fig. 327.

For instance, Fig. 326 shows an arch on the point of collapse by pushing down of the crown and spreading of the haunches, which might occur in a flat arch or with a heavy load at the crown. The part bab' corresponds to a minimum line of resistance, and the parts bc and $b'c'$ to a maximum line. If, as in this case, only one line is possible in the arch, the structure is on the point of collapse.

Figure 327 shows an arch on the point of collapse by pushing in of the haunches and pushing up of the crown, which might occur with

heavy loads on the haunches and little at the crown. The part bab' is a maximum line and the parts bc and $b'c'$ a minimum line.

12. Stability vs. Strength.—In what has preceded, stability has been considered, not strength. In steel structures, strength is the basis of design, and stability plays a small part. In masonry structures, stability is perhaps preponderant. One is independent of the other. A structure may be stable but not strong enough, or strong but unstable. The strength of an arch depends upon the maximum stress at the edge of a joint; its stability upon the position of the center of pressure, independent of the amount of pressure.

Maximum and minimum lines of resistance have, in the above discussion, been referred to the full arch ring, assuming stability as the criterion. If the criterion is to be that no joint is to tend to open at the edge, they may be referred equally well to the middle third of the arch ring, and all that has been stated will apply *pari passu.*

13. Criterion of Stability.—From considerations analogous to those in the above discussion it was for many years concluded that an arch would be stable *if a line of resistance could be drawn inside the arch ring*, for, it was said, if any line can be drawn, the maximum and minimum lines can be drawn, and the true line will not be at the same time the maximum and minimum. This conclusion, however, assumes that the true line is intermediate between the so-called maximum and minimum, and is fallacious, because the so-called maximum and minimum lines are not really extremes, but only extremes *supposing the arch to be stable*, or extremes *lying within the arch ring.* It would be perfectly possible, unless some other limitation can be found, for the true line to lie outside the maximum or inside the minimum. Clearly, no conclusion can be drawn until *something*, at least, can be determined regarding the true line.

If it be true, as the writer believes, that an arch will stand if it is statically possible for it to do so, then it will stand if a line of resistance can be drawn inside the arch ring. It was for a long time believed by many that the true thrust was the minimum consistent with stability, that is, that the minimum line was the true line;[1] but this is clearly incorrect. Others have maintained that the true line was the one for which the maximum stress on any joint was a minimum (Culmann, Du Bois). Speaking somewhat loosely and generally, this means that the true line is the one lying nearest the axis.

By the admittedly correct principle of least work, the true line is the one for which the total work of the internal stresses is a minimum. This does not define its position, but again it points to the conclusion that the true line is the one lying nearest the axis.

[1] Moseley's Principle of Least Resistance, so-called; see MOSELEY, "Mechanical Principles of Engineering and Architecture," 1837.

In 1879, Winkler showed that, for an arch of constant section, under vertical loads, the true line of resistance is approximately the one which lies nearest the axis of the arch ring, in the sense that the sum of the squares of the vertical deviations is a minimum.[1] The demonstration assumes that the line of resistance coincides with the equilibrium curve or string polygon, which is not quite true, but nearly so.

Remembering now the location of the maximum line of resistance, it is clear that at or near one end of the axis this line of resistance is as far from the axis as possible; that it approaches the axis, crosses it, and at or near the crown is as far from it as possible on the other side; then it approaches, crosses, and at the other end of the span is again as far from the axis as possible. A similar statement holds for the minimum line, the maximum deviations being in the opposite directions from those of the maximum line.

Suppose the maximum and minimum lines drawn, as in Fig. 323. If we draw a new line from a point in the crown midway between the two, and make it pass at the springing through a point midway between the two, this line will cross each of the others, and is quite sure to be nearer the axis than either of them.

If, therefore, any line of resistance is drawn in the arch ring, and if it is not at the same time the maximum and the minimum, then the maximum and minimum lines may be drawn, and it is reasonably certain that there is a line nearer the center line (axis) than either of these, and that the arch is stable.

The writer therefore believes the statement to be true that *if any line of resistance can be drawn within the arch ring the arch is stable;* and if a line of resistance can be drawn within the middle third, the true one will also be within the middle third, and there will be compression over the whole of every joint.

Professor Baker says that the writer's "method of applying Winkler's theorem is practically nothing more or less than an application of the conclusions derived from the hypothesis of least pressure." With this statement the writer must strenuously disagree. The "hypothesis of least pressure," as stated by Professor Baker, is that the true line of resistance is that which gives the smallest absolute pressure on any joint. But the writer's argument says nothing about absolute pressure. It simply says that, as we pass gradually from the maximum line to the minimum line, there is sure to be a line which is nearer the axis than either of those. And Winkler's result is proved by means of the elastic theory, which Professor Baker advocates, using some reasonable assumptions.

Professor C. C. Williams, in his excellent book on "Masonry Structures," lumps together Moseley's, Winkler's, Scheffler's. Rankine's

[1] *Zeit. d. Arch. u. Ing. Ver. zu Hannover*, p. 199, 1879.

and others as "line of thrust theories, since they were concerned chiefly with locating the line of thrust," and adds: "At best they are very indefinite and unsatisfactory, most of them being based on hypotheses which were more or less conjectural." To lump these theories together is not very discriminating. It is true that all the theories before Winkler's were unsatisfactory, because none of them said a single word about the true line of resistance. Winkler's surely stands in a different category, because its whole object is to find out enough about the true line, based on the elastic theory, to serve as a basis. Of course, all theories, even the elastic theory, are concerned, not chiefly, but entirely, with "locating the line of thrust" (resistance). Winkler's theorem, based on the elastic theory, states a simple concrete result after making a few not unreasonable assumptions. As a matter of fact, the strict elastic theory, which so many engineers today love to use because it is complicated, is itself based on assumptions that are not true (see Art. 22). The writer believes Winkler's principle a sound and sufficient basis for the theory of the arch, not only the stone arch but the concrete and the reinforced concrete arch as well.

The Winkler principle bears upon its face the inherent stamp of probability and plausibility. If a line of resistance coincides with the axis, there is uniform compression over each cross-section, and the work done is that of a central compression. If the line of resistance does not coincide with the axis, then, in all sections where it does not, the compression is unequally distributed, and even if the total compression is the same as in the first case, the work done on a thin slice is greater. By the principle of least work, the condition must be such that the work done is the least possible. Two lines of resistance, one coinciding with the axis and one not doing so, will not have exactly the same total compression or the same section, but this compression will be very nearly the same. The excentricity, however, would increase the work done in the case of the second line. It can easily be proved that if W is the work done on a slice of a rectangular rib of width unity and depth b by a central compression, and W_1 the work done by the same compression acting with an excentricity e_1, along the depth,

$$\frac{W_1}{W} = 1 + \frac{12e^2}{b^2}$$

If

$$\frac{e}{b} = \frac{1}{6}; \frac{W_1}{W} = 1.33$$

If

$$\frac{e}{b} = \frac{1}{12}; \frac{W_1}{W} = 1.08$$

There is inherent probability, then, that the least work will be done by the line nearest the axis. The difficulty in proving this mathemati-

cally to be exactly true is because of the difficulty of taking account of the change of the total compression on a section, for different lines of resistance, and of the fact that the line of resistance is not the equilibrium polygon.

It is common today to deny the correctness of the above reasoning, but the writer believes it valid. Professor Baker says:[1]

> To apply Winkler's theorem, it is necessary to (1) construct a line of resistance, (2) measure its deviations from the axis of the arch, and (3) compute the sum of the squares of the deviations; and it is then necessary to do the same for all possible lines of resistances, the one for which the sum of the squares of the deviations is least being the "true" one.

If this statement is correct, it is of course foolish to use the principle. But it is not necessary to do all this. It is only necessary to observe that, starting with the maximum line and gradually reducing the thrust and raising the point of application in the crown, the line will gradually change from the maximum to the minimum line, and that surely there will be some line that will be nearer the center line than either maximum or minimum, in order to conclude that if any line can be drawn in the arch which is not at the same time maximum and minimum, the true line will be in the arch and the arch will be stable. If it were necessary to find accurately the true line, it would be necessary to do what Professor Baker says; and of course the *stresses* at the edges of a joint cannot be found accurately unless the true line is found. But this is unnecessary in order to judge of *stability*. The stresses can be computed with quite sufficient accuracy without doing all this. The writer considers it a useless expenditure of time to try to find the true line anyway, considering the many uncertainties of the problem, regarding loads, their distribution, the material and workmanship, and even the so-called accurate elastic theory itself (see also Arts. 18–22). The writer has had occasion to study and design a good many stone arches, and concrete arches too, and he has never found it necessary to do what Professor Baker says. He has found the principles of this chapter sufficient, even for concrete arches. His judgment has not proved wrong in a single case. It is true that his computed maximum stresses were probably not accurate, but they were accurate enough, and their error was probably no greater than would be due to using another type of locomotive on a railroad, or another type of vehicle on a highway, or another brand of cement in the concrete, or a little different mix of concrete.

Professor Baker also says that the assertion in italics on page 412 "is disputed by Winkler himself." If Winkler ever disputed this in print, I am not aware of it. In my article on the subject,[2] in which I for the

[1] "A Treatise on Masonry Construction," p. 619, 1909.

[2] "The Stability and Strength of the Stone Arch," *Van Nostrand's Engineering Magazine*, October, 1880.

first time in this country called attention to Winkler's paper, I stated that the assertion was disputed by Winkler himself, and perhaps this was the basis of Professor Baker's statement. The circumstances were as follows: When Winkler's paper was published, I was one of Winkler's students, and having always been dissatisfied with the current theories of the arch, and having always realized that the true basis of the arch theory must depend upon some knowledge of the position of the true line of resistance, I thought I could see that Winkler's principle furnished that knowledge and the clue to the whole matter. I went to Winkler and told him so, and stated that I thought his paper proved that if any line of resistance could be drawn inside the arch ring, the arch would be stable. He disagreed with me, and that was the basis for my statement in the Van Nostrand paper. I did not have time at the moment to argue the matter with Professor Winkler, nor did I have the opportunity to do so at any subsequent time. I believe I could have convinced him, though he was a good deal of a theorist, and often inclined, in my opinion, to carry theory too far. My own mind, however, was satisfied, and that was enough for me.

14. The Process of Designing an Arch. Formulæ for Thickness at Crown.—The span, rise, and shape of intrados and extrados must first be assumed, having reference to appearance, area of waterway required, and all other conditions.

For the determination of the thickness at the crown and springing there are various approximate formulae. Thus, calling t the crown thickness in feet, the following are among those that have been proposed:

Rankine, for a single arch: $t = \sqrt{0.12 \text{ radius at crown}}$

Rankine, for an arch of a series: $t = \sqrt{0.17 \text{ radius at crown}}$

Perronet: $t = 1.066 + 0.035$ span (for larger spans this gives too great values)

Dupuit, for segmental arches: $t = \sqrt{0.074 \text{ span}}$

Dupuit, for semicircular arches: $t = \sqrt{0.13 \text{ span}}$

Trautwine, for first-class cut stone: $t = 0.2 + 0.25\sqrt{\frac{\text{span}}{2} + r}$

where r is the radius of the circle drawn through crown and springing.

For second-class work, take $\frac{9}{8}$ of above

For brick or rubble, take $\frac{4}{3}$ of above

These formulae are defective because they take no account of the loads, shape of intrados, or ratio of rise to span, though Trautwine's takes some account of the last.

Croisette-Desnoyers gives the following:

$$\text{For highway bridges: } t = 0.492 + c\sqrt{6.56r}$$

$$\text{For railway bridges: } t = 0.656 + c_1\sqrt{6.56r}$$

r being the radius of the circular segment with the same rise and span as the arch, and the factors c and c_1 having the values in the following table:

Curve of intrados, $S = \frac{\text{rise}}{\text{span}}$	Highway c	Railroad c_1
Semicircle, ellipse, or segment with $S \leq \frac{1}{4}$	0.15	0.17
Segment of circle, $\frac{1}{4} > S > \frac{1}{6}$	0.14	0.16
Segment of circle, $\frac{1}{6} > S > \frac{1}{8}$	0.13	0.15
Segment of circle, $\frac{1}{8} > S > \frac{1}{10}$	0.12	0.14
Segment of circle, $\frac{1}{10} > S > \frac{1}{12}$	0.11	0.13

Séjourné, in his monumental work on arches, in six quarto volumes, deduces from a study of 562 works, comprising about 3,300 arches, the empirical formula:

$$t \text{ (in feet)} = \alpha\mu(3.28 + \sqrt{3.28l})$$

where l is the span in feet.

α depends upon the loads. Séjourné says:

For highway bridges:

A good mean value is 0.15
It is cowardly to use above 0.18
It is reckless to use less than 0.12

For standard-gage railroad (French):

A good mean is . . . . . 0.18 to 0.19
Above 0.21 is cowardly; below 0.15 reckless.

For narrow-gage railroads:

A good mean is 0.17
Above 0.20 is cowardly; below 0.14 reckless.

μ is another constant, depending upon the shape, and the ratio S of rise to span, such that:

For semicircular arches, $\mu = 1$

For flat elliptical arches, $\mu = \dfrac{4}{3 + 2S}$

For segmental arches, $\mu = \frac{4}{3}(1 - S + S^2)$

For hinged arches (three hinges) the same formula is used, with the last value of μ.

The thickness at the springing is ct, where the average value of c for highway bridges is 1.46, varying from 1.0 to 2.22.

Séjourné recommends:

For full centered arches:

Thickness at midheight = twice thickness at crown.

For elliptical arches:

Thickness at midheight = thickness at crown $(1 + 2S)$

$$\left(S = \frac{\text{rise}}{\text{span}}\right)$$

For segmental arches, with central angle $> 120°$:

Thickness at 60° from crown = twice thickness at crown with central angle $< 60°$.

Thickness at springing = thickness at crown $(1 + 12S^2)$

The following formula,[1] devised by J. P. Schwada, City Engineer of Milwaukee, is said to give close results for reinforced concrete arches:

$$t = \frac{l^2}{57.6 f_c K(R - t)}\left(\frac{B}{20} + R + 8t + 6F + \frac{w}{20}\right)$$

where l is the span, R the rise, and F the depth of fill at the crown, in feet; B the weight of track and ballast or pavement and w the uniform live load, in pounds per square foot, f_c the allowable unit stress in pounds per square inch, and K the ratio of the average stress at the crown to f_c. This must be solved by trial, since t occurs on both sides of the equation, and K depends upon circumstances, a table of its values being given. The apparent lack of homogeneity of the formula is due to the cancellation of certain constants. The formula is derived by assuming the line of resistance to pass through the center of the crown and the springing, the vertical depth at springing being three times that at the crown; and finding the crown thrust by taking moments.

15. Having tentatively assumed the arch ring, it must next be divided into a sufficient number of voussoirs, not too many, and the dead load on each computed, together with its point of application. This is best done by reducing the spandrel filling to an equivalent mass of masonry. If the weight of fill is 110 and of masonry 150 pounds per cubic foot, and the actual top of the fill is ab in Fig. 328, reduce each depth of fill bc or de in the ratio $^{110}/_{150}$, and the curve so obtained, $a'b'$, will be the top of the equivalent masonry fill. The dead weight on any voussoir fg will be the weight of the masonry $fghknef$, 1 foot thick perpendicular to the paper, if any other dead load, such as track or pavement, has been included with the fill, or the earth pressure may be computed by the usual methods.

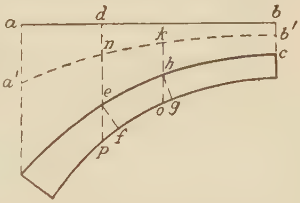


Fig. 328.

The live load per linear foot on a strip 1 foot thick may be taken as a uniform load or as disconnected portions of a uniformly distributed load representing the effect of concentrated loads distributed through the

[1] *Eng. News*, Nov. 9, 1916.

ballast and fill. See Art. 38 of Chap. XX for this distribution. In a highway bridge it is sufficient to consider the live load as uniform per square foot, and to consider three cases of loading, namely, over the whole span, over one-half, and over the central half. In a railway bridge, the track live load must first be distributed over a width equal to the distance apart of tracks center to center, which is generally 12 to 14 feet. If the span is, say, 50 feet, and if heavy engines are run, the table on page 46 shows that the uniform load per foot may be 10,000 pounds, with 15,000 or 16,000 over a smaller length of, say, 15 feet. If the width between tracks is 13 feet, these weights correspond to 770 and, say, 1,230 pounds per square foot respectively, or to a depth of masonry of $^{770}/_{150} = 5.1$ and $1{,}230/150 = 8.2$ feet. The depth of 5.1 feet, or a rectangle of masonry of this depth, may cover the entire span of 50 feet, or one-half, or the central half. The depth of 8.2 feet over a length of, say, 15 feet on the rail will be distributed to some extent longitudinally (it has already been distributed transversely) by the filling; if the depth of fill is 2 feet at the crown, this load may be considered distributed over $15 + 4 = 19$ feet there, giving a load per foot of $^{15}/_{19} \times 8.2 = 6.5$ feet of masonry at the crown, covering 19 feet; and if the depth of fill at the springing is 12 feet, it may be considered distributed over $15 + 2 \times 12 = 39$ feet (at 45° slope), giving at the springing a load equivalent to a depth of $^{15}/_{39} \times 8.2 = 3.15$ feet of masonry. The live load may therefore be a depth of masonry of 5.1 feet covering any part or the whole of the span, or a depth of 6.5 feet covering about 19 feet at the crown.

It is obvious that the loads are uncertain. If there is no fill, but if the track is carried on a solid floor supported at intervals by columns resting on the arch, the live load would be 770 pounds per running foot over any part of the span, or this load *plus* a load of $1{,}230 - 770 = 460$ pounds per foot over any 15 feet of track.

In dividing the arch into voussoirs, it is better to take the joints of the ring radially, and it is often sufficient to consider the extrados of each voussoir to carry the vertical load above it. Sometimes, to make the computation of the center of gravity easier, the arch ring is divided by vertical and not radial lines, but this is not advisable. The area of dead load carried by each voussoir will be a trapezoid *kopn* in Fig. 328 plus the triangle *ogh*, minus the triangle *pfe*.

16. Drawing the Line of Resistance.—Having found the loads and decided on the live load to be assumed, it is very easy to draw a line of resistance. Lay off the vertical loads to scale, choose a pole O, draw the rays, start at some point in the crown joint, draw the equilibrium polygon, and find the intersection of each string with its corresponding joint, string $O1$ with joint 1, and so on.

It is necessary, however, to choose the pole and start the polygon at the point in the crown joint (joint O) so that the line of resistance may

fulfil certain conditions, such as passing through certain points. This has all been thoroughly explained in Chap. XI.

The best method of procedure, *supposing that the arch is symmetrical and symmetrically loaded*, is first to draw a line of resistance which shall pass through a and b (Fig. 329). The thrust H at a will be horizontal, and the pole O is a horizontal line through o in the force polygon. The value of H which will cause the line of resistance to pass through b may be found analytically by taking moments about b, or by first drawing any

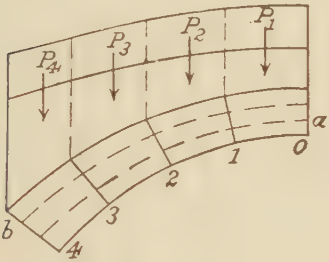


FIG. 329.

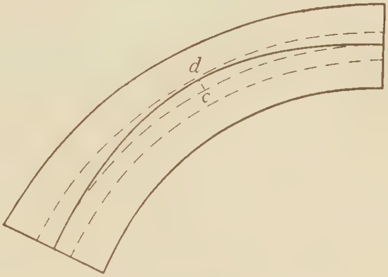


FIG. 330.

equilibrium polygon through a, and then, by the graphical method of Chap. XI, finding the new pole for which the last string will go through b. If this line of resistance lies wholly within the middle third, it is safe to conclude that the true line of resistance also lies within those limits, that there is no tension anywhere, and that the arch is *stable*. The actual stress at the edge of any joint, however, cannot be calculated from the line drawn, because it is not the true line.

A second line may now be drawn, passing through the centers of the crown and springing joints (Fig. 330). This may or may not lie entirely within the middle third. If it deviates from the axis most at the point c, but is within the middle third, a new line may be drawn passing through the middle of the deviation cd, finding analytically the point of application of the same horizontal thrust at the crown which will make the line do this (which may be done by a single equation of moments). This line will then be inside the axis at the crown and springing and outside at c, and will be very near the true line.

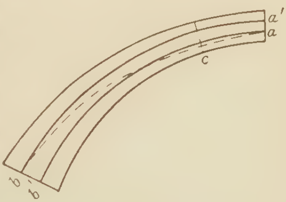


FIG. 331.

With a little initiative, by a few trials such as this, a line may be found which is sufficiently near the true line to allow the maximum stresses at the edges of the joints to be computed with sufficient accuracy.

If, however, the line through a and b (Fig. 331) does not lie entirely within the middle third, but lies inside of it near the crown, deviating from it most at joint c, the following reasoning is obvious. By keeping

the same thrust and raising the point of application in the crown, a new line may be drawn which will lie entirely above the old one; by raising it sufficiently, it may be made to pass through the inside middle third point at joint c, but it will now pass outside the middle third at the springing. Find, now, the value of the thrust H and its point of application at the crown, which will give a line passing through b and the inside middle third at joint c. If this line lies entirely within the middle third, there is no tension. If, on the other hand, the point of application at the crown is above the middle third, while the line remains within the middle third except near the crown, not passing inside at joint c but being tangent to the middle third there, then no line of resistance can be drawn within the middle, and the thickness must be increased if there is to be no tension.

Instead of beginning by drawing a line through a and b in Fig. 331, that is, drawing the maximum line of resistance and seeing whether the maximum line is also the minimum line, we might begin by drawing a line through a' and b', that is the minimum line.

17. A symmetrical arch without hinges has thus far been considered, and the load has been considered symmetrical. An unsymmetrical load must next be studied, as, for instance, a live load on the right half of the span. In this case the thrust at the crown is not horizontal, and the entire arch must be considered, and not merely one-half. Ordinarily, the only unsymmetrical load which it is necessary to consider is a live load on one half; but in some cases it may be desirable to consider a live load over one-quarter of the span; or, for an electric railroad bridge, a live uniform load covering a certain distance at and near the center of the right half, representing the case when one truck of a heavy car has its center at the quarter point.

The process of investigating a case of unsymmetrical loading on a symmetrical span, or of any loading on an unsymmetrical span, is the following:

Suppose the live load to cover the right half. Draw a line of resistance passing through the inside of the middle third at the springings, and through the center of the middle third at the crown. This may be done analytically or graphically. Analytically, the point of application at the crown being assumed at the center of the arch ring, the horizontal and vertical components of the thrust there are easily found, by two simultaneous equations of moments about the springings. For a load on the right, this line of resistance will probably lie above the axis for some distance to the right of the crown, and below the axis on the left. If this line of resistance lies within the middle third, there is no tension anywhere; if it goes outside or inside the middle third, the points can be seen at which it deviates most, and a new line can be drawn passing through the edge of the middle third at these points. A line of resistance

can easily be found which will touch the extrados at some point near the crown, and the intrados at two points, one near each springing; and this will be the minimum line of resistance, as in Fig. 332. If the line so drawn passes outside the middle third near the springing, no line is possible inside the middle third, as in Fig. 333.

By a few trials such as these it may be found, for any loading, whether a line of resistance may be drawn within the middle third. If one can be so drawn, the arch is stable, with no tension anywhere; and a close approximation may be found to the true line of resistance, or the one lying nearest the axis, and so the unit stresses may be found, and the angle which the pressure on any joint makes with the normal to that joint. If the minimum line of resistance, as drawn in Fig. 332, is far from being also a maximum line of resistance, the thickness of the arch ring may be reduced, if desired, and if the unit stresses will permit.

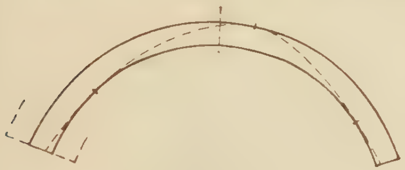


Fig. 332.

Fig. 333.

The Joint of Rupture.—If an arch springs vertically from the abutments, and perhaps if it does not, it may obviously be built up for a certain distance and, after the mortar has set, would stand up to that distance even if the central part were omitted. In other words, up to a certain point the arch is really a part of the abutment or pier, and the real arch is only the part between these points. This is the *joint of rupture* of the actual arch; it is perhaps best defined as the point at which the real line of resistance approaches nearest to the intrados, or at which there is the greatest tendency to open at the extrados. It is often, however, as by Baker, referred to the theory of least crown thrust, and defined as being the joint requiring the greatest thrust which will make the line of resistance pass through the inner edge of the joint. Anyway, up to a certain joint of rupture the arch may be considered as a part of the abutment or pier.

18. The Elastic Theory.—By this theory, which is very popular at present, and generally considered the most reliable, the arch is considered as a homogeneous elastic rib of continuous section. If such a rib is supposed to be fixed at the left end, as it would be if imbedded in the abutment, and free at the other, as in Fig. 334, the outer forces are the known loads, the three unknown reaction elements at the right, namely the vertical and horizontal components of that reaction and the couple necessary to maintain the unchanging inclination of the axis, and the

reactions at the left, which can be found by the principles of statics from the other forces. There are thus three unknown outer forces or elements: two forces and a bending couple. Three elastic conditions are therefore necessary. These are that the vertical and horizontal deflections at the right end, and the change of inclination of the axis there, are zero. These deflections and this rotation of the axis may be found by the theory of curved beams. The arch is a curved cantilever free at the right end, and the deflections and change of slope at that end must be found in terms of the known loads and the unknown reaction elements at the right, and the expressions for these placed equal to zero, from which simultaneous equations the three unknown reaction elements may be found. This is one way of treating the problem. There are others.

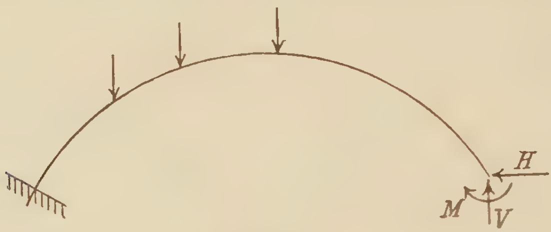


Fig. 334.

In order to express the deflections and the rotation at the right end, the effect must be found of the deformation of each infinitesimal slice of the arch, ds in length, and with a moment of inertia I, and these changes added or integrated over the whole length. An integral is obtained containing the factor ds/I.

In a solid steel arch rib, I is usually assumed constant and the integrals found. In a stone on concrete arch, in which the depth is sometimes much greater at the springing than at the crown, it is generally considered too inaccurate to assume I constant, since it varies as the cube of the depth of a rectangular section. The sums are therefore not obtained by integration, but the rib is divided into parts of varying lengths S along the axis, such that each S divided by the average I along that length is constant, and the integrals are obtained by a summation, which is somewhat long and tedious.

If this elastic theory is applied to the stone arch, in which the line of resistance is everywhere within the middle third, it is evidently only approximate. The arch is not homogeneous, the modulus of elasticity of the stone being different from that of the mortar joints; and although the latter are thin, there is unquestionably an error of unknown amount here. The writer believes that the method of treatment in this chapter, based on Winkler's principle, to be equally accurate and much easier of application.

If the elastic theory is applied to the reinforced concrete arch, it assumes the section which resists stress to be a rectangle, varying in depth from the crown to the springings. This is not correct. Even the usual theory of the simple reinforced concrete beam is not correct. It may be safe, and probably is, but it is self-deception to imagine it accurate. If the steel at any section of a reinforced concrete beam or arch is acting in tension, then unless there is a crack in the concrete there, or unless the steel slips in the concrete, the concrete *must also be acting in tension.* There is no question about this. It is *assumed* that the concrete carries no tension, *because there may be a crack there* since the tensile strength of concrete is small, but the assumption is untrue unless there is actually a crack. But making this assumption, what is the cross-section? Clearly a compression rectangle of concrete and the steel rods that are in tension (and the steel that is in compression on the other side, if there is any). Suppose, now that in a reinforced concrete arch (Fig. 335), the line of

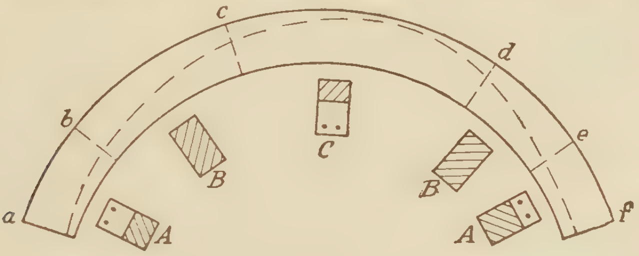


FIG. 335.

resistance is as indicated, the steel at the extrados acting in tension near the springings, and the steel at the intrados acting in tension near the crown. Then from a to b, and from e to f, where the extrados rods are acting, the section is like A, but varying as the neutral axis varies; from b to c and from d to e, where the whole section is in compression, the section is like B (neglecting here the steel in compression, which would complicate the matter still more), and from c to d, where the intrados steel is acting, the section is like C, under the assumption that is made. Is it not rather credulous and impractical to imagine that the section is continuous and rectangular, and to employ fine-spun mathematical theories in this case? The writer is convinced that to apply the elastic theory even to the reinforced concrete arch is illusory, and a vain seeking after exactness where exactness is impossible.

If the elastic theory is applied to the plain concrete arch, or to the reinforced concrete arch in which the line of resistance is everywhere within the middle third, the case is similar to that of the stone arch. Yet this seems to be the case to which the theory is best applicable, because we have a material entirely of concrete, and not of stone with mortar joints. But in that case the reinforcement is unnecessary,

except as a safeguard against possible stresses of tension due to errors of computation or in the loads assumed, or to accidents. A small increase in the thickness of the arch ring would provide against these things just as well, and more cheaply.

Such arches may be treated by the methods of this chapter, with complete reliance on the results, if the following are admitted:

1. The true line of resistance is the one lying nearest the center line.
2. As we pass gradually from the maximum to the minimum line, some line is sure to be found which is nearer the center line than either the so-called maximum or the minimum.
3. It is not necessary to compute the stresses at the edges of the joints with extreme exactness.

Temperature stresses will be caused in arches of stone or concrete. They are not taken account of by the methods of this chapter. They may, however, be computed approximately without much difficulty by elastic formulae.

19. The elastic theory, however, seems to be firmly intrenched in American engineering literature. Perhaps some who use it do not realize its defects and assumptions, and like it because it is complex and mathematical. It seems to be a curious characteristic of the human mind that it so often prefers complexity to simplicity, and mistakes obscurity for profundity. Those who use the elastic theory no doubt think it accurate; but the writer has grave doubts whether it is really more accurate than the methods explained in this chapter. Certainly it is not exact. It has not been proved which of the two is more accurate, and it cannot be proved. Therefore those who like it may go on using it, and believing that they approach exactness. The writer believes in elastic methods, if they are necessary; not if they are unnecessary and if a simpler method is just as good. He has had occasion, during a somewhat long experience, to treat a number of arches of stone and of reinforced concrete, and he has found the methods of the present chapter, compared with the elastic theory, fully as satisfactory, much simpler, and, he believes, as accurate.

The elastic theory belongs with advanced structures, and is given in a later volume.

20. If the steel in a reinforced concrete arch acts in tension, it is because the resultant on a section acts outside the kernel (practically the middle third) of the section. If this is allowed, the arch ring may of course be thinner than if no tension is allowed, and some saving of concrete results. The writer, however, doubts the advisability of allowing tension in an arch. He believes that it should not be allowed in an arch over salt water, and perhaps not in any arch in a damp locality, for the reason that if the steel acts in tension there must be tension in the concrete also (though this is neglected), and any cracks in the concrete may lead to the rusting of the rods. Many engineers, and of course all who are

interested in promoting the use of concrete, will disagree with this, and it is true that many slender arches of reinforced concrete exist, in which the tensile action of the steel is depended on.

21. These remarks must not be construed as an endeavor to dissuade anybody from using the elastic theory. The writer is merely pointing out the inaccuracies in that theory, and presenting arguments to show that simpler and less mathematical methods will give results quite accurate enough for practice, and perhaps as accurate as the elastic theory. The word "perhaps" is used because the matter is necessarily uncertain, and accuracy impossible of attainment. Nobody can ever know which method is the more accurate, but we should realize that more mathematics does not mean more accuracy. It is suggested that the reader reflect upon the wise remark of Aristotle, quoted in the last paragraph of "Strength of Materials."

22. Assumptions Made in the Elastic Theory.—The elastic theory is often termed "exact." The assumptions made in it are the following:

1. That the ends are rigid and do not rotate (this is untrue).
2. That the span does not change at all (this is untrue).
3. That the material is homogeneous (this is untrue).
4. That the modulus of elasticity is constant, not changing with the pressure (this is untrue, though perhaps close).
5. That the terms with r in the denominator may be neglected (this may be far from true).
6. That the integrals may be replaced by summations (this is approximate).
7. That the formulae for flexure are exact (this is untrue).
8. The stresses due to shrinkage are neglected.
9. That the section is a rectangle (this is untrue; see Art. 18).
10. That the loads may be determined accurately (this is untrue; both the loads and their distribution on the arch ring are quite uncertain).

Possibly to some minds, not too mathematical, these facts may justify some of the conclusions in the present chapter. Perhaps the reason may be perceived for the Avant-Propos to M. Séjourné's great work, quoted at the beginning of this chapter.

23. Best Shape of Arch Axis.—It has been shown that the least material and the greatest stability would result if the axis had the shape of the linear arch for the loads. If the loads are unchangeable, this condition may be realized, and is discussed in the chapter on Linear Arches, though such a condition would result in a polygonal arch for a strict series of concentrated loads. If the load consists of the dead load and a uniform live load which may cover the whole or any part of the span, the question arises as to the most favorable shape of the axis, considering the variability of the live load. If the axis coincides with the linear arch at a given point, the moment about that point of all the loads on one side of

it is zero. If the moment at a point of the axis is positive, the center of pressure lies outside the point; if negative, inside. The best shape for the axis would seem to be that for which at each section the total maximum positive and negative moments are equal.

Let M_1 = maximum live load positive moment.

M_2 = maximum live load negative moment (a negative quantity).

M_3 = maximum dead load moment.

The above condition requires that

$$M_1 + M_3 = -(M_2 + M_3)$$

$$M_3 = -\frac{M_1 + M_2}{2}$$

$$= -\tfrac{1}{2} \text{ (moment for live load over entire span)}$$

$$= -\text{moment for one-half live load over entire span} \qquad (6)$$

This condition will be fulfilled, if the axis is the linear arch for a loading consisting of the dead load plus a load equal to one-half the live load per foot, extending over the entire span. Such a shape cannot always be realized, but the condition is a useful one to remember.[1]

24. Skew Arches.—The skew arch has already been described as one in which the roadway makes an angle with the axes of the arch, so that

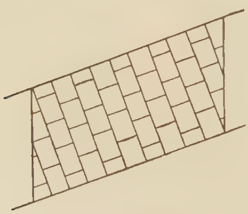


FIG. 336.

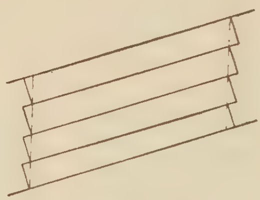


FIG. 337.

the faces are oblique with the axis (Fig. 336). It is possible to build this arch as a series of right arches parallel to the face (Fig. 337), but in this case the intrados is not a cylinder, but a series of cylinders with parallel axes, and there will be no masonry bond between the rings, though they may be connected inefficiently by dowels. A load on any ring should therefore be considered as carried entirely by that ring, which, while all right for a uniform load, is not favorable for concentrated loads.

If the intrados and extrados are made cylindrical, it would be possible to make the coursing joints all planes perpendicular to the faces, and the ring joints all planes parallel to the faces, with a bond as in a square arch, as in Fig. 336. In this case, all the intersections of joints with the intrados would be portions of ellipses, and would not meet at right angles

[1] See a valuable paper by VICTOR H. COCHRANE on "The Design of Symmetrical Hingeless Concrete Arches," *Proc. Eng. Soc. Western Penna.* vol. XXXII, pp. 647–713, 1916.

(on the intrados or extrados), though the joint surfaces themselves would be at right angles and all would be plane surfaces. This has led to another construction by which the joints intersect the intrados in curves which cross at right angles, and the joints are helicoidal or warped surfaces, and not at right angles to each other. Such arches are difficult to construct, and are seldom or never built now, since the advent of reinforced concrete, which enables the arch to be built as a monolith.

While reinforced concrete structures will be considered in a later chapter, it is desirable to give here the general principles of monolithic skew arches. In such an arch a single load tends to be carried by a ring perpendicular to the abutments, and not by one parallel to the faces, which would have a longer span. The load on *abcd* (Fig. 338) would then be carried as on a square arch, but the load on *ade* could not be carried by rings perpendicular to *de*, because these rings would be incomplete. Therefore the load on *ade* must be carried over, by means of

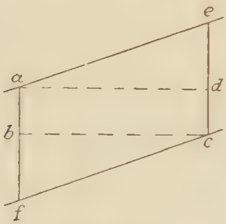


Fig. 338.

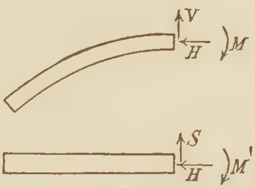


Fig. 339.

shearing stresses on the section *ad*, into the part *abcd*, and the load on *bcf* similarly. This obviously produces an unequally distributed reaction on *af* and *ce*, even with a uniformly distributed load, with a greater reaction at *a* and *c* than at *f* and *e*. Indeed, there may even be downward reactions at *f* and *e*, as in some tests made in the laboratory on a model arch, by Prof. Clyde T. Morris.[1] This may be otherwise expressed by saying that on a ring section (Fig. 339) there are not only, at the crown, two forces H and V, and a moment M, but there are also, in plan, a transverse shear S and a moment M'.[2]

The statement is made in some books that a skew arch may be treated like a right arch, taking rings parallel to the forces. This is obviously incorrect.

In a skew arch which failed in Australia, about 1903, the concrete near *a* and *c* was found "literally pulverized"; and Prof. W. C. Kernot, who studied the subject with rubber models, found excessive pressures at those points.

[1] *Eng. News-Record*, p. 638, Apr. 20, 1922.

[2] See, for a complicated mathematical analysis, a paper by Prof. J. C. Rathbeun on "Analysis of the Stresses on the Ring of a Concrete Skew Arch," *Trans. Am. Soc. C. E.*, February, 1924.

Dr. E. Fischer,[1] gives an analysis of the stress in a skew arch which seems reasonable and simple.[2] He likens it to an excentrically loaded pier. Let the skew arch $ABA'B'$ (Fig. 340) be completed to a right arch $ADA'D'$. The load is only on $ABA'B'$, the parts ABD and $A'B'D'$ being without weight. For a full load on the entire span, the thrust at

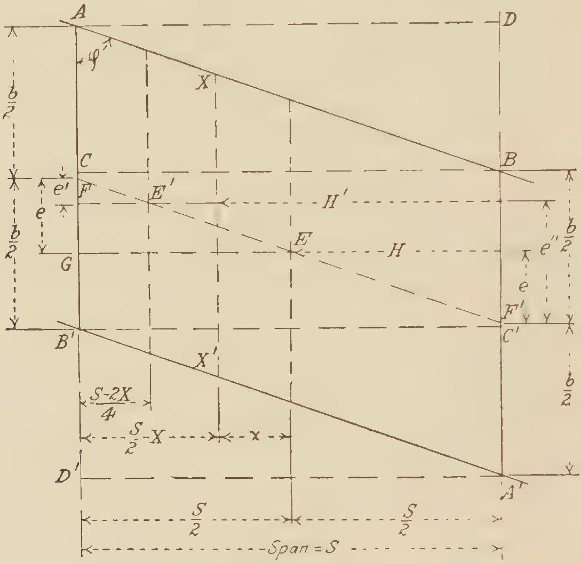


Fig. 340.

the crown will go through E, the center of the crown coursing joint, on account of symmetry. F and F' are at the centers of AB' and BA'. The excentricity of the thrust on the abutment is

$$FG = \frac{S}{2} \cot \varphi = e$$

The thrust H is distributed along AB', in a planar manner, with the following thrust per unit length p:

$$\left.\begin{aligned} \text{At } A: p &= \frac{H}{b} - \frac{6He}{b^2} \\ \text{At } B': p &= \frac{H}{b} + \frac{6He}{b^2} \end{aligned}\right\} \qquad (7)$$

At any section XX', distant x from the crown, replace e by $e' = x \cot \varphi$, to find the distribution on XX'. The total H and V at each abutment may be found as for a right arch of equal area, $AB'C'D$. The vertical reaction will be distributed in the same manner as the horizontal thrust, by Eq. (7) with V instead of H.

[1] "Beton und Eisen," p. 391, 1911.

[2] See Professor Morris' article, *Eng. News-Record*, Apr. 20, 1922.

As an illustration of the procedure with partial loading, suppose the area $AXX'B'$ to be loaded with a uniform load. The total thrust and vertical load at each abutment H' and V' are found by considering the right arch of equal area $AB'C'D$, and the thrust must act opposite the center of the load, at E'. The excentricity on the left abutment is

$$e' = \frac{S - 2x}{4} \cot \varphi$$

and on the right abutment

$$e'' = \frac{3S + 2x}{4} \cot \varphi$$

If the resultant thrust on an abutment lies outside the middle third of its length AB', there will be tension and a downward reaction at A, as in Professor Morris' tests. In a series of skew arches there will thus be a *horizontal torsion* on the piers, which can be calculated.

The fact that the pressure at B' exceeds that at A proves that there is a longitudinal shear in the arch on vertical planes perpendicular to the abutment.

This theory is not exact. No structural theory is. There may be a small component of thrust *along* AB'. But on the whole, the theory is no doubt *safe*.

The best rule to remember is: *Avoid skew arches if possible.*

25. The Groined Arch.—This arch is formed by the intersection of two arches M and N (Fig. 341), which may have the same or different form and shape. BD and AC will be the lines of intersection of the intrados. ADE and BCE will be parts of arch M, and ABE and DCE parts of arch N. If the ring joints are parallel to the lines shown, the load on ABE will cause thrusts on BE and AE, and the load on BCE similarly, as indicated; and, strictly speaking, there should be ribs at AC and BD to carry the resultant thrust. Generally there are no ribs; the two arches are perfectly cylindrical; in that case the material along the diagonal lines does act as such ribs. Or the load on an area such as BCE may be carried by shearing and cantilever action back to the part of M below BC, and the structure may be considered as four arch faces such as BC, each carrying a triangular projecting cantilever. The theory of such an arch is very uncertain, and no doubt there is a combination of arch and cantilever action. The engineer must merely design so that there will be safety under some one reasonable assumption.

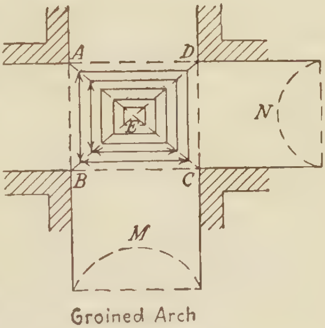


FIG. 341.

Groined arches are most frequently used to cover a rectangular area, being supported on columns, as in Fig. 342. They are often so used as the roofs of covered reservoirs, with several feet of earth above them. The extrados is often parabolic, as shown.[1] This construction was probably first used for this purpose by William Wheeler, C.E., at Somersworth, N. H., and at Ashland, Wis. It allows free circulation of air, and is economical. The arches are generally not reinforced. The floor may also be groined.

The load on $ABCD$ is on a simple arch. It has often been assumed, without analysis, that the loads on ADE and similar parts take care of themselves somehow. The thickness is generally guessed at, or taken from existing similar structures. Metcalf, in *Engineering News*,

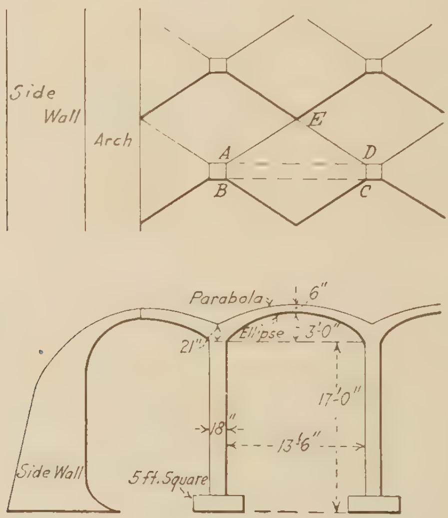


FIG. 342.

Dec. 24, 1903, gave a list of groined arches, but Wiggin extended this list. A minimum crown thickness of 6 inches is common, as in Fig. 342. When used to cover reservoirs, the side row of arches abut at the crown against half of a simple arch springing from the side wall, as in the figure. A roof of this kind, at Baltimore, collapsed in 1913,[2] said to be on account of premature loading of the arches nearest the side wall without provision for taking of the thrust at the end of the completed section, which might have been done by diagonal braces from the tops of one row of columns to the bottom of the next row.

[1] See WIGGIN, T. H., "The Groined Arch in Filter and Covered Reservoir Construction," *Eng. News*, p. 398, Apr. 7, 1910; and *Eng. Record*, p. 298, Mar. 10, 1910; METCALF, LEONARD, "The Groined Arch," *Trans. Am. Soc. C. E.*, vol. XLIII, p. 37, 1900.

[2] See *Eng. News*, p. 1100, Nov. 27, 1913.

FIG. 343.—Philadelphia covered filter bed. (*Courtesy of Alexander Murdock, Chief of Bureau of Water, Philadelphia, Pa.*)

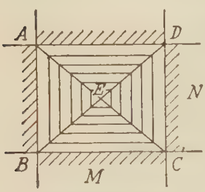
FIG. 344.—Philadelphia covered filter bed. (*Courtesy of Alexander Murdock, Chief of Bureau of Water, Philadelphia, Pa.*)

From Wiggin's table, which gives a list of 56 groined arches, with dimensions, concrete proportions, loads, location, and date of construction, it appears that the span varies from 10 to 17 feet, the rise from 1 foot 6 inches for 10-foot span to 4 feet 3 inches for 17-foot span, the crown thickness 6 to 12 inches (generally 6), and the piers about 2 feet square.

Fig. 345.—Philadelphia covered filter bed.

Figure 343 shows the interior of a groined-arch structure of one of the Philadelphia filters, and Fig. 344 the span next to the side wall, while Fig. 345 shows the centers.

26. Other Forms of Arches.—There are many forms of arches, with reference to the shape of the intrados or soffit. The Gothic or pointed arch has the intrados of two curves meeting at an angle at the crown, and is only structurally suitable when there is a concentrated load at that point. A *dome* is a form of arch with a spherical or otherwise double curved soffit; it will be considered in a succeeding volume.



Cloistered Arch
Fig. 346.

A *cloistered* arch resembles a groined arch in being formed by the intersection of two cylindrical arches (Fig. 346); but here the portions ABE and CDE are parts of arch M (not of arch N as in the groined arch), and the portions ADE and BCE are parts of arch N. There are walls AB, BC, CD, and DE, and the construction is suitable for covering a single chamber or dungeon.

27. Hinged Masonry Arches.—We have seen that the arch with fixed ends is statically undetermined, as regards the outer forces, because there are six unknown quantities and but three statical conditions, so that three elastic conditions must be found. These three conditions are as follows:

If the arch is considered to be a cantilever curved beam fixed at one end, the vertical and horizontal movements and the rotation at the other end are zero. This is never exactly true, because piers and abutments will always yield *to some extent*. The amount of such yielding is unknown, but the so-called exact methods are clearly not exact. A committee of the A.S.C.E. has made and is making observations on the actual movements of arches, and possibly may obtain results which will indicate how far the so-called exact theory is in error.

The arch may be made statically determined with respect to the outer forces by inserting three hinges, one at the crown and one at each springing. This will remove the uncertainties stated in Art. 22. Yielding of the piers and abutments will simply allow the arch to rotate about the hinges. Each half of the arch will be a curved beam acted upon by known outer forces. There will be no temperature stresses.

The arch may be *partially hinged* or *completely hinged*.

A *partially hinged arch* is one in which the hinges act under the dead load alone; after the centers are lowered, the joint at the hinges is filled with mortar, so that for the live load the arch is without hinges, though even then the action will be something between that of a hinged arch and that of a rigid one. The advantages of partially hinging are as follows:

a. Appearance.—Some engineers believe that an arch looks better if it appears without hinges. This is a matter of opinion. I do not agree with this view.

b. Metal hinges will rust. This cannot be prevented, and hinges are impossible to replace. Filling the joints with mortar protects the hinges against rusting.

But hinges need not be of metal, as will be seen.

A *completely hinged arch* is one in which the hinges act for both dead and live loads, and show as seen from the side, although they may be masked or concealed by a thin facing, which is often done.

28. Types of Hinges.—Four types of hinges may be distinguished:

a. Lead Plates.—A strip of lead, or, if a harder metal is desired, of an alloy of lead and antimony, is placed in the joint (Fig. 347). This will concentrate the pressure nearly at the center of the joint. The thickness of the lead has almost always been taken as about ¾ inch (20 millimeters); its width w has a ratio to t which has been as follows:

$$\text{At crown, values of } w/t = \begin{cases} 0.50 \text{ maximum} \\ 0.13 \text{ minimum} \\ 0.237 \text{ mean} \end{cases}$$

$$\text{At springing, values of } w/t = \begin{cases} 0.50 \text{ maximum} \\ 0.10 \text{ minimum} \\ 0.242 \text{ mean} \end{cases}$$

The length of each strip is 40 to 60 inches, with about 4 inches between the strips extending along the joint from face to face of the arch. An average pressure allowed on the lead is 850 pounds per square inch, the maximum being assumed to be 1,700 pounds per square inch.

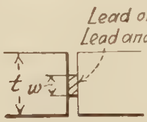


FIG. 347.

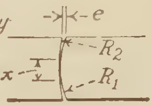


FIG. 348.

FIG. 349.

FIGS. 347–349.—Lead plate: rolling hinges.

b. Rolling Contact Hinges.—This type (Fig. 348) has two surfaces of different radii which roll upon each other. Originally, with no load, the contact will be along a line, but a load will flatten this to a width $2x$, depending upon the material and the load. The contact may be between granite blocks, or concrete blocks, plain or reinforced, and dressed smooth, placed at the hinges; or between metal surfaces. One surface may be plane and one curved (Fig. 349). Figures 350, 351, and 352 illustrate different forms of metal hinges of this type.

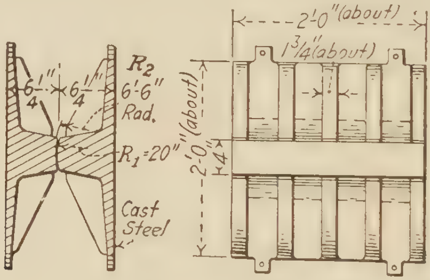


FIG. 350.—Rolling contact hinge. Bridge at Kempten (Bavaria).

The Maximilian bridge at Munich (Fig. 352), in which the hinges were lubricated to secure more free rotation, slipped off its hinges. It was replaced, and the dowels shown in the figure, which fit loosely in one casting, were inserted. If they had been inserted in the beginning, the accident probably would not have occurred.[1] *Every hinged connection, in any structure, should have provision against any movement except rotation, either by dowels as above shown, by collars on pins, by overlapping plates, or in some other manner.*

The design of rolling hinges requires a knowledge of the stresses in a rolling contact. Formulae for these have been given by Hertz, Köpcke,

[1] See *Eng. News*, p. 373, Oct. 27, 1904; also GODFREY, "Engineering Failures."

and Barkhausen. They are far from agreement. Hertz's formulae, which Séjourné accepts as the most reliable, are, if $e \times 10^5 = E$, and N = normal pressure per inch length,

$$2x \text{ (inches)} = \frac{0.0096}{\sqrt{e}} \sqrt{\frac{N}{\frac{1}{R_1} - \frac{1}{R_2}}}$$

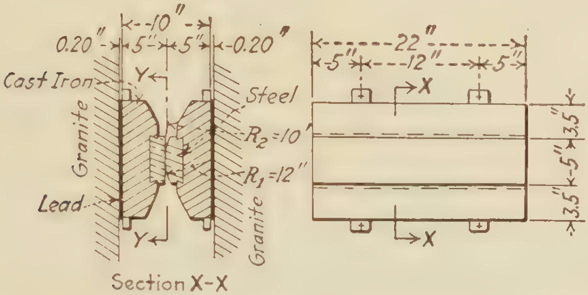


FIG. 351.—Rolling contact hinge. Cornelius bridge at Munich.

p = maximum pressure (pounds per square inch)

$$= 1.273\frac{N}{2x} = 1.273 \text{ average}$$

$$= 132\sqrt{e}\sqrt{N\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

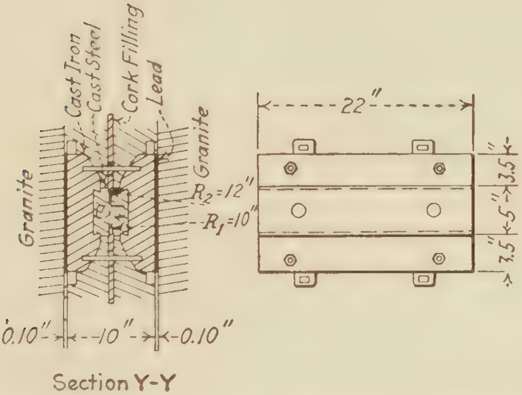


FIG. 352.—Rolling contact hinge. Maximilian bridge at Munich.

For a cylinder rolling on a plane, $R_1 = R$ and $R_2 = \infty$

$$2x = \frac{0.0096}{\sqrt{e}}\sqrt{NR}$$

$$p = 1.273\frac{N}{2x} = 132\sqrt{e}\sqrt{\frac{N}{R}}$$

For $e =$	20 (concrete),	40 (granite),	300 (steel)
$0.0096/\sqrt{e} =$	0.00215	0.00152	0.000554
$132\sqrt{e} =$	590	835	2,286

The formulae are not applicable unless R_1 and R_2 are quite different.

It is desirable that the center of the area of contact between the surfaces should be at the center of the joint; but as the parts will roll, this

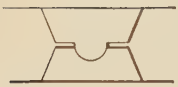


FIG. 353.

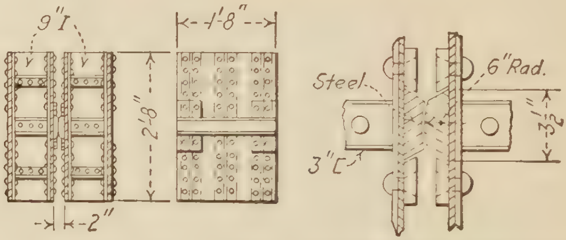


FIG. 354.—Knee hinge. Bridge at Munderkingen (Würtemberg).

can occur for only one loading. By computing the movement of the joints under the dead load, however, the opening at the joints at the extrados under no load at all, that is to say, the distance e in Fig. 348, may be computed in such a way that if the opening is held at this distance by wedges during erection, and the wedges removed just before the centers are struck, the joints will revolve so that the bearing will be

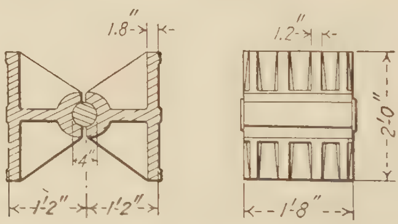


FIG. 355.—Pin-connected hinge. Bridge over the Danube at Inzigkofen.

central under the dead load alone. With steel bearings, the distance between the upper edges of the castings may be adjusted by screw bolts in a similar way.

c. Knee Hinges (*"Articulations á Genou"*).—This type consists of two curved surfaces of the same radius turning on each other, as in Fig. 353. The bridge at Munderkingen has hinges of this type (Fig. 354); these were partial hinges, for dead load only.

d. Pin-connected Hinges (*"Articulations Tournantes"*) (Figs. 355, 356, and 357).—These do not differ essentially in principle from the previous class. There is less danger of this class slipping off than of the others, and they are preferred by Max Leibbrand to the rolling hinges.

29. Advantages and Disadvantages of Hinging Arches.—The advantages of hinging arches are these:

(*a*) The stresses may be computed practically with accuracy, without tedious mathematical formulae.

(*b*) Settling of piers and abutments has no effect.

(*c*) There are no temperature stresses.

(*d*) The arch ring will be lighter.

(*e*) The unhinged arch may crack anywhere; the hinged arch is not likely to.

The disadvantages, as given by Baker, are these, with my remarks:

(*a*) The hinges add some cost and complication.

(*b*) The maintenance of the hinges is difficult. (There is no maintenance of the hinges. They are not touched. But they cannot be replaced, and, if of steel, they will rust. This last is a valid objection; it may be met by partial hinging.)

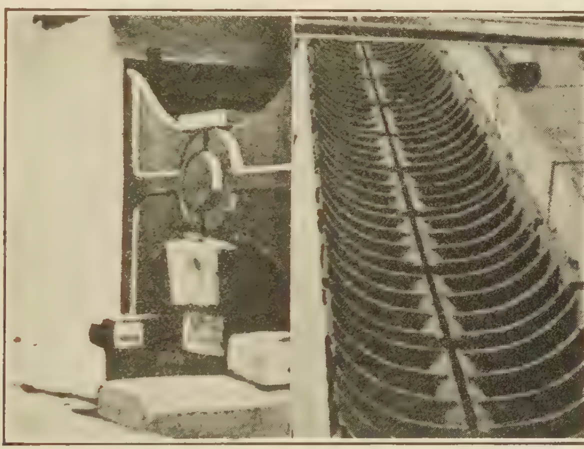


FIG. 356.—Pin-connected hinge. Bridge at Munich.

(*c*) There is not perfect freedom of rotation. (This is no objection at all. It is curious reasoning to infer that because there is not perfect freedom, it is better to have no freedom at all.)

(*d*) Movement about the hinges changes the stresses. (But so does any deflection of the unhinged arch. The objection is not valid.)

(*e*) The solid spandrels will interfere with rotation at the hinges. (This need not be so. *There must be a joint in the portion of the structure above the ring, at the crown and springing, to allow free rotation.* This can be provided, though it may be a little inconvenient.)

(*f*) The saving of material in the ring is small compared with the material in the entire structure. (Does this mean that small savings are inadvisable?)

(*g*) The structure is less rigid. (This simply means that it has hinges. The unhinged arch will deflect. *Séjourné says that many unhinged arches, such as the one at Valence, deflect quite as much as a comparable hinged arch.* Deflection depends upon many causes independent of hinges.)

(*h*) The durability of the hinges is questionable. (This is a valid disadvantage. Hinged arches have been used too short a time to be able to answer the objection from experience. But it may be overcome by partial hinging, or by using rolling hinges of granite.)

(*i*) The movement of a hinged arch due to changes of temperature are greater than the movements of an unhinged arch. This is because the arch can freely rotate, without causing stresses. In the rigid arch the movement is less because it is resisted by stresses. (This disadvantage, if it is one, is not given by Baker.)

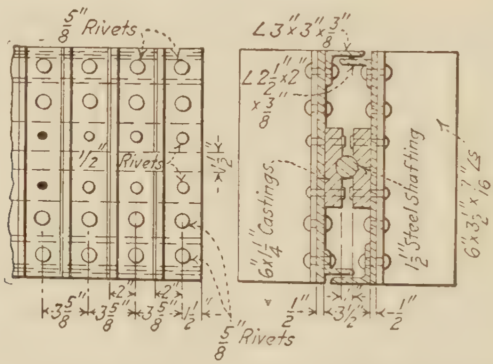


Fig. 357.—Pin-connected hinge. Bridge at Brookside Park, Cleveland, Ohio.

30. Conclusions Regarding Hinged Arches.—(*a*) It is perfectly practicable to hinge stone or concrete arches.

(*b*) There is less need of hinging a stone voussoir arch than of hinging a concrete arch, because a stone arch has no tension anywhere and is thus less likely to crack, and it if does crack it will be at a joint, which can easily be pointed.

(*c*) The arches which it is most desirable to hinge are those in which there is the most danger of movement of the supports, and reinforced concrete arches in which the tension in the steel is depended upon.

(*d*) I believe that it is prudent and desirable at any rate to hinge arches whose supports may settle, and especially large arches and thin arches, and those which are very flat, whether large or small.

(*e*) Experience must be awaited as to the best form of hinge, especially of metal hinges. Stone rolling hinges are hard to dress smooth to the required radius, but require no maintenance, give no trouble due to moisture, and may be completely hinged.

(*f*) The advantages of hinged arches deserve more consideration than has been given to them by American engineers.

The first suggestion to hinge arches of stone or concrete was made in 1870 by the French engineer Dupuit.[1] Curiously enough, the idea has not been adopted in France, and there are no hinged masonry arches in that country unless some have been built since 1914. Neither has the idea found favor in England or America. In the United States there are, according to Baker, three hinged arches:

1. A 40-foot arch of plain concrete at Mansfield, Ohio.

2. An 83-foot arch of plain concrete at Brookside Park, Cleveland, Ohio.[2] This had hinges made of a 1½-inch steel pin between cast-iron bearings bolted to frames of plates and angles (Fig. 357). The joint was filled with asphalt. When the centers were struck, the crown dropped 1½ inches.

3. A 135-foot skew ribbed arch of reinforced concrete in Denver, Colo.[3] "Rocker plates" were used for hinges, placed between two cast-steel shoes. The rocker plate had one face cylindrical with 8-inch radius and axis of cylinder horizontal, and the other face the same except that the axis of the cylinder was vertical and contained the axis of the arch rib. At each abutment one of the steel castings was anchored into the abutment by round 1¼-inch U-bolts reaching 3 feet 9 inches into the abutment. Thus the hinges permitted rotation both horizontally and vertically. It is not generally necessary or desirable to allow horizontal rotation; but as this was a skew arch, it may have been desirable in this case. The structure was designed by Charles W. Comstock.

In other countries on the continent of Europe, however, and particularly in Germany, the suggestion to hinge arches was soon adopted, and many such arches have been built.

Séjourné devotes Vol. IV of his great work to hinged arches. He gives a list of 81 concrete arches, of which 42 are completely hinged and 9 are partially hinged. Some stone arches have also been hinged.

Köpcke, of Dresden, was the first engineer to build a hinged masonry arch, in 1880. He used rolling hinges.

Max Leibbrand, of Württemberg, was the first to build a hinged arch with lead hinges, in 1885, and also the first to use rotating hinges, in 1895, in the bridge at Inzigkofen. He prefers the rotating hinge. Only since 1895 have large arches been hinged.

Further remarks on hinged concrete arches will be made in the chapter on Reinforced Concrete Structures.

31. Construction of Arches.—The ring of a stone arch almost always consists of a single ring or barrel, with two face walls and a fill between.

[1] Baker erroneously ascribes the first suggestion to the German engineer Köpcke, in 1880.

[2] See *Eng. News*, vol. LV, pp. 507–508.

[3] See *Eng. Record*, vol. LVII, pp. 336–339.

Provision must be made for draining off any water that may percolate through the fill. This is done by carrying it through pipes in the haunches of the arch (Fig. 358), or through openings in the ends of the piers. If the rise of the arch is large, the spandrels may be hollow, with longitudinal walls resting on the arch rings, carrying transverse arches or slabs supporting the roadway. The roadway is often carried on slabs or arches supported on transverse columns or walls as in Figs. 359 and 360. Often circular or oval openings extend transversely through the arch, for appearance or to lighten the load on the haunches, or to afford a passage for water in times of flood.

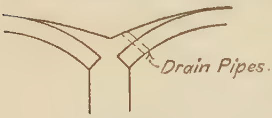


Fig. 358.

Great care must be taken to provide sufficient waterway in times of flood, and to prevent obstruction by trees or debris which may be carried downstream. An arch which rises vertically from the supports may not have the waterway much constricted even if the water level rises to above the springing, while a flat segmental arch would reduce the water-

Fig. 359.

way much. In flat arches, the passage of floods is sometimes facilitated by shaving off a triangular piece at each end of the span (so-called cow-horn) (Fig. 361).

It is quite common, instead of building the arch as a single barrel, to make it of two or more separate ribs (Fig. 362), the opening between them spanned at the top by a slab supporting the roadway.

The Walnut Lane bridge in Philadelphia has two ribs; the Haverhill bridge in Massachusetts, built under the direction of R. R. Evans, County Engineer, with the writer as Consulting Engineer, has four ribs (Fig. 363). The construction of piers and abutments must be such as to distribute the load of these ribs over the entire foundation. The use of ribs is a decided improvement and economy. Half the bridge

FIG. 360.

longitudinally may be built at a time, and the forms or centers used again for the other half.

An arch is an engineering and not an architectural construction, and should be designed by engineers, not architects. But the services of an architect to advise in matters of architectural detail is advisable, such as the treatment of the piers, the ornamentation, railings, or the towers which are frequently placed over the piers. The sidewalks frequently have recesses or refuges extending out over the ends of the piers, or on rounded cantilevered supports at the top of the pier.

FIG. 361.

Full details regarding all these matters may be seen in the illustrations in the works cited in Art. 35, particularly in that of Séjourné. See also Figs. 364 and 365.

32. Piers of Arches.—Piers supporting arches on both sides must resist the thrust from both in addition to their own weight. The thrust

Fig. 362.

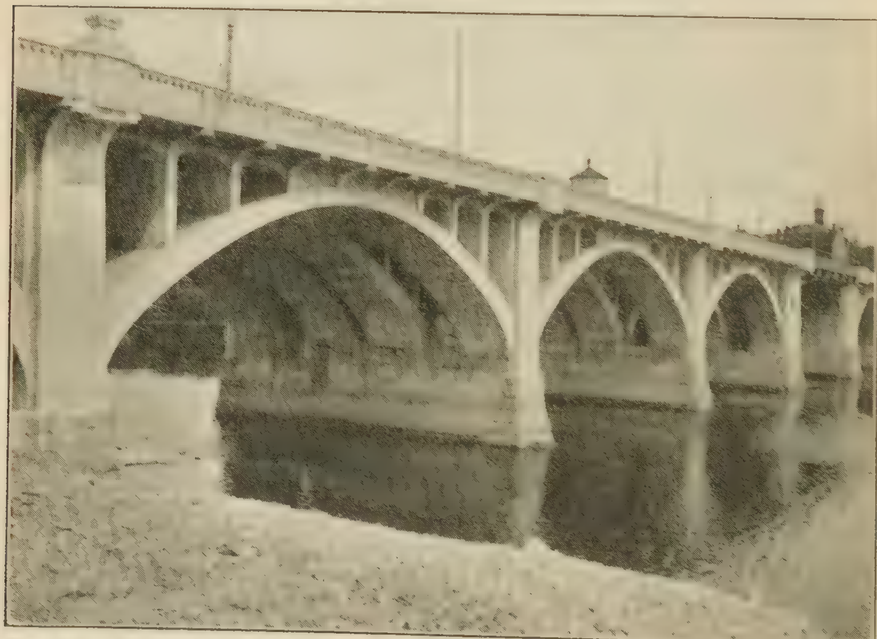


Fig. 363.—Highway bridge at Haverhill, Mass. (*Courtesy of R. R. Evans, County Engineer.*)

is applied at the springing joint, or perhaps more properly at the joint of rupture. The most unfavorable condition for the pier is when the thrust from one arch is as large as possible and that from the other as

FIG. 364.—Pont Neuf at Paris.

FIG. 365.—Pont Neuf at Toulouse.

small as possible; this will be when one arch is fully loaded (or nearly so) and the other with no live load. In this case, however, the true line of resistance will not be the same, in either arch, as if it stood alone on

immovable abutments. In fact, in a bridge with several arches, the whole structure, with all the arches and all the piers, must be considered in applying the principle of least work and finding the true lines of resistance. The piers will deform as well as the arches, and any yielding of the foundations, by settling or tipping, will also affect the result. This case, of a multiple arch, is very complicated, and really more incapable of an exact solution than even a single arch. The pier will tend to tip away from the fully loaded span, even if the line of resistance in that span is the minimum line of resistance; this will increase the resistance of the unloaded span till its line of resistance may be the maximum line of resistance, and it will diminish the thrust from the loaded span till it may become the minimum line of resistance.

Attempts have been made of late to treat a series of arches by the elastic method. These attempts will be further referred to in the next volume, which will deal more in detail with the theory of the elastic arch.

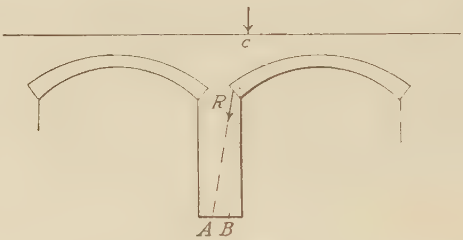


FIG. 366.

Leaving this aside for the present, the following principles are sufficient for the design of piers:

1. The pier will be safe if, taking one span fully loaded and the other without live load, and using in each the true line of resistance for an arch on rigid supports, the line of resistance in the pier lies within its middle third. This presupposes, of course, that the pressure on the foundation and on the pier joints is within safe limits. It is a criterion of stability.

2. The pier will be safe even if, taking one span fully loaded and the other without live load, and using in the loaded span the minimum line of resistance within the middle third, and in the unloaded span the maximum line of resistance for dead load alone within the middle third, the line of resistance in the pier lies within its middle third.

Strictly speaking, one of the spans should not be fully loaded, but the load should cover the part where the reaction due to a single load would pass outside the middle third of the base of the pier. Thus, in Fig. 366, A is the critical middle third point of the pier base for a load

on the right span. Now if a load at C causes a reaction R which passes through A, then a load at any point to the right of C will cause a reaction which passes to the left of A, tending to overturn; and a load at any point to the left of C will cause a reaction which passes to the right of A, tending to prevent overturning. Under these circumstances, the most unfavorable load to produce tipping of the pier to the left would be a load covering the span to the right of C but not to the left of C. This refinement, however, may be generally neglected in stone and concrete arches, and it may be assumed that one span is fully loaded and the other unloaded.

If the two spans adjacent to a pier are unequal, or if they spring from different levels, tipping of the pier in both directions must be investigated. The most favorable condition for the pier will be if the longer span (or the one which has the greater horizontal thrust) springs from the lower level. It is not the span alone, but also the rise, which determines the horizontal thrust.

33. Advantages of Arches of Stone or Concrete.—The advantages claimed for masonry or concrete arches over structures of steel are the following:

a. Better Appearance.—A stone arch is no doubt more beautiful than a steel structure. The same may be true of a concrete arch; Séjourné, however, says that they have a poor appearance, with large monotonous surfaces, often stained and cracked; much depends upon the treatment of the surface.

b. Greater Durability.—A stone arch is the most durable structure known, if built of proper material, such as granite. Such arches nearly 2,000 years old are still in existence and serviceable. The durability of concrete arches is still uncertain, but is probably far less than that of stone. Man cannot compete with Nature in the manufacture of concrete.

c. Greater Solidity.—A masonry arch is no doubt more solid than a steel structure; that is to say, its dead weight is greater, it is subject to less vibration, and its margin of strength for a possible increase of live load is greater, though this last may not be true of a reinforced concrete arch.

d. Greater Simplicity.—Relatively speaking, an arch is a simple structure with no complicated parts.

e. Easier Maintenance.—A masonry arch, since it deteriorates less, is easier of maintenance than a steel structure. It does not need painting. The joints may need pointing, and the surface of a concrete arch may occasionally need attention, but an arch is more permanent and needs little attention.

On the other hand, since an arch exerts a thrust on its supports, it may be inadvisable where the foundations may yield considerably. Also, it

generally costs more than a steel structure, and if it collapses, the whole structure is wrecked. As between stone and concrete arches, stone arches are no doubt superior. The advantages of concrete are:

(*a*) It is more economical if suitable materials are at hand.

(*b*) It requires less skilled labor than stone masonry.

(*c*) The mixing is done by machines, and the arch can be erected more quickly.

(*d*) It allows the construction of a skew arch as a monolith.

(*e*) It is lighter than stone, and, if reinforced, the arch ring may be thinner.

On the other hand:

(*a*) Concrete is more pervious to moisture than sound stone, like granite. If water reaches a stone arch, it can attack only the joints, which may be pointed; in a concrete arch it may attack the whole structure. It is therefore much more necessary to protect a concrete arch against water than to protect a stone arch.

(*b*) There is much more danger of getting poor concrete than of getting poor masonry. It is easy to get very poor concrete.

(*c*) Concrete is likely to crack anywhere and irregularly, and to become unsightly or stained. If a stone arch cracks, it will be at the joints, which may easily be repointed.

The stone arch is undoubtedly a much superior and more beautiful structure, but more costly.

34. Lists of Arches.—Baker gives, on pages 648–649, a list of large voussoir arches; on page 703 a list of plain concrete hingeless arches; and on page 722 a list of three-hinged masonry and concrete arches. In the first table, the masonry voussoir arch with the largest span is that at Plauen, Saxony, a highway bridge with a span of 295.3 feet and a rise of 59.5 feet.[1] Fifth on the list is the Cabin John bridge at Washington, D. C., a segmental granite arch with a span of 220 feet and a rise of 57.3 feet, built in 1859, and carrying a roadway and an aqueduct; it was for many years the longest stone-arch span in the world.

In the second table, the longest concrete arch is that at Cleveland, Ohio, carrying Detroit Avenue over Rocky River, built in 1910, with a span of 280 feet and a rise of 80.8 feet, 6 feet thick at the crown and 11 feet at the springing. Second on the list is the Walnut Lane bridge at Philadelphia, built in 1908, with a span of 232 feet and a rise of 70.2 feet, 5.5 feet thick at the crown and 9.5 feet at the springing. Both of these arches are three centered.

Séjourné gives more extended lists of arches.

35. References.—While the books in English, such as Baker's and Williams', give good chapters on arches, the great works on the subject

[1] See *Eng. News*, vol. LI, pp. 73–77, 1904; vol. LIV, pp. 155–157, 1905.

are in French.[1] France is the country which has done most for the development of the arch, both of stone and of concrete, and where, on the whole, the greatest structures of this kind have been built, as remarked by Séjourné.

[1] An excellent work is that by C. GAY, "Ponts en Maçonnerie," 1924, one of the Encyclopédie du Génie Civil et des Travaux Publics, published under the direction of Mr. Mesnager, Paris, J. B. Bailliere et Fils, Paris.

A shorter work is that of the same title by M. A. AURIC, one of the Encyclopédie Scientifique, 1911, O. Doin et Fils, Paris.

The greatest work on the subject in any language, however, is that of PAUL SÉJOURNÉ, "Grandes Voûtes," published in 1914 in six magnificent volumes. It gives detailed discriptions, beautifully illustrated, of almost every important structure built in any country up to that date. It is a veritable monument of research and learning. Its price is 600 francs, which, at the present low price of the franc, makes its cost, unbound, less than $25. Now is the time for anybody interested in this subject to procure this monumental work.

CHAPTER XXV

LINEAR ARCHES

1. Definitions.—By *linear arch* is meant the shape that a flexible cord or chain would assume under given loads. A true linear arch could not actually exist, for no structure could be absolutely flexible, having no moment on a section; but a suspended cord or steel cable comes so near it that it may be considered as a linear arch inverted, for of course it is not an arch in the proper sense, for an arch is in compression and has reactions directed inward, while a suspended cord is in tension and has reactions directed outward. A linear arch is an equilibrium polygon for the given loads, nothing more. A series of straight bars connected by frictionless hinges would take the shape of a linear arch for loads acting at the joints.

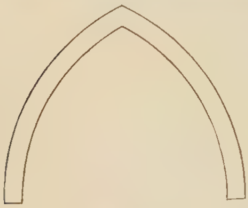


Fig. 367.

Fig. 368.

The study of the linear arch is thus the study of the equilibrium polygon. Its significance as applied to the actual arch is that if the axis

of an arch had the shape of the linear arch, there would be no bending moments, but only a central compressive force on each section. The arch most economical of material would thus be one whose axis is the linear arch for the loads; but this condition can be used effectively only if the loads are unchanged, which is seldom the case, except, perhaps, in some buildings. Structurally, and aesthetically also, the axis should be the linear arch for the loads as nearly as practicable; aesthetically, because beauty requires adaptation to conditions. Thus, a Gothic or pointed arch (Fig. 367) is justified structurally only when there is a concentrated load at the crown, because an equilibrium polygon cannot have an angle in it except at a concentrated load. Figure 368, from *Engineering News*, Jan. 15, 1914, shows a French viaduct having a Gothic arch properly employed.

2. But the loads on an arch are uncertain, as has been shown in the previous chapter, and any tendency of the arch ring to fail may bring into action outer forces which are not usually present. If a tunnel arch tends to bulge or spread horizontally, it brings into action the passive earth pressure. If a roadway arch tends to spread outward at the haunches, and if the arch is one of a series, with solid masonry or concrete backing between it and the adjacent arches as in Fig. 369, there may be a large horizontal load generated below the point a.

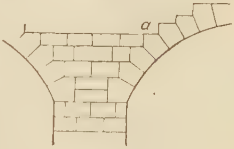


Fig. 369.

Hence the study of linear arches is of some importance as well as interest.

3. We have learned in Chaps. XI and XVI how to find graphically the equilibrium polygon or linear arch for given loads, and also how to find the distribution of load which will hold any given linear arch in equilibrium. The same problem may be studied analytically.

If the load is unchanging, the theory of the linear arch may be applied. This is the case, approximately, for tunnel arches or other arches supporting earth, as some sewer sections, and would be accurately applicable if there were an arch under water, supporting constant hydrostatic pressure. In the latter case the pressure at any point would be normal to the arch; and in other arches the pressure at some point may be normal to the arch, as at the crown of an arch exposed to loads from a roadway.

4. Relation between Normal Load and Stress in a Linear Arch, Sometimes Called Navier's Principle.—This is an important principle which has frequent applications, and should be clearly perceived. It may be stated as follows: *At any point of a linear arch where the pressure is normal to the arch, the stress in the arch is the normal load per unit length multiplied by the radius of curvature.* This is demonstrated as follows: Let AB in Fig. 370 be an infinitesimal length ds of a linear arch of radius

of curvature r, the normal load per unit being p, and the load on AB being pds acting radially. The stresses P and P' at A and B act along tangents at those points, as shown, and meet on the load pds. Taking moments about A,

$$pds \cdot r \sin \frac{d\alpha}{2} = P \cdot AB \cdot \sin \frac{d\alpha}{2} = 2Pr \sin^2 \frac{d\alpha}{2}.$$

But since $d\alpha$ is an infinitesimal,

$$ds = 2r \cdot \sin \frac{d\alpha}{2}$$

Hence

$$P = pr \tag{1}$$

By taking moments about B, it would be found that

$$P' = pr$$

Hence

$$P = P' = pr$$

It follows that if a linear arch is exposed throughout to a normal load, even if varying from point to point, the stress in the arch must be constant throughout. The same thing is seen by taking moments about D.

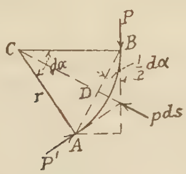


FIG. 370.

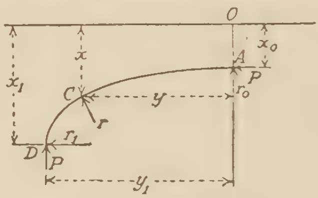


FIG. 371.

5. Hydrostatic Arch.—An arch under water has been called a hydrostatic arch. In such an arch (Fig. 371) if the water surface were at O, the depth of the crown x_0, the depth of any other point c being x, the stress is the same at every point of the arch. If w is the weight of a cubic unit of water, r_0 the radius of curvature at the crown, r_1 where the arch is vertical, and r at any other point,

$$P = wx_0r_0 = wxr; \quad \therefore x_0r_0 = xr,$$

so that r decreases as x increases, and r is least at the springing. Such an arch is therefore flatter than a semicircle.

An arch to sustain earth pressure has been called a *geostatic arch.* Some attempt has been made to use the principle of the linear arch in the design of sewer sections, but not with great success. A simple shape is more economical.

6. If a linear arch or rib of any figure is given, under vertical loads distributed in any manner, it is always possible, by the principles explained,

to determine a system or set of horizontal or sloping pressures which will keep it in equilibrium. These last may be called *conjugate pressures.* Thus, in Fig. 372 let 16 be a semiarch, horizontal at 1 and vertical at 6. Divide it into any number of parts, at 2, 3, 4, 5, and draw tangents at each point. From O draw lines parallel to these tangents. If the only load on 12 is vertical, find the point 1 in the right-hand figure where the

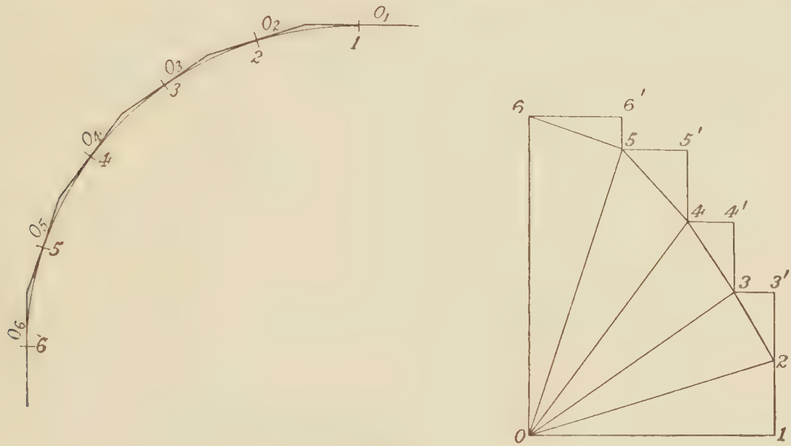


FIG. 372.

vertical 12 will represent the load on the part 12, to the scale of loads. Draw 23′ to represent the given vertical load on the part 23, and draw the horizontal 3′3, which will represent the horizontal load on the part 23. Similarly if 34′ is the vertical load on 34, 4′4 is the horizontal load, and if 45′ is the vertical load on 45, 5′5 is the horizontal load. The same process may be used if the conjugate pressures are inclined at any given angles.

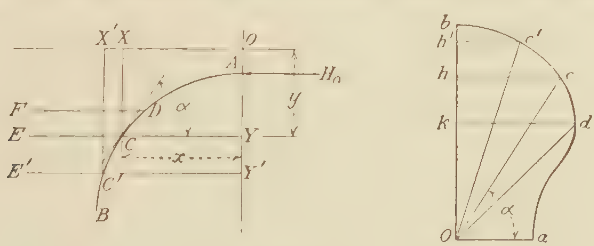


FIG. 373.

In Fig. 373, let AB be a half of a symmetrical arch, loaded symmetrically on each side of the crown A. The arch is horizontal at A and vertical at B. Let the conjugate pressures be horizontal. Several problems will now be solved.

Problem 1.—*To find the total horizontal pressure against the rib below any given point.* This may be solved graphically as follows: Let C be any

point of the rib. In the diagram of forces, at the right, draw Oc parallel to the tangent to the rib at C. On the vertical line Ob, which represents the load line, take $Oh = V$ to represent the known vertical load on AC, and draw the horizontal hc. Then Oc will equal T, the thrust along the rib at C, and hc will equal H, the horizontal component of that thrust, *which will be the total horizontal load on the rib below* C, that is, on CB, because the thrust at B is vertical.

This solution is expressed analytically as follows: The coordinates of C being x and y with reference to OX and OY, if α is the inclination of the rib to the horizontal at C, $dy/dx = \tan \alpha$,

$$H = V \cot \alpha;\ T = V \operatorname{cosec} \alpha = \sqrt{V^2 + H^2} \tag{2}$$

Problem 2.—*To find the thrust at the crown.* This is found by the principle of Art. 4,

$$H_0 = w_0 r_0 \tag{3}$$

if w_0 is the load at the crown per unit length, and r_0 is the radius of curvature of the crown.

Problem 3.—*To find the mean intensity of the conjugate horizontal pressure in any given layer.* To find the horizontal pressure on CC', draw

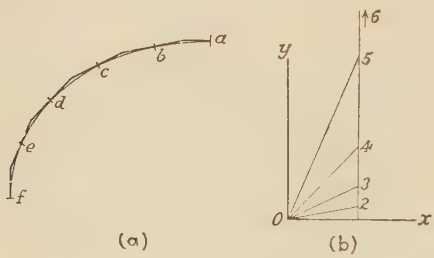


FIG. 374.

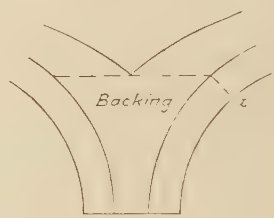


FIG. 375.

Oc' parallel to the tangent to the rib at C', take Oh' equal to the vertical load on AC', and $h'c'$ will be the total horizontal pressure below C'. Then $hc - h'c'$ will be the horizontal pressure on CC', and by dividing it by YY' the average pressure may be found. If $hc > h'c'$ there is a pressure from the outside on CC'.

By finding the pressure below various points and drawing the curve $bc'cda$, it may be found that at some point D corresponding to d, where the tangent to the rib is parallel to Od, that pressure is a maximum. If such is the case, above D a tension from outside is necessary to keep the linear rib in equilibrium. This is the *point or joint of rupture*. See Art. 17, Chap. XXIV, for the joint of rupture of an actual arch.

It has been shown how to find the *vertical* loads that will hold any linear arch in equilibrium. Thus in Fig. 374*a*, at the points a, b, c, d, e, f

of the linear arch draw tangents; in Fig. 374*b* draw OX horizontal or parallel to the tangent at the crown, and OY vertical. At any point 1, draw a vertical line, and draw lines radiating from O parallel to the tangents in Fig. 374*a*. Then 12 will represent, to some scale, the load on ab, 23 will represent the load on bc, and so on; and $O1$ will represent the thrust at the crown. If the tangent at springing is vertical, it will be impossible to hold the linear arch in equilibrium by vertical loads. Hence in such a case the backing should be able to resist a horizontal force. In a series of thin arches springing vertically from the abutments and piers (Fig. 375), it is thus especially important to have solid masonry or concrete backing filled in over the piers, and carried up to such a height that the axis of the arch, as a linear arch, will have a considerable slope at the joint a, opposite the lowest point of the backing.

As already stated, the only actual linear arch is a suspended cord, such as the cable of a suspension bridge. When that subject is reached, the shape of the linêar arch, inverted, will require further discussion. There are many other problems to be solved, including the study of the deformation of the cable.

7. Navier's Principle Applied to Double Curved Surfaces.—In the derivation of Eq. (1) in Art. 4, using Fig. 370, the surface is supposed to be single curved; *i.e.*, AB is a section of a cylinder whose axis is perpendicular to the paper. Then $P = pr$ is the hoop tension or compression per unit length perpendicular to the paper. If the surface is double curved, that is, if a plane through CD perpendicular to the paper cuts the surface in a curve, then the inward or outward load p on a unit area is resisted not only by the radial components of the forces P acting on a ring parallel to the paper, but by the radial components of similar forces acting at the ends of a small part of another ring perpendicular to the paper.

Equation (1) applies to a thin cylinder exposed to uniform normal pressure p inward or outward. The total pressure on a diameter one unit high is $2pR$, and this is resisted by the hoop tensions at the ends of the diameter, each being P. Hence

$$2P = 2pR; \quad P = pR$$

In the case of a sphere, the total pressure on a diametral section is $\pi R^2 p$, and the diametral section of the shell has a length $2\pi R$; hence on each unit of circumferential length the stress is

$$\frac{\pi R^2 p}{2\pi R} = \frac{pR}{2} \tag{4}$$

or the unit intensity, if t is the thickness, is

$$\frac{pR}{2t}$$

Here the pressure p on each unit area is resisted equally by two equal diametral rings, each carrying $p/2$, and the tension or compression on either ring, per unit of circumferential length, is $pR/2$, as above.

If the surface is not spherical, the two diametral rings will not each carry $p/2$. One will carry p_1 and the other p_2, such that

$$p_1 + p_2 = p \tag{5}$$

If the radius of curvature of the ring which carries p_1 is R_1, and that of the ring which carries p_2 is R_2, what is the relation between p_1 and p_2? It seems reasonable to assume

$$\frac{p_1}{p_2} = \frac{R_2}{R_1}; \text{ or } p_1R_1 = p_2R_2 \tag{6}$$

whence

$$\left.\begin{aligned} p_2 &= p\frac{R_1}{R_1 + R_2} \\ p_1 &= p\frac{R_2}{R_1 + R_2} \end{aligned}\right\} \tag{7}$$

This means that the load p is divided between the two rings in such a way that in each the hoop tension is the same. The ring with the smaller radius of curvature carries the larger proportion of p. It also means that

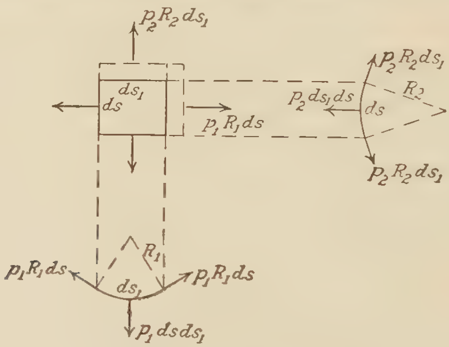


FIG. 376.

the two hoop tensions are principal stresses at the point, and these being of equal intensity, there is no shear on planes containing the normal to the surface, or no tendency to shearing distortion of a little area of the surface. The rectangle in Fig. 376 deforms so as to remain a rectangle and similar to itself.

The relation assumed above holds true in the two extreme cases of a single curved surface (cylinder), and a sphere. If the side ds has no curvature, or $R_2 = \infty$, $p_1/p_2 = \infty$, or $p_2 = 0_1$ $p_1 = p$, and the entire normal force $p \cdot ds \cdot ds_1$ is balanced by the radial components of the hoop tensions p_1R_1ds. If $R_1 = R_2$, as in the sphere, $p_1 = p_2 = p/2$, and the correct formula for a hollow sphere results.

At any point of a double curved surface, imagine the normal. Any plane containing that normal will be normal to the surface, and will cut the surface in a curve which will have a certain radius of curvature at the point. The planes containing the greatest and least radii of curvature will be at right angles; and for any two planes at right angles the sum of the radii of curvature is the same.

If the surface considered is a thin shell subjected to normal pressure, in applying the formulae given above the two rings taken should be those containing the maximum and the minimum radius of curvature respectively, and the assumption made is that the stresses at any point, on these two rings, are two principal stresses at the point, and, being equal, there is no shear on any other plane. There are cases in which this assumption is certainly true, as in the case of a surface of revolution, where symmetry proves it true.

Navier's principle for a single curved surface applies to a linear arch under normal pressure. There is no shear or bending in the arch. It thus applies to a thin circular cylindrical shell exposed to uniform normal pressure. If a thin elliptical ring were exposed to uniform normal pressure, then taking a little arc ds in Fig. 370 the hoop tension would be $P = pr$. Taking the adjoining arc ds', it would have a different radius r', and its hoop tension would be $P' = pr'$, which could not equal P as it would have to if there were no shear or moment in the ring. But since P could not equal P', there must be shear and bending in the ring. The circle is the curve of equilibrium for a thin cylindrical shell exposed to uniform normal pressure, and the elliptical ring would tend to be bent by the shear and moment into a circular form.

Similarly, a sphere is the shape of equilibrium for a uniform internal pressure, and a surface of any other shape, under uniform normal pressure, would tend to be bent into the spherical form. And a sphere, if under a normal pressure which is *not uniform*, would have shears and moments which would tend to change its shape. The bottom of a circular cylindrical tank, if hemispherical in shape, would be exposed, it is true, to a normal pressure at every point, but that pressure would vary with the head, and there would be bending which would tend to distort the bottom. See the chapter on Tanks, in the next volume, for a consideration of this case.

8. In Art. 23 of Chap. XXIV the question has been discussed as to the best shape to give to the axis of an arch in order that this axis may be as nearly as practicable the line of resistance for some set of actual loads.

CHAPTER XXVI

ARCH CENTERS

1. The framework which supports an arch during erection is called a *center*. It consists of a series of parallel ribs, only a few feet apart, on which is laid a continuous layer of boards, parallel to the axis of the arch, called the *lagging*, which has the shape of the intrados. The center or centering is left in place until the mortar in the joints has become sufficiently set, when the center is gradually lowered, without shock, and removed. Often the center is slightly lowered when the keystone has been set, in order to allow all joints to come to a full bearing. Lowering the center is called *striking the center.*

The design of the center will depend upon circumstances, such as the shape and dimensions of the arch, and whether the space beneath it must be left open or not. The center is a temporary structure. It may therefore be designed with a smaller factor of safety than that suitable for a permanent structure. But, on the other hand, it must be very rigid, so that it will not deform appreciably during erection, for if it did, the shape of the intrados would be altered and the stability of the arch might be endangered. Rigidity is best insured by having many points of support at short distances apart. Being temporary, centers are generally of wood, unless the space beneath the arch must be kept open, in which case the centers may be steel trusses. It is desirable to design the structure so that the same centers may be repeatedly used, and so that there may be as much salvage of material as possible.

To design the center, the loads upon it must be determined. After this is done, its design is that of any trestle or truss, and need not be further considered here.

2. The Loads on the Center.—The following theory was first given by Culmann, in his "Graphische Statik." The arch is supposed to be built from the springing up to any given point, and it is desired to find the pressures on the center under this condition.

Any voussoir, as *abcd*, (Fig. 377) is acted upon by four forces, which must be in equilibrium, *viz.*, its weight W, the reaction of the center R, the pressure P_1 exerted upon it from above, and the pressure P_2 exerted upon it from below. Of these forces we know simply W: all the others are unknown both as regards direction and magnitude. The problem of finding R is therefore, in its general aspect, indeterminate, and we must make some assumptions. In the first place, leaving out of account the

cohesion of the mortar, we know that P_1 and P_2 must not make angles with the normals to their joints greater than the angle of repose of masonry on masonry; we assume that these forces make *just* this angle, or, in other words, that the center exerts the *minimum force* R sufficient to hold up the stones. We further assume that R acts normal to the intrados and at the center of the stone. We can now find the value of R for each and every stone. Starting at the top stone, we know for that stone W_1 and the assumed directions of R and P_2, while $P_1 = 0$; hence we may find P_2. For every other stone, in order, we know W and P_1 and the directions of R and P_2, and these last two forces may therefore be determined. Thus, let $ab = P_1$, $bc = W$; draw EF parallel to ac till it meets R, and the force P_2 must act through F. Draw cd parallel to P_2 and ad parallel to R; then $ad = R$ and $cd = P_2$, and FG parallel to cd gives the line of action of P_2, which is the force P_1 for the next stone below.

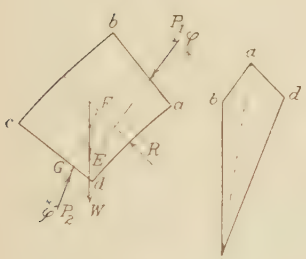


FIG. 377.

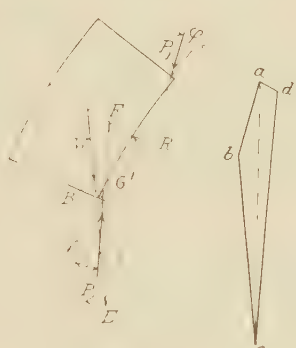


FIG. 378.

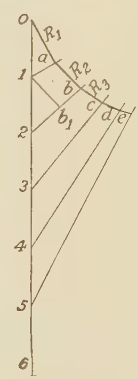


FIG. 379.

This only holds, however, as long as the point G falls within its joint. If it falls outside the joint, as at G' (Fig. 378), then this stone will not tend to *slide* on the one below it, but to tip, with all the arch above it, about the inner edge of the joint. Hence we must in this case draw the line FG, *not* making an angle φ with the normal to AB, but passing just inside the point B. If the stone were incompressible, we should draw it through B, but the yielding of the stone will clearly throw the center of pressure a little in from the edge.

The arch being built up from the springing to a certain point, we may now find the pressure of each stone upon the center (Fig. 379). Lay off 01, 12, etc., equal to the weights of the successive stones, beginning at the top. Draw $1a, 2b, 3c, 4d$, etc. parallel to the lines making angles φ with the normals to the successive joints, and draw oa, ab, bc, etc. parallel to normals to the center at the middle of the stones. Then will oa, ab, bc, etc. represent the values of R for the successive stones. Draw, on the figure showing the arch, the lines FG, etc., so as to find the actual line of action of

each force P_2, as in the previous figures. Suppose that for the fifth stone, P_2 must be drawn through or near the inner edge of the joint, and not making an angle φ with the normal, since the latter construction would throw G' inside of B as above explained. Then we must draw $5e$ making a greater angle with the vertical than if it were drawn making the angle φ with the normal to its joint. From the above figure it is now evident:

1. The pressure R on the center is greatest at the top stone and diminishes as we go downward, if each stone has the same weight. This holds as long as the stones tend to slide.

2. At the stone where the tendency to tip first occurs, there is a sudden increase in R.

3. Below this stone, R again decreases.

4. For some stone, the value of R may be negative or a tension; and the same for all stones below it. For these stones there is no pressure on the center.

The stones of the arch may therefore be divided as follows going down from the top:

(*a*) Those which tend to slide.

(*b*) Those which exert no pressure on the center. (This group may not occur.)

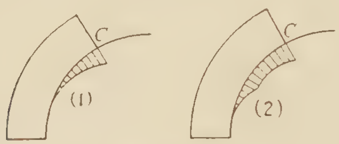


FIG. 380.

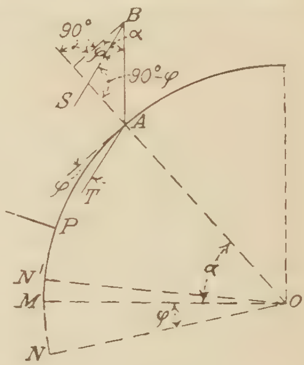


FIG. 381.

(*c*) The stone which first tends to tip, exerting a greater pressure on the center than those above or below.

(*d*) Those stones below which tend to tip.

(*e*) Those which exert no pressure upon the center.

This will be the order of the stones from the top down; but in any particular case some of these classes may be absent; the upper stones are either sliding or they exert no pressure on the center, though sometimes they may have a tendency to tip. Class (*b*) will sometimes be absent.

The distribution of pressure (supposed distributed) on a center, the arch being built up to C, is therefore as shown in Fig. 380. The effect of the sudden change from a sliding to a tipping stone will be a bulge as shown in 2 of Fig. 380.

The following conclusions are obvious:

The maximum pressure at any point on the center occurs when the arch is built up just to that point; for, in Fig. 379, the value of R_2 if the first or top stone 01 is removed, is $1b'$, parallel to ab. But since the lines $1a$, $2b$,

etc. converge, $1b' > ab$; hence the value of R_2 is diminished when the next stone above is added. Hence, *the pressure on the center at any point is diminished by building up the arch above that point.* We may easily construct this *maximum* pressure at any point A (Fig. 381). Lay off AB vertically, equal to the vertical load per running foot of intrados at A: draw BS parallel to the line AT which makes the angle φ with the tangent to the intrados at A. AS being normal to the intrados at A, the distance AS represents the pressure on the center per running foot at A, as is easily seen by comparing this figure with Fig. 379. If $AB = w$, we have

$$R:w::\sin(\alpha - \varphi):\cos\varphi$$

$$R = \frac{\omega \cdot \sin(\alpha - \varphi)}{\cos\varphi}$$

For $\alpha = 90°$, $R = w$, as it clearly should.
For $\alpha = \varphi$, $R = 0$

The value of φ should be taken less than the angle of repose of masonry on masonry, as the moist mortar in the joints will act as a lubricant.

If the joints are radial, to draw a line making an angle φ inward from the normal to a joint P, in Fig. 381, lay off the angle $MON = \varphi$, then lay off the distance $PN' = MN$, and a line perpendicular to ON' will make the angle φ with the normal to joint P.

3. The above theory as already stated, was given by Culmann, in 1857. Other theories take account of sliding only and not of tipping; the first theory of this kind being probably that of Navier, in his "Resumé des Leçons," 1826. In other theories the pressure between the arch stones is taken as normal to the joints.

4. A center must be computed so that each part is capable of sustaining the maximum pressure to which it can be subjected, and so that the

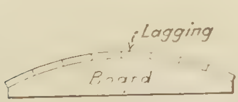


FIG. 382.

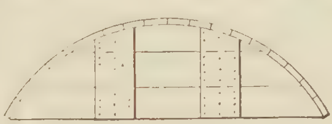


FIG. 383.

entire center shall fulfil the same condition. If the center is built as a series of isolated or nearly isolated supports, the question of total loading will not enter into consideration. But if it is a truss, the total load must be considered. The circumstances of each particular case must show what loadings must be considered, and the above principles enable us to solve any problem which may arise. An important thing being that the center should be stiff, it should if possible *not* be built as a truss, but intermediate supports should be frequent and rigid. Generally, rigidity being the controlling factor, centers have a large excess of strength.

Fig. 384.

Fig. 385.

FIG. 386.

FIG. 387.

5. Concrete arches must be supported on centers or forms, but the theory of these is quite different from that of the stone-arch center. The pressure on a concrete form will depend upon the rapidity with which the concrete is poured, and may be the hydrostatic pressure of the material. Some remarks about forms will be found in the chapter which is to come, on Concrete Structures.

FIG. 388.

6. Examples of Centers.—For a very small span and rise, the center may be a series of boards having the top shaped to the curve of the intrados (Fig. 382); or, if the rise is greater than the width of one board, two or more may be used, held together by cleats nailed to them, as in Fig. 383. For longer spans, the center may be a wooden arch made of boards.

Generally, in order to have many points of support close together, the center is composed of a series of trestle bents, with braces radiating in fan shape from them, as in Fig. 384; or a framework with many supports,

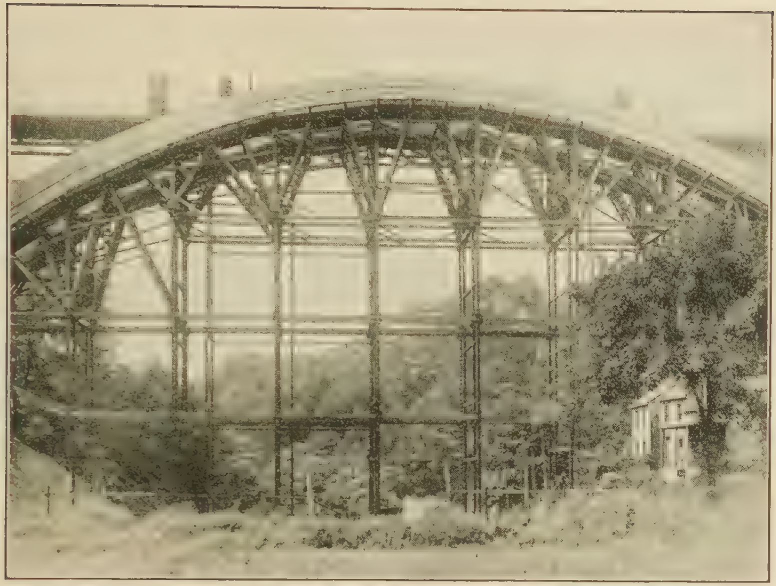


FIG. 389.

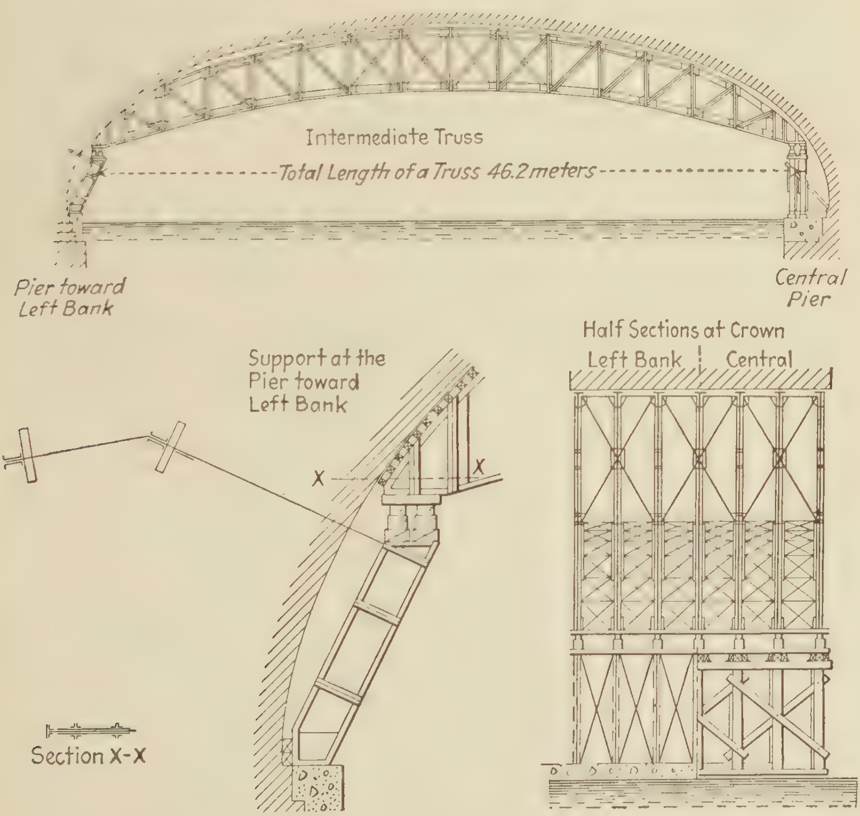


FIG. 390.

carried on trestle bents, as in Fig. 385. Figures 386 to 389 show other centers, and Fig. 390 shows a steel-truss center.

7. Camber.—Bridge trusses are generally built with a *camber*, that is, an initial upward curve, so that under full load it will be horizontal. Centers, however, rarely if ever are built with a camber, but are made so stiff that the deflection under the dead load will be negligible.

8. "Striking" or Lowering the Centers.—The centers must be so supported that at the proper time they may be lowered as one piece. This is commonly done by supporting the centers at both ends on wedges. All the ribs may rest at each end on a stringer or timber parallel to the springing line. Beneath this, and between it and the cap of the end trestle bent, there may be wedges under each rib. To lower the center, all these wedges must be driven out uniformly and slowly, by making a mark on each wedge and driving to this mark in succession. The wedges should have a flat taper.

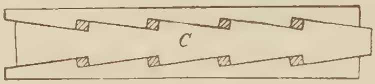


Fig. 391.

Instead of wedges under each rib, the ribs may all rest upon a compound wedge (Fig. 391) which extends parallel to the springing line; when the center is to be lowered, the small shaded wedges are loosened, and the piece C of the compound wedge is then driven back so as to lower the center slightly at each end of the span. In this way the center is gradually lowered.

In some French arches, the centers have been supported on pistons of cylinders filled with clean, dry, fine sand. In the bottom or side of the cylinder is a plug which can be removed when the center is to be lowered, and the sand is forced out by the pressure on the piston, which gradually sinks. This "sand box" cannot be used if the sand is likely to become wet. Wedges are more reliable.

CHAPTER XXVII

DAMS

1. A dam is a structure built across a stream to hold back the water and form a pond or reservoir, for purposes of water supply, water power, irrigation, navigation or stream control. It may be of earth, wood, steel, concrete or stone, or of a combination. It may be fixed or movable, a movable dam being so arranged that it can be lowered or raised, if necessary, so as to control the water.

This chapter will be devoted almost entirely to dams of masonry or concrete.

MASONRY DAMS

2. A masonry dam is similar to a retaining wall, holding back water instead of earth. Such a dam may be straight or curved, and occasionally a broken line. If straight, it retains the water by virtue of its weight alone, and is called a *gravity dam.* If convex upstream, it acts in part as an arch in a horizontal plane, and is called an *arch dam.* If composed of a succession of horizontal arches, with piers between, it is a *multiple-arch dam.*

Most of this chapter will deal with gravity dams.

3. Outer Forces.—The outer forces acting on a dam are (1) its weight; (2) the hydrostatic pressure on the upstream face, and perhaps also on the base; and the pressure of silt on the upstream face; (3) ice pressure; (4) wind pressure, and (5) the impact of waves or of floating objects. If the dimensions are known, the first two are definitely known (see Art. 5 for pressure on base). Ice pressure is uncertain and may be large. In the Quaker bridge dam it was taken at 43,000 lbs. and in the Wachusett dam at 47,000 lbs. per linear foot; if care is taken to break up the ice, it may be small, and in many dams it has been neglected. Wind pressure is generally neglected, because with the reservoir full there is practically no wind pressure downstream, while wind pressure upstream would increase the stability. In some cases it may be desirable to consider wind pressure upstream with reservoir empty. The impact of waves may generally be neglected, especially if ice pressure is assumed.

4. Conditions to Be Fulfilled.—Two cases are to be considered: (1) with the reservoir full, and (2) with the reservoir empty. In the first case the line of resistance will pass nearest to the lower face, and in the second case it will pass nearest to the upper face. In each case there are, as in retaining walls and arches, three conditions:

1. The resultant on any joint should lie within the middle third.
2. The maximum normal pressure at the edge of any joint must not exceed the allowable.
3. The angle between the pressure on any joint and the normal to that joint must not exceed a safe limit.

5. Uplift on Base, and Ice Pressure.—A very important matter is that of the possible upward pressure on the base, and on a horizontal joint, due to the percolation of water. Possible ice pressure is also important. These matters will be discussed in Art. 17, after the profile has been considered.

6. Elementary Shapes of Section.—The three elementary shapes of section are those in Fig. 392. Let us find the width b which will make the

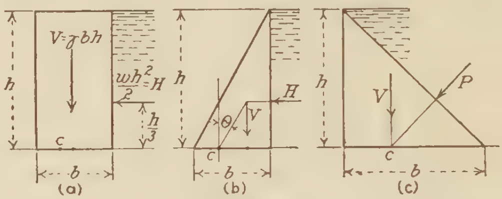


FIG. 392.

resultant on the base go through the lower middle third point C, with the water level at the top of the dam.

$$\left.\begin{aligned}
&\text{In case } (a)\text{: } \frac{\gamma b^2 h}{6} = \frac{wh^3}{6};\ b = h\sqrt{\frac{w}{\gamma}}\\
&\text{In case } (b)\text{: } \frac{\gamma b^2 h}{6} = \frac{wh^3}{6};\ b = h\sqrt{\frac{w}{\gamma}}\\
&\text{In case } (c)\text{: } P \text{ must pass through } C \text{ or}\\
&\qquad \frac{wh^3}{6} = \frac{whb^2}{6};\ b = h
\end{aligned}\right\} \quad (1)$$

Case (b) has the same stability and strength as case (a) with one-half the material, except for the consideration mentioned in Art. 9; while, as w, the weight of water per cubic unit, is less than γ, the weight of masonry per cubic unit, case (c) is less economical. Case (b) is the most economical of the three; for this case

$$\tan\theta = \frac{H}{V} = \frac{wh}{\gamma b}$$

or, substituting for b its value from Eq. (1),

$$\tan\theta = \sqrt{\frac{w}{\gamma}} \quad (2)$$

Since w/γ is about 0.426, $\tan\theta = 0.65$, which, even for pure friction, is not in general too great.

7. Actual Section.—The proper section for a masonry dam is essentially the triangle of case (*b*), modified as follows:

1. The dam must extend above the highest water level, unless it is an overflow dam, and must have a top width sufficient for a walk or a roadway; both faces may be vertical down to a depth given by the first of Eq. (1), because it is safest to assume that the water may reach the top. W. P. Creager thinks that the economical top width is between 10 and 17 per cent of the height.[1]

2. Below the rectangular part the downstream face should be curved, while the upstream face may be vertical down to a point at which, for reservoir empty, the line of resistance will pass through the upper middle-third point.

3. Below this point, both faces should be curved, the lower face more than the upper face.

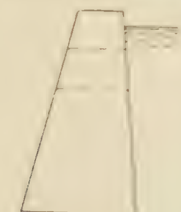


Fig. 393.

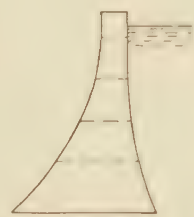


Fig. 394.

For a low dam, the section may be a trapezoid (Fig. 393). For a high dam it will have a section similar to Fig. 394.

It is a principle in masonry construction that there should be no continuous vertical joints where the load is vertical or nearly so. Since in a dam the forces are considerably inclined, there should be no continuous joints in any direction. The masonry above any assumed horizontal plane, or any plane, should act as a monolith. Hence a masonry dam generally has thin faces of cut stone, with joints perpendicular to the face, and between these faces it is of irregular rubble or of cyclopean concrete, that is to say, concrete with large stones imbedded in it. The face walls are built up first, and serve as forms for the concrete, being kept always a little above the concrete. All joints in the faces, vertical joints as well as horizontal joints, should be well filled with mortar and made as impervious as practicable (see Fig. 424).

8. The Conventional or Usual Method of Analysis.—By this method, the resultant force above any assumed *horizontal section* is found, and the point where that resultant cuts the section. The *normal component* on the section is then assumed to be distributed linearly over the section,

[1] *Trans. Am. Soc. C. E.*, vol. LXXX, p. 723, 1916.

and the three conditions stated in Art. 4 are fulfilled. It will be shown that this method is erroneous, but it is generally sufficient for practical purposes, although the fact demonstrated in the next article should always be considered.

9. Pressure at a Sloping Face.—The first reason why the usual method is erroneous is that the actual pressure at the sloping face is greater than that found by considering the normal component on the joint to be distributed in a linear manner.

It was shown in Arts. 5 and 6 of Chap. V of "Strength of Materials" that *at the free surface of a body the stress on any plane is parallel to the surface.* Therefore at a horizontal joint AB of a masonry dam (Fig. 395), the stress is not vertical or at right angles to the joint, but parallel to the surface. After finding the total normal component N on the joint, that force may be considered to be distributed over the joint in a planar manner, and the maximum vertical intensity f_0 may be found. On any small

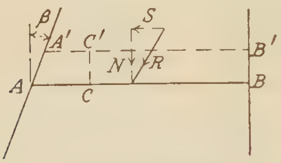


Fig. 395.

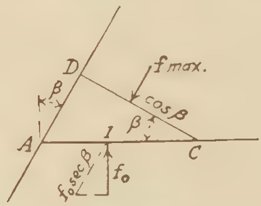


Fig. 396.

distance AC (Fig. 396) which may be called unity, the total stress f_0 will not be the real stress on AC, but will be the normal component of that stress. The real stress on AC will be $f_0 \sec \beta$. But drawing CD perpendicular to the face AD, $CD = \cos \beta$. The real stress on CD must be parallel to the surface; there is no shear on CD. The real intensity on CD will thus be

$$f_{\max} = \frac{f_0 \sec \beta}{\cos \beta} = f_0 \sec^2 \beta \qquad (3)$$

which may be considerably greater than f_0.

This important fact seems to have attracted insufficient attention in American engineering literature.

Another reason why the usual theory is or may be wrong is discussed in Art. 12. Meantime we will proceed with it.

10. Trapezoidal Dam (Fig. 397).—For this section, with the upstream face vertical and given top width c, the width b at the base which will make the resultant pass through the lower middle third of b is given by the equation

$$\frac{wh^3}{6} = \gamma ch\left(\frac{2}{3}b - \frac{c}{2}\right) + \gamma h\frac{b - c}{2}\left(\frac{b}{3} - \frac{2}{3}c\right) \qquad (4)$$

A similar but more complicated equation may be written if the upstream face is sloped.

11. Section of a High Dam (Fig. 398).—In a high dam, if the top width b_1 is known, the depth h_1 of the rectangular portion is found from the first of Eq. (1).

$$h_1 = b_1\sqrt{\frac{\gamma}{w}} \tag{5}$$

Below this point let the outer face be sloped, while the inner (upstream) face is vertical. Assume any convenient value for h', and let it be pro-

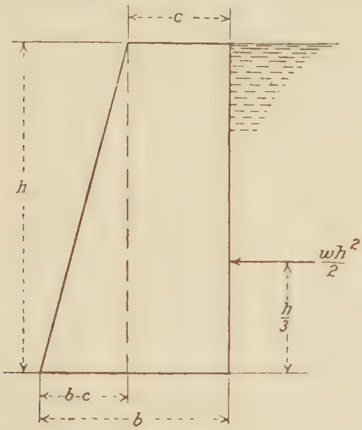


FIG. 397.

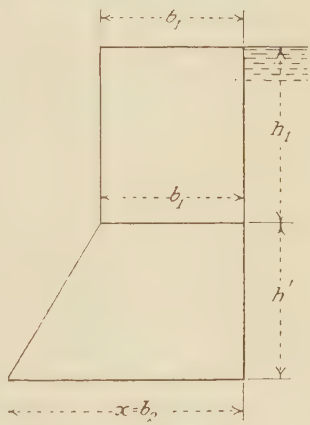


FIG. 398.

posed to find the width $b_2 = x$, such that the resultant with reservoir full will pass through the lower middle-third point of x. The equation of condition will be

$$\frac{w(h_1 + h')^3}{6} = \gamma b_1(h_1 + h')\left(\frac{2}{3}x - \frac{b_1}{2}\right) + \gamma\frac{h'(x - b_1)}{2}\left[\frac{2}{3}(x - b_1) - \frac{x}{3}\right] \tag{6}$$

From this equation x may be found. In a similar way, another joint may be taken and its width found; and this may be continued to a depth at which, for reservoir empty, the line of resistance passes through the *upper* middle-third point, or until the pressure at the upstream edge, without wind blowing, equals the hydrostatic pressure, if M. Levy's advice is to be followed (Art. 17). Below this point, the upper face may be inclined, and the widths may be found by trial.

The criteria for sliding, and for maximum intensity of pressure, observing Eq. (3), must also be fulfilled at each joint.

When the section has been determined in this way, or when it has been assumed in accordance with one of the standard profiles given in Weg-

mann's book, it should be drawn to a large scale, and the lines of pressure drawn by the aid of the equilibrium polygon. Thus, let P_1 be the weight of the first block or layer, P_2 the water pressure upon it, P_3 the weight of the second layer and P_4 the water pressure on it, and P_5 and P_6 the weight and water pressure on the third layer. Lay off, in Fig. 399, $01 = P_1$, $12 = P_2$, $23 = P_3$, and so on. Take the pole at O and draw the equilibrium polygon as shown. The center of pressure on the base is at C. If it is desired to allow an upward pressure P_7 on the base, prolong $O6$ till it meets P_7, draw the string $O7$, and the center of pressure on the base will be shifted to C_1. This shows clearly how an upward pressure always shifts the center of pressure downstream. If an upward pressure is to

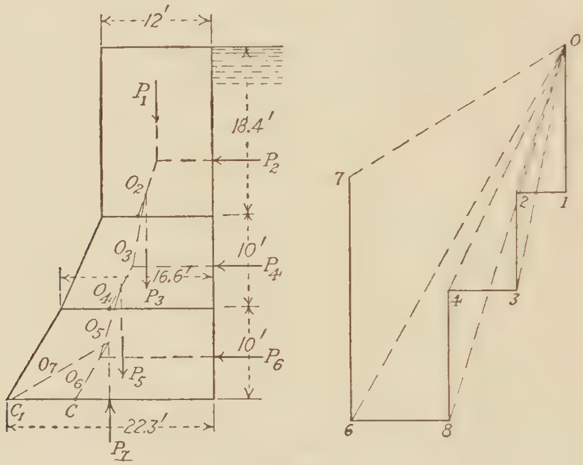


FIG. 399.

be allowed on the part above joint 2, its effect would be found in the same way; but this would not change the determination of C and C_1 on the joint next below.

The maximum pressures at the edges of the joints should then be found.

By studies like this the profile of a dam may finally be determined. The lines of resistance should be drawn for reservoir full and for reservoir empty.

It is better to make the faces curved rather than polygonal, because at an angle there is a complication of stress.

If $f_{\max}$ is the real maximum stress at the edge of a joint perpendicular to the surface, it is a principal stress, and there is a shear of $f_{\max}/2$ on two planes at right angles to each other, and at 45° to the surface.

The greatest oversight in the usual design of masonry dams appears to be the failure to use Eq. (3). The effect of this is that the actual maximum stress greatly exceeds in many cases that usually found and stated, dou-

bling it if the slope of the face is 45°. Rankine realized that the pressure acted parallel to the surface, but so far as I know, he did not deduce nor use Eq. (3), and contented himself with reducing arbitrarily the allowed stress at a sloping face. I have found this formula mentioned in but two places in American engineering literature: first, in Professor Cain's paper[1] on "Stresses in Masonry Dams," and, second, in the chapter on Dams in Turneaure and Russell's "Public Water Supplies," but its importance is not sufficiently emphasized. The equation, however, is used by the U. S. Reclamation Service in the design of dams.

It follows that when we see it stated that the maximum pressure in a dam is a certain figure, it would be well to inquire how it was calculated.

12. Criticisms of the Usual Method.—By the usual method, the joints are taken as horizontal and the normal pressure is supposed distributed in a planar manner. The stress at the curved face should then be

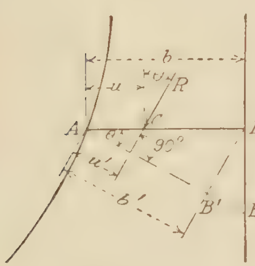


FIG. 400.

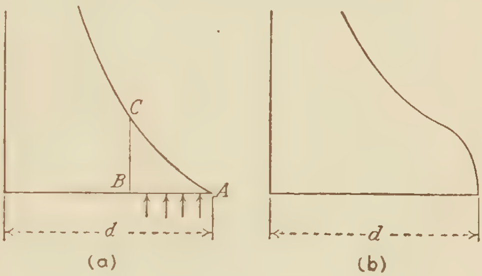


FIG. 401.

computed by finding f_0 from planar distribution and then using Eq. (3), though this last step, which should be taken, is often neglected, as just stated.

But why should the joints be taken as horizontal? The axis is not vertical, nor is it straight. It was explained in "Strength of Materials" that the sections should be taken at right angles to the axis, and that the axis passes through the centers of gravity of the sections, so that the two are interdependent. There seems no convincing reason, therefore, why the sections should be taken horizontal. How, then, should they be taken? French engineers have discussed this question, and various propositions have been made, none entirely satisfactory; but the subject has not received the attention it deserves in American literature, though Wegmann gives some space to it.

Bouvier proposed, that after finding the resultant force R on any horizontal joint AB (Fig. 400), the joint should be projected at right angles to R, at AB' and that R should be considered as distributed in a planar manner on AB', neglecting $BB'B''$. The normal component of R on

[1] *Trans. Am. Soc. C. E.* vol. LXIV, p. 215, 1909.

AB is $R \cos \theta$ and if $AB = b$ and $AC = u$, by planar distribution of $R \cos \alpha$ on AB,

$$f = \text{maximum stress at } A = \frac{2R \cos \theta}{b}\left(2 - \frac{3u}{b}\right),$$

while if R is distributed in a planar manner on $AB' = b' = b \cos \theta$,

$$f' = \text{maximum stress at } A = \frac{2R}{b \cos \theta}\left(2 - \frac{3u}{b}\right) = f \sec^2 \theta$$

If R happens to be parallel to the tangent to the lower face at A, this result agrees with Eq. (3), but otherwise this is not true. There seems no rational ground for assuming that R causes no pressure on $B'B''$.

13. There is no good reason why *only* horizontal planes should be considered, and it has been suggested by *Guillemain* that, instead of contenting ourselves with distributing the vertical component of R over AB, various planes radiating from A should be considered, the total on each and its normal found, and the maximum stress at A. L. W. Atcherley, in a paper on "Some Disregarded Points in the Stability of Masonry Dams," published in 1904, maintains that vertical sections should also be considered, where large tensions will be found to exist. His tests, he thought, proved that "A dam collapses first by the tension on the vertical sections of the upstream (downstream?) toe,"[1] and that the vertical sections are more important than the horizontal sections as subjects of investigation.

That vertical sections may need consideration is shown in Fig. 401(a). According to the accepted theory, the maximum pressure with reservoir full is at A. On the vertical section BC, then, there is, by this theory, a shear, and also a bending moment causing tension at B. If the toe is pointed, the critical section may be a vertical section near the toe, which tends to break off. It may be stated definitely that a sharp toe is bad, and that by the usual theory it might crack away long before the maximum stress usually computed is reached at A. As Mr. Van Buren remarks, "the toe should be bold, with a nearly vertical profile," or as shown by Fig. 401(b), in which the curve is rounded. The pressure on AB is not vertical, however, but inclined, which reduces the possible tension at B. James B. Francis thought that the maximum pressure was not at the toe, but was "mainly transferred from the toe to the interior of the base, where the resistance to compression is much greater." He erroneously thought that all that is necessary is to have the curve of stress distribution enclose an area equal to the total vertical pressure, and neglected the necessary statical relation which fixes its center of gravity.

[1] Wegmann, *op. cit.*, pp. 8–10.

Various writers have recognized that since at a point on a horizontal section there is shear as well as normal stress, the plane of principal stress is inclined to the horizontal. Since the maximum normal stress, however, is probably at one edge, and the real maximum stress there is parallel to the surface, it follows that, if Eq. (3) is used, no further consideration need be given to the principal stresses at other points.

14. Regarding Guillemain's suggestion that we should take various planes through A, find the normal force on each and by planar distribution the maximum, at A, and take the plane for which this maximum is greatest, it is clear that if α_3 is the angle which this plane makes with the normal

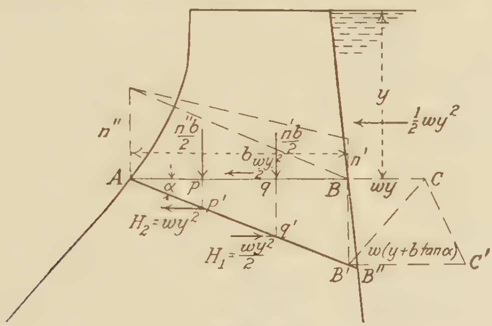


Fig. 402.

to the surface, and f_3 the maximum normal stress on this plane, at A, we should still, by Eq. (3), be obliged to consider

$$\text{maximum stress} = f_3 \sec^2 \alpha_3$$

Also, by Bouvier's method if the plane AB'' makes an angle α_2 with the normal to the surface at A and f_2 is the maximum found at A by his method, we should have

$$\text{maximum stress} = f_2 \sec^2 \alpha_2$$

By the usual method of distributing the normal force on AB over AB, if f_1 is the maximum at A and α_1 is the angle which AB makes with the normal,

$$\text{maximum stress} = f_1 \sec^2 \alpha_1$$

If all these methods are correct, then

$$\text{maximum stress} = f_1 \sec^2 \alpha_1 = f_2 \sec^2 \alpha_2 = f_3 \sec^2 \alpha_3 \qquad (7)$$

Certain it is, since at any point A the stress on any plane is parallel to the surface, the maximum intensity must occur on a plane perpendicular to the surface; and it is possible that all three methods, by using the equations just given, may give the same maximum stress. If this is true, then we may as well continue to use the simple method of taking horizontal sections.

Maurice Lèvy[1] has investigated this question. He had a computation made of the three values in Eq. (7) for the New Croton dam, and found that they differed by only 2 per cent. He therefore concludes that the usual treatment, by horizontal sections, which is the simplest, is correct provided Eq. (3) is used to find the real maximum stress.

Guillemain's method may be worked out and compared with the usual method in the following manner, as given by Lèvy:

Let AB (Fig. 402) be any horizontal joint at a depth y. By the usual method the normal force on AB is distributed in planar manner on AB, giving a stress n'' at A and n' at B. The distribution is represented by a trapezoid, and the method may be called the trapezoidal method. The normal stress on AB consists of two forces, $n''b/2$ and $n'b/2$ at the middle-third points p and q, as shown. If θ is the angle between the resultant on AB and the normal, the shearing force on AB, $wy^2/2$, is the same as two concentrated forces at the middle-third points, $\frac{n''b}{2}\tan\theta$ at p and $\frac{n'b}{2}\tan\theta$ at q. Let AB'' be any plane through A making an angle α with AB. By Guillemain's method we are to find the maximum normal stress at A on AB''. The resultant force above AB'' is that of the force on AB, consisting of the two normal and two shearing forces just defined, added to the weight of ABB'' and the pressure on BB''. It will be close enough to draw BB' vertically, and to consider only the weight of ABB' and the pressure on BB', and this will be accurate if the upstream face is vertical. Planar distribution is assumed on AB'.

The force $n'b/2$, at q has a normal component on AB' which gives zero stress at A, since q is at the middle third.

The force $n''b/2$, at p, gives a normal stress at A of $\frac{2n''b\cos\alpha}{2\ AB'}$ $= n''\cos^2\alpha$.

The weight of ABB' gives no normal stress at A.

The horizontal force $wy^2/2$ on AB may be resolved into a force $wy^2/2$ at q' and a force wy^2 at p', acting as shown; for $H_1 + H_2 = wy^2/2$ acting to the left, and taking moments about p, the moments of H_1 and of H_2 are equal and opposite. H_1 has a normal component on AB', but it causes no stress at A. H_2 has a normal component on AB' equal to $wy^2 \sin\alpha$, which causes a compression at A equal to

$$\frac{2wy^2\sin\alpha}{AB'} = \frac{2wy^2\sin\alpha\cos\alpha}{b}$$

The water pressure on BB' is represented by the trapezoid $BCC'B'$, which may be resolved into two triangles $BB'C$ and $B'CC'$. The force represented by the latter acts at q' and causes no normal stress at A on

[1] *Annales des Ponts et Chaussées*, 1897.

AB'. The force $wyb \tan \alpha/2$ represented by $BB'C$ acts at p', has a normal component $\frac{wyb \tan \alpha}{2} \sin \alpha$, and produces a normal compression at A equal to $wyb \tan \alpha \sin \alpha / AB' = wy \sin^2 \alpha$.

Hence the maximum normal pressure on AB' at A, f_n, is

$$\begin{aligned} f_n &= n'' \cos^2 \alpha + \frac{2wy^2 \sin \alpha \cos \alpha}{b} + wy \sin^2 \alpha \\ &= n'' \cos^2 \alpha + \frac{wy^2}{b} \sin 2\alpha + wy \sin^2 \alpha \\ &= \frac{n'' + wy}{2} + \frac{n'' - wy}{2} \cos 2\alpha + \frac{wy^2}{b} \sin 2\alpha \qquad (8) \end{aligned}$$

while by planar distribution on AB the maximum is n''.

By Guillemain's method we must now find the maximum value of f_n as α varies. By differentiating, it will be found that

$$\text{maximum } f_n = \frac{n'' + wy}{2} + \sqrt{\frac{(n'' - wy)^2}{4} + \frac{w^2y^4}{b^2}}$$

The real maximum stress on a plane normal to the surface at A (Fig. 403) will be by Guillemain's method,

$$\text{maximum } f_n \sec^2 (i - \alpha'),$$

if α' is the value of α corresponding to maximum f_n, or given by the equation

$$\tan 2\alpha' = \frac{2wy^2}{b(n'' - wy)}$$

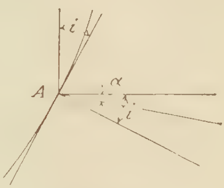


Fig. 403.

by the usual method, $n'' \sec^2 i$.

As stated above, Mr. Lèvy considers that the usual theory is so close to the truth that its use is justified.

15. Distribution of Pressure on a Section in a Dam.—In the preceding article it was assumed that the stress on the sections AB and AB' was distributed in a planar manner. Is this actually the case?

M. Lèvy has studied this subject, and states definitely that planar distribution on a *horizontal joint* is *exactly true* for a triangular dam having the water level at the crest (Fig. 404), but not otherwise.[1] Professor F. Keelhof, of Ghent, makes the same statement.[2]

They prove this by showing that planar distribution agrees in this case with the differential equations of the mathematical theory of elasticity. To the writer this does not appear sufficient. To prove that the distribution is planar, it is not only necessary to show that such distribution agrees with the theory of elasticity, but *that it is the only distribution that does so agree;* and this does not appear to have been done. M. Keelhof

[1] *Comp. Rend.*, vol. CXXVI, pp. 1235–1240, May 2, 1898, and vol. CXXVII, pp. 10–15, 1898.
[2] "Cours de Stabilité des Constructions," Part II, pp. 25–28, Ghent, 1924.

admits that the solution ignores the deformation of the base, and that this makes the value of the solution uncertain.

The general problem of finding the distribution for any form of section has never been solved. Lèvy proves to his own satisfaction that planar distribution for a rectangular dam gives results more unfavorable than the truth, and thinks it probable "but not certain" that the same is true for shapes intermediate between the rectangle and the triangle.

One result arrived at by M. Lèvy by a long mathematical process can be *seen* at once by taking what may be called the *magnifying-glass* or "similarity" point of view. It may be worth while to explain this point of view. It found its application in studying the distribution of the pressure on a retaining wall, the earth being granular and homogeneous and with a plane upper surface. Then taking any distance x from the top of the wall along the back, the total pressure is found by combining certain forces which vary as the square of x, by a geometrical construc-

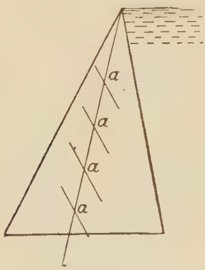


Fig. 404.

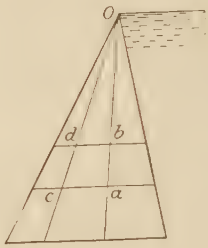


Fig. 405.

tion. Now if the distance along the wall is $2x$, the situation must be the same as if we looked at the first construction through a magnifying glass; hence the *direction* of the pressure must be the same as before, and the forces are merely on a different scale; and, since the weights are four times what they were in the first case, the total pressure must be four times what it was. That is, the intensity of pressure varies as x.

In a similar way it may be *seen* that in a triangular dam with the water level at the crest, if on any straight line through the crest a number of points be taken, and at each point parallel planes a (Fig. 404) be imagined, the stress on all these planes must be parallel and the intensity must be proportional to the distance from the crest. This would not be true if the water level were not at the crest, or if the dam were not triangular. From the above it also follows that in the case of the triangular dam with water level at the crest (Fig. 405) if any two straight lines be drawn through the crest, and any two horizontal lines,

$$\frac{\text{Stress at } a}{\text{Stress at } b} = \frac{\text{stress at } c}{\text{stress at } d}$$

therefore, if normals to the paper be imagined at each point of *ca* representing the distribution of stress over *ca*, the stress at any point in the dam will be represented by the normal from the paper to a surface generated by a straight line which passes always through O and passes through the upper ends of the normals on *ca*.

In Arts. 30 to 35 of Chap. X of "Strength of Materials" the question of the actual distribution of stress over a plane section was discussed, and this discussion should now be reviewed. It was there shown that, aside from the effect of shear, there was good reason to believe, *a priori*, that *for a prismatic piece*, a plane section should remain plane after bending, even above the elastic limit. But this may not hold for *any plane section.*

Let SS (Fig. 406) be a horizontal section of a dam, and suppose we are studying the distribution on this section of the total normal force which acts upon it. Imagine the two parts A and B separated, each having the actual stresses acting on SS. At c on the upper part there will be a certain intensity, and the same at c_1 on the lower part; at d a different intensity and the same at d_1. The two parts A and B will be deformed by these stresses. SS in the lower part will take some shape S_1S_1, and SS in the upper part will take some shape $S_1'S_1'$. The deformations will be such that the two deformed surfaces S_1S_1 and $S_1'S_1'$ must fit together. There seems no escape from this conclusion, for otherwise the stress would cause a separation of the parts. Let $c_1'c_1'$ be the total deformation of the fiber at c_1 of the lower part, not the unit deformation, but the total deformation; and let d_1d_1' be the total deformation of the fiber of the lower part at d_1. Suppose $c_1c_1' > d_1d_1'$. Then in order that the parts may fit, the total deformation cc' of the fiber of the upper part at c must be less than the total deformation of the fiber at d. The parts A and B will in general not be alike, in shape or in dimensions; hence they will deform unequally under equal stresses, as pointed out in the chapter referred to. The total deformation of B at c_1 being greater than at d_1, let us suppose the stress intensity at c_1 to be greater than at d_1. Then in order for the parts to fit, it follows that in A, although the stress intensity at c is greater than that at d_1 the total deformation at c must be less than that at d. If this can be so, then the parts may fit even though S_1S_1 is a curve, that is, if the section SS does not remain plane; but if it cannot be so, that is, if in both parts A and B the total deformations must increase as the stress intensities increase, then the only way in which the two parts can fit is to have S_1S_1 a straight

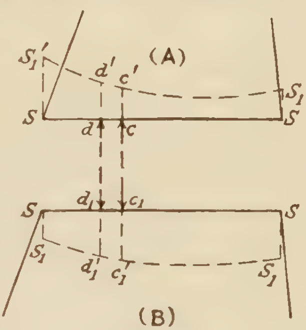


FIG. 406.

line and $S_1'S_1'$ also. There will then be a rotation of the section, but the parts will fit.

I am therefore not convinced, notwithstanding the weighty authority of M. Lèvy, that in a triangular dam, and only then, the normal stress on a horizontal plane is distributed *exactly* in a planar manner, though it may be true. I cannot disprove it.

This discussion, however, is academic, because the only thing we can do is to assume planar distribution, and experience shows that it gives a safe design. This also is the conclusion arrived at by the French engineers, notwithstanding their elaborate mathematical studies. *But Eq.* (3) *must not be neglected.*

16. Center of Gravity of a Triangle or Trapezoid.—In studying dams, it is often convenient to use a method of finding the center of gravity of a triangle or a trapezoid which has not hitherto been mentioned in this

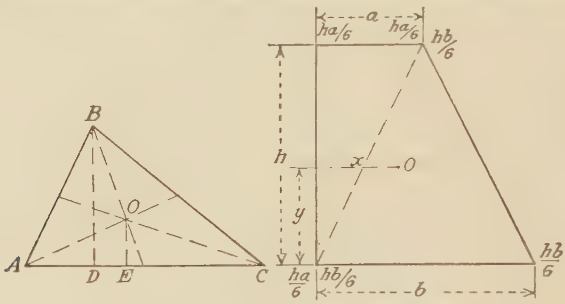


Fig. 407. Fig. 408.

work. Let ABC (Fig. 407) be any triangle. Then it is well known that the center of gravity is at O, the intersection of the lines drawn from the angles to the centers of the opposite sides. Draw OE and BD perpendicular to AC; then $OE = \frac{1}{3} BD$, and

$$\text{Area } ABC \times OE = \frac{1}{3} \text{ area } ABC \times BD$$

From this, and similar relations for the other sides, it follows that *the center of gravity of a triangle is at the point where the paper is cut by the resultant of three equal forces, each equal to one-third the area of the triangle, acting perpendicular to the paper at A, B, and C.*

To find the center of gravity of a trapezoid (Fig. 408), divide it into two triangles, and consider one-third the area of each concentrated at the angles. Then there will be six forces as shown. To find the distance of the center of gravity O above b, take moments about b,

$$y = \frac{\frac{h^2a}{3} + \frac{h^2b}{6}}{h\frac{a+b}{2}} = \frac{h}{3} \cdot \frac{2a+b}{a+b} \tag{9}$$

when

$$a = 0;\ y = \frac{h}{3}$$

when

$$a = b;\ y = \frac{h}{2}$$

To find the distance of O from h, take moments about h.

$$x = \frac{\dfrac{ha^2}{6} + \dfrac{hab}{6} + \dfrac{hb^2}{6}}{h\dfrac{a+b}{2}}$$

$$= \frac{1}{3}\,\frac{a^2 + ab + b^2}{a + b} \qquad (10)$$

when

$$a = 0;\ x = \frac{b}{3}$$

when

$$a = b;\ x = \frac{b}{2}$$

17. Upward Pressure on the Base, and Ice Pressure.—If water percolates under the dam, it will exert an upward pressure on the base, which, as seen from Fig. 399, will throw the line of pressure downstream, and so increase the pressure at the toe and diminish the safety against overturning.

The amount of this upward pressure will depend upon circumstances. The two extremes would be (1) to allow no upward pressure and (2) to allow the full hydrostatic pressure over the entire base. Neither extreme should be assumed. In the case of a dam on solid rock, without seams, if the joint between the dam and the foundation can be made tight, there will be no upward pressure. This has probably been assumed in the majority of dams on rock. But the joint will not always be tight and the rock generally has seams, so that there will in almost all cases be *some* upward pressure. It may be reduced by care in preparing the foundation, by stripping the rock below its natural level, removing all loose rock, uncovering all seams and springs and filling them with concrete or grout, by carrying a cutoff wall at the heel deeper than the rest of the foundation, by carrying the foundation some distance below the natural rock surface, and by making the joint between dam and foundation rough and irregular, so as to retard continuous percolation, and make the path of the percolating water long and tortuous, or by laying drains in the foundation, to carry any seepage to below the dam. If the water percolates, and the percolation is stopped at any point, the tendency is to produce the hydrostatic pressure on the area above that point; it is better to let the water have a free outlet. Thus a cutoff wall at the toe may be

bad, and may increase the upward pressure, while a cutoff wall at the heel is desirable, and should be made as tight as possible; that is, Fig. 409(*e*) is better than Fig. 409(*f*), as shown in Art. 28 of Chap. XX.

In the interest of economy, it is clearly desirable to prevent upward pressure if possible, but it is also clearly wise to assume some upward pressure. The question is as to its amount. If the rock is porous or seamy, it is desirable to lay drains in the foundation to conduct any seepage below the dam; but in cold weather these drains may freeze up, and they cannot be considered to eliminate all upward pressure.

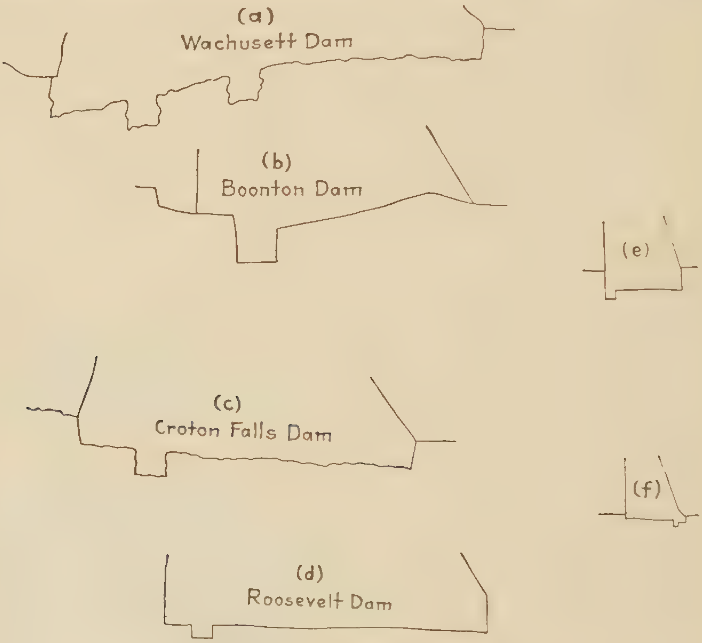


FIG. 409.—Bases of dams.

The other extreme assumption, taking full hydrostatic pressure over the entire base, is quite inadmissible, though it has been urged by some engineers.[1] The pressure at the toe can be no greater than the backwater head. The pressure need not be taken as greater than a uniformly varying pressure, from the full hydrostatic pressure at the heel to zero or the backwater or groundwater pressure at the toe; and even this, with proper care in construction, will be too great, and it will be sufficient to assume two-thirds of this, as has been done in the case of several dams by the Board of Water Supply of New York. The same upward pressure was also assumed in the design of the Wachusett dam of the Metropolitan Water Works of Boston, largely because of the location of the dam and

[1] VAN BUREN, JOHN D., "Notes on High Masonry Dams," *Trans. Am. Soc. C. E.*, vol. XXXIV, p. 493.

the great loss which would follow a failure, since the city of Clinton is but half a mile below the dam.

As a matter of fact, the upward pressure cannot act over the entire area of the base. The dam is not floating; it must be in contact with the rock. Hence the above-described uplift—two-thirds the hydrostatic head at the heel, diminishing to zero or the backwater or groundwater head at the toe—is probably too great in almost every case, and one-half might well be used instead of two-thirds.

A dam on sand, gravel, or other permeable materials is always exposed to uplift. If there is a cutoff line of sheet piling at the heel, the percolating water will follow down this sheet piling if it is tight, up on the back side, and then along the base of the dam. The total distance traveled is called the *creep*, and in this distance the pressure head falls from a maximum at the heel to zero or the backwater head at the toe. The pressure at any intermediate point is determined by proportion. Some tests, however, have shown the pressures much less than according to the theory of the line of creep.[1] The pressure will not be the hydrostatic pressure, even at the heel, but the hydraulic pressure, for the percolating water is in motion. The pressure at the heel may be much less than the hydrostatic pressure, and will tend to diminish as the channels become silted up. In some soils, there will be practically no pressure. There will be a sudden drop from one side to the other of the sheet piling. The pressure will be less the greater the losses of head in following the channel. This is the reason for making the bottom of the dam irregular, so that the path may be crooked. If any obstruction or layer of solid material stops the flow, the pressure above that point may be increased to the hydrostatic head.[2]

If the water percolates under a dam and issues below it, *the important thing is that the velocity should not be great enough to wash out the finest particles of earth*, for if it does, its channel will gradually become enlarged, its velocity will increase, it will wash away larger particles, and failure will be likely to result. This is called *piping*. To reduce the velocity, the length of the channel followed by the percolating water—the *creep*—may be increased by lengthening the apron, or by having a rear apron of concrete or of puddle covered by paving (Fig. 410). The resultant upward pressure on the rear apron will not exceed that of a head equal to its thickness. The apron should be impervious, and particular care must be taken to have no vertical joint at A. If there are two or more cutoff walls, only the one at the heel should be tight; those below should, by means of weep holes, allow percolating water to pass.

[1] *Eng. News-Record*, May, 20, 1920.

[2] For experiments, and a good discussion of uplift generally, see an interesting article by Capt. W. A. Mitchell, U. S. A. in *Professional Memoirs of the Corps of Engineers, U. S. A.*, vol. VII, pp. 32–83; see also C. L. Harrison, "Provision for Uplift and Ice Pressure in Designing Masonry Dams," *Trans., Am. Soc. C. E.*, vol. LXXV, pp. 142–225, 1912.

Provision for uplift cannot properly be made by taking the weight of the masonry as its weight in water, for the section of the dam is not a rectangle.

It may be laid down as a fundamental principle for masonry dams that *a high masonry or concrete dam should be founded throughout on rock.* If low, it may be admissible to found it on hardpan; and with proper precautions a low dam may be built on gravel or sand. It is particularly bad to found a masonry dam partly on rock and partly on earth. This is illustrated by the Puentes dam, in Spain, which failed in 1802. It was intended to carry the foundation to rock; but a deep pocket of earth was found at the center of the valley, and it was decided to found the dam at this place on piles. The water forced its way through the soft material, and undermined the dam. The Bouzey dam, near Epinal, France, rested on red sandstone, but the rock was fissured and quite permeable, and the dam failed by overturning in 1895, probably on account of the upward pressure, which had not been provided for.

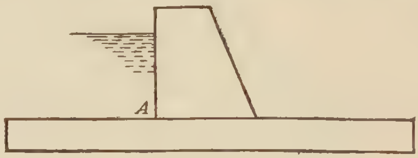


FIG. 410.

The examination and preparation of the rock foundation is one of the most important matters in the construction of a masonry dam, and may well justify the employment of a competent geologist.

There may be an upward pressure on the part of a dam above a horizontal section above the base. Water may enter a horizontal crack in a concrete dam, or a joint in a masonry dam; but such cracks will generally be of small area, and the upward pressure should not be assumed to act over an entire horizontal section. Following the failure of the Bouzey dam, Maurice Lèvy[1] suggested, to meet this possibility, that instead of allowing the line of resistance for reservoir full to pass through the outer middle-third point of a joint, which would mean zero pressure at the upstream face, it should be kept within the outer middle-third point far enough to produce the hydrostatic pressure at the upstream edge, which would prevent water from entering a crack. This amounts to allowing the hydrostatic upward pressure at the upstream edge of any joint for reservoir full. He also suggests that if the dam is curved horizontally and the horizontal arch pressure at the upstream face is made to equal the hydrostatic pressure, the water would not enter any vertical joint. In fact, if this pressure existed, no crack would occur, or it would be closed by the pressure.

[1] *Comp. rend.*, vol. CXXI, pp. 288–300, Aug. 5, 1895.

But concrete and masonry are porous, especially concrete; and water can pass through it. It has generally been assumed that the water is drawn in by capillary action, and exerts no hydrostatic pressure. J. B. Francis showed by experiment that hydrostatic pressure could be transmitted through 15 inches of Portland cement mortar 1:2.[1] He took a cast-iron pipe reducer 8 to 6 inches, placed inside of it a shorter length of 3-inch pipe, bolted both to a base, placed a layer of gravel at the bottom, then about 15 inches of mortar, with water above connected to the city supply at a pressure of about 77 pounds. The inside of the 3-inch pipe, and the outside were provided with outlet pipes, with cocks and gages. The pressure below the mortar reached as high as 73.5 pounds when both cocks were closed, but was gradually reduced by the filling of the pores in the mortar by a fine sediment from the city mains. The mortar was 42 days old when the tests began, having set mainly in water. When the cocks were open, there was no pressure beneath the mortar, but there was percolation at the rate of 10 or 11 pounds of water per 24 hours. This indicates that, if the water is allowed a free outlet, the hydrostatic pressure will be very small. To provide against hydrostatic pressure in the body of the dam, it is quite common now to place vertical or inclined drainage tubes a few feet from the upstream face, leading to a horizontal gallery near the base, from which any seepage is led off by pipes. In the Olive Bridge and Kensico dams (Figs. 424 and 425), these tubes are 16 inches in diameter and about 12 feet apart.

To sum up, then, as regards upward pressure, some upward pressure should be assumed in most cases; to reduce it, the foundation should be carefully prepared, with a cutoff wall in general at the heel, the heel made as tight as possible, the upstream face should be made as impervious as possible by pointing all masonry joints and in concrete by a wash of rich mortar, and any percolating water should be allowed a free exit. By proper construction, hydrostatic pressure in the body of the dam may be rendered negligible. In concrete work, great care should be taken, where there are horizontal joints between layers of concrete, to clean and roughen such joints and wash with grout, to make as perfect a bond as possible and to break the continuity of the joint.

C. L. Harrison states the following principles as governing uplift:[2]

For convenience in discussing this subject, reference is made particularly to masonry dams on rock foundations. The principles involved will apply equally to other foundations and to dams built of other materials. The upward pressure may be due to water getting into the foundation of the dam or into the dam itself.

Foundations vary so much in character that it is necessary to study each particular site before deciding to what extent water may get into them.

1. In the case of a foundation of hard, sound rock, without either horizontal or vertical seams, there is no reason to expect that water will get into it and produce an

[1] "High Walls or Dams to Resist the Pressure of Water," *Trans. Am. Soc. C. E.*, vol. XIX, p. 147.
[2] *Trans. Am. Soc. C. E.*, vol. LXXV, p. 142, 1912.

upward pressure, and, in the design, no allowance should be made for it. In such cases the junction between the masonry and the foundation can easily be made watertight.

2. In the case where the foundation is stratified with well-defined horizontal seams, and the dam is located near a fall or rapids in the stream, so that the water may flow from the seams at the toe of the dam as freely as it enters them from the reservoir, the upward pressure will be approximately equal to the static head at the heel and gradually decrease to zero at the toe of the dam.

3. Take a foundation similar to the foregoing in every respect except that the water in the seams of the rock cannot escape freely near the toe of the dam, but must flow some distance downstream through rock or other materials before it reaches the surface of the ground, or must rise vertically to the surface. Then the upward pressure at the heel will be equal to the static head, and that at the toe will be equal to the head required to overcome the resistance to the water escaping at that point.

Generally, it will be found cheaper to make large expenditures to provide a cutoff in the foundation, which will not only reduce the uplift, but will also save the water. Such a cutoff should be located at the heel of the dam. If it is located under the middle of the dam, there would be an upward pressure under the upstream half of the dam, due to the full head of the water in the reservoir.

In order to determine what allowance to make for pressures due to water which gets into the dam itself, one must first decide on the character of the construction. With suitable stone, sand, and cement, it is possible to build a masonry dam which will have no horizontal cracks or seams, and it is also possible to provide against vertical cracks, to a large extent, by expansion joints. Water in vertical cracks, however, does not produce an upward pressure. In such structures very little, if any, allowance should be made for the upward pressure due to water getting into the masonry.

He further concludes:

An analysis of the discussion indicates the following conclusions:

1. For any stable dam the uplift in the foundation cannot act over the entire area of any horizontal seam, and in the masonry it cannot act over the entire area of any horizontal joint.

2. The intensity of uplift at the heel of the dam can never be more, and is generally less, than that due to the static head. Also, this uplift decreases in intensity from the heel to the toe of the dam, where it will be zero if the water escapes freely, and will be that due to the static head if the water is trapped.

3. The uplift in the foundation should be minimized by a cutoff wall, underdrainage, and grouting when applicable; and in the dam itself by using good materials and workmanship, and by drainage when advisable.

4. The design should be based on the conditions found to exist at each site after a thorough investigation by borings, test pits, and otherwise, and modified if found necessary after bedrock is uncovered.

After all, determining the proper allowance to be made for the uplift and ice pressure is not an exact science, but the engineer, after considering all the obtainable facts bearing on the case, must use his best judgment, based on experience and observation, in making these allowances; and the final design should be made so as not to involve undue risks on the one hand, or unnecessary cast on the other.

Ice Pressure.—Water expands in freezing, and, *if confined*, will exert a pressure on the confining walls. It is possible, but not probable, that such pressure may be exerted against a dam. It may be prevented by

keeping the ice sheet broken above the dam. Harrison[1] suggests the following conditions under which it is not necessary to provide for ice pressure:

1. For the ordinary storage reservoir with sloping banks, in climates where the maximum thickness of ice is 6 inches or less; for dams with southern exposure this limit may be placed as high as 1 foot. None of the discussions fixes this limit, but it is what the writer has in mind as a reasonable provision.

2. For reservoirs which are filled during the flood season and from which all the stored water is drawn off each year during the low-water season. This would include even the large reservoirs on the headwaters of the Mississippi River, where the ice has a thickness of more than 4 feet, and the atmospheric temperatures reach 50° below zero.

3. For storage reservoirs where the water will be drawn off each year during the winter to a level where the dam is strong enough to resist the ice pressure.

4. For reservoirs where the contour of the ground at the high-water level is such that the expansive force of the ice will not reach the dam.

No allowance for ice thrust has been made in the dams built by the U. S. Reclamation Bureau.

Probably ice pressure has not been considered in most cases; in others, it has been taken at about the crushing strength of an ice sheet 1 foot thick. In the Olive Bridge and Kensico dams this principle was followed, and the pressure was assumed as 47,000 pounds per linear foot; in other dams of the New York water supply it has been taken as 30,000 and 24,000 pounds, and in the New Croton dam no ice pressure was assumed.

18. Construction of Masonry Dams.—The foundation has already been discussed. The masonry of the dam usually has a facing of cut stone on both faces, the backing being of rubble masonry, or of concrete with large stones imbedded in it (so-called *cyclopean masonry*). The use of large stones diminishes the voids, and gives a good bond between successive layers if the large stones project above the layer in which they are placed. It is also economical. Each layer of concrete should be cleaned of all loose material or laitance, generally with a wire brush, before the next layer is placed. In some cases it may be necessary to roughen the surface with a pick, to get a better bond, but with the use of large stones this should not be necessary. Each concrete surface should be given a wash of grout before the next layer is poured. The joints of the stone facing should be perpendicular to the surface.

In Germany, many dams have been built according to the plan of Professor Intze, who places a tight earth embankment on the upstream side up to about midheight of the dam, with paving on it. It is questionable if this is advisable, for it makes the lower part of the upstream face inaccessible, and, if the earth dam becomes saturated, the pressure on the masonry dam will be increased. This last will also be true if silt collects above the dam.

[1] *Loc. cit.*, p. 219.

The water is discharged from the reservoir by pipes through the dam, regulated by valves at the upper end placed in a gatehouse. Generally there are two pipes, and the gatehouse has two compartments with a vertical wall between, with openings in it which may be closed by gates. Each compartment has usually three openings from the reservoir, one at the bottom, one near the top, and one midway. These have screens, and grooves for stop planks or gates. The floor of the gatehouse is level with the top of the dam, and there is hoisting apparatus for raising or lowering the gates which control the inlets, and the valves which control the flow through the pipes.

Percolation along the pipes should be prevented by surrounding them with rich concrete. Sometimes the pipes are carried through an open conduit, so as to be accessible.

In a concrete dam it is almost certain that there will be cracks due to contraction.[1] These cracks will be nearly vertical, and there should be vertical expansion joints at intervals of about 50 feet.

19. Overflow Weirs.—A dam must have an overflow or waste weir to discharge surplus water. The crest of this waste weir should be placed at the height above which the level should not normally rise. The maximum flood discharge must be carefully estimated, and the length of the waste weir made sufficient to discharge this flood without rising too near to the top of the dam. The cross-section of the waste weir is different from that of the main part of the dam, because the water flows over its crest and lower face. Sometimes a dam in a river is designed entirely as an overflow weir. If it is desired, in times of drought, to hold the water at a higher level than the crest, flashboards may be placed along the crest, supported against steel pins inserted in holes drilled at close intervals along the crest. Figure 411 shows the cross-section of the dam across the Connecticut River at Holyoke, Mass., which is an overflow dam. At the bottom of the overflow weir, care must be taken to protect the bed of the stream from scour. The bottom is sometimes paved with stone blocks for this purpose, for a considerable distance. In some cases the downstream face of the overflow weir is stepped, for a vertical fall best destroys the energy of the falling water. The face of the main dam is also stepped in some cases, as in the Gilboa dam. If the sheet of water passing is deep, steps are not generally advisable, for, unless wide, the sheet would clear them.

In computing overflow dams, the head of water on the upstream face should be taken to the highest level to which the water may rise. The weight of water on the downstream face should be neglected for safety and because the water may in fact partly clear the face as it shoots over. If the face does not conform to the shape of the falling sheet, there may be a

[1] See paper by Charles S. Gowen, on "The Effect of Temperature Changes on Masonry," *Trans. Am. Soc. C. E.*, vol. LXI, p. 399, 1908.

partial vacuum under the sheet, which would increase the tendency to overturn. There should be provision for the air to enter freely under the sheet, which can be arranged at the ends. If this is not done, the rapid forming and release of the pressure may give rise to vibrations and produce the phenomenon known as that of "trembling dams." This may in some cases be felt several hundred feet away, and may be sufficient to cause windows to rattle. The face, however, should be so shaped as to lie slightly above the parabola for a jet at the top, so that there will be no tendency for a vacuum to form unless the water rises unexpectedly high.

The theoretical curve of the sheet for a sharp-crested weir is a parabola, but it will be different for an actual dam, depending upon the shape of the crest. The bottom of the slope may be turned slightly upward, in order to throw the water up from the bed of the stream.

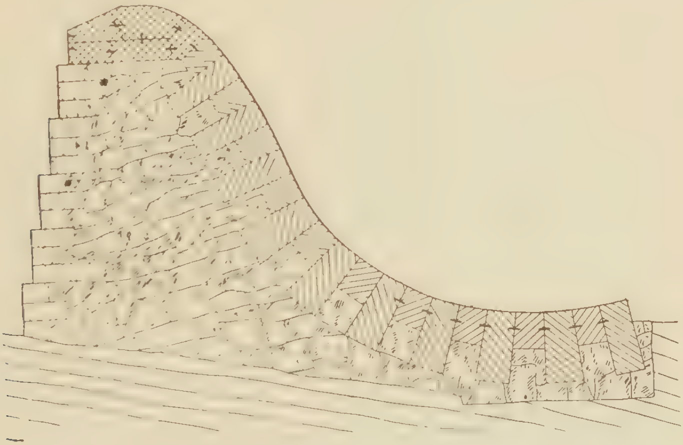


FIG. 411.—Holyoke dam across the Connecticut River.

It is often preferred to make the spillway separate from the main dam, discharging into a channel of its own which is led back below the dam to the main channel.

Circular spillways have been used in some cases, consisting of a hollow vertical tube placed just above the dam. The water discharges over the rim, falls through the tube, and is carried through the bottom of the dam by pipes or other channels.[1] The circular spillway there described, called the *morning-glory* spillway, was located 350 feet upstream from the center line of the dam and on one side of the valley. It was 22.5 feet in diameter for the lower 100 feet, flaring at the top to a diameter of 160 feet.[2]

If a dam is built on erodible soil or rock, the problem of the spillway is to destroy the excess energy of the falling water before it reaches the

[1] See paper by A. C. EATON, *Jour. Boston Soc. C. E.*, p. 9, January, 1925.
[2] See also paper by FORD KURTZ, *Trans. Am. Soc. C. E.*, vol. LXXXVIII, p. 1, 1925.

erodible material. By excess energy is meant that in excess of what it would have as a part of the water flowing in the stream below the dam. This may be done by steps. A long apron will not destroy it, for the friction loss is small, and the effect may only be to transfer the erosion to the lower end of the apron. The smooth "ogee" shape of overfall does not destroy the energy of the water, but guides the water smoothly to the apron, where it is discharged in a horizontal direction. Baffles or other obstruction on the apron would destroy energy. A vertical fall, instead of the smooth curved face, would destroy energy but would tend to erode the apron. With the smooth face and apron, a hydraulic jump would occur at the end of the apron, with erosion and undermining. Erosion will occur if the excess energy of the water is not destroyed before

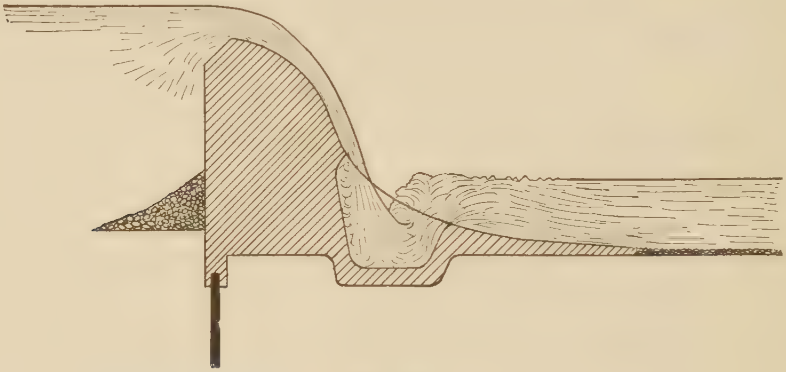


Fig. 412.

it reaches the erodible bed. These considerations have led in some cases to sections like Fig. 412.[1]

The energy may be destroyed by having a pool of water below the main dam, and this is sometimes artificially made by building a low dam below the main dam.

In a symposium before the American Society of Testing Materials, A. P. Davis said,[2] referring to observations made by the U. S. Reclamation Service:

> These observation are very convincing upon two points:
>
> 1. That where clear water can be made to glide over concrete without disturbing its velocity or abruptly changing its direction, there is no practical limit to the velocities that can be permitted without harm.
>
> 2. That concrete which is subjected to the impact of water under high velocity is rapidly eroded and that under such conditions the velocities must be very carefully limited.
>
> Water carrying sand or grit will gradually erode good concrete if the velocity is high.

[1] See an interesting article by Adolph F. Meyer, in *Eng. News-Record*, p. 597, Apr. 9, 1925.

[2] *Proc. Am. Soc. Testing Materials*, Part II, p. 185, 1923.

Wegmann gives two sheets of drawings showing profiles of overflow weirs.

Further details of appurtenances, valves, gates, gate chambers, outlets, etc., belong in works on hydraulic engineering.

20. Siphon Spillways.—Instead of separate overflow weirs, the discharge over which depends upon the depth of water flowing over, as given by one of the weir formulae, there may be siphon spillways in the dam itself, by which, as soon as the water rises slightly, or enough to put the siphon in operation, the atmospheric pressure on the reservoir surface becomes available to force the water through the siphon. The

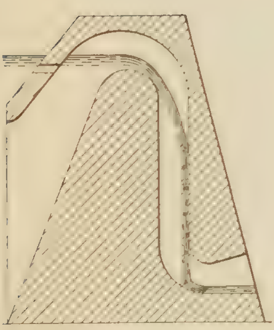


Fig. 413.—Siphon spillway.

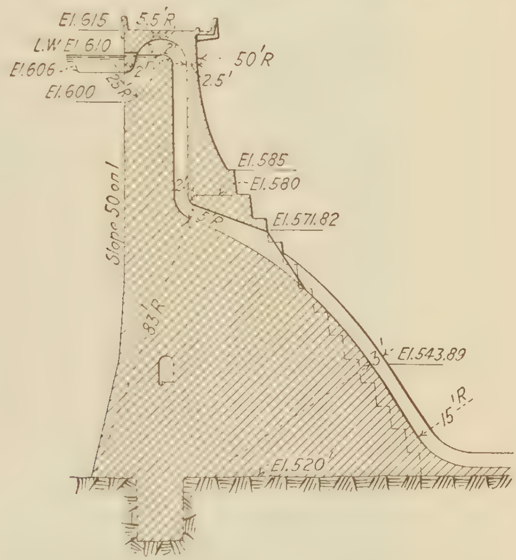


Fig. 414.—Siphon spillway at Alpine Dam, in California. (*Stickney.*)

theoretical velocity of discharge at the outlet then becomes the difference of level between the outlet and the reservoir level, and the actual velocity may be 60 to 80 per cent of this, provided that the difference of level between the outlet and the highest point of the siphon (the throat) must, if the siphon is of uniform section, not exceed the head due to the atmospheric pressure plus the losses of head between those points, and must be somewhat less than this to prevent the accumulation of air at the throat. If the siphon is not of uniform section, but decreases in area toward the outlet, so that the velocity increases toward the outlet, the elevation of the throat above the outlet may be greater.

Siphon spillways have been used in Europe since about 1870, and lately have been used in this country,[1] largely through the efforts of G. F.

[1] *Eng. News*, p. 467, Apr. 20, 1911; *Eng. Record*, p., 489, May 3, 1913; *Eng. Record*, p. 567, May 6, 1914; *Eng. Contr.*, Apr. 14, 1920.

Stickney, of Albany, N. Y., who has taken out several patents for the device.[1] An overflow weir, unless very long, may discharge little water until the level rises to a considerable height above the crest; while a siphon spillway may discharge a large quantity for a slight rise of level, and hence may be much shorter than the overflow weir.

If the water rises above the lowest point of the throat sufficiently to close the air inlet, and if the siphon is sealed at the lower end, or pref-

Fig. 415.—Siphon spillway at Alpine Dam in California. (*Stickney.*)

erably at any point between by the flowing sheet crossing the lower leg (as in Fig. 413), the air above the sheet will be gradually carried out. This will reduce the pressure and cause the water to rise still farther. When the water reaches the highest point of the throat, the siphon is said to be "primed," and true siphonic action begins. The flow may be stopped by letting more air in above the sheet, if there is a pipe and valve at the throat, than can be absorbed and carried away by the sheet. A small reduction of air pressure above the sheet may prime the siphon and start it in operation quickly. When the level falls so low as to open

[1] See his paper on "Siphon Spillways," *Trans. Am. Soc. C. E.*, vol. LXXXV, pp. 1098–1151, 1922.

the inlet, the flow abruptly stops. There are various devices to hasten priming.

Figures 414 and 415 show Stickney's siphons in the Alpine dam, in California. These have a total length of 64 feet, but with a rise of water of 1 foot, they discharge as much water as would pass over an overflow weir 1,140 feet long with the same rise. Where, as in this case, there are several siphons in a dam, by placing the crests or the air vents at slightly different levels, the discharge may be controlled as desired.

A siphon spillway used by the U. S. Reclamation service is shown in Fig. 416. Combined with it, but not a necessary part of it, is a sluice closed by a gate.

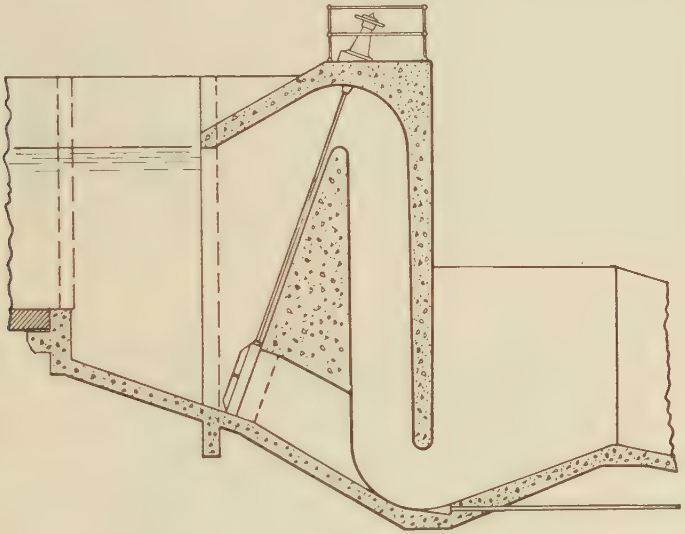


FIG. 416.—Siphon spillway of U. S. Reclamation Service.

21. Arched Dams.—If the dam is curved horizontally, it may act partly as a gravity dam and partly as a horizontal arch. It should not be considered to act as an arch if the thickness is too great in proportion to the radius. According to Delocre, in order for it to act as an arch, the thickness at the crown should not be greater than one-third the radius of the upstream face; according to Pelletreau, one-half. The arch action can only transmit the hydrostatic pressure, or part of it, to the sides of the valley, leaving the rest, and the entire weight of the dam, to be carried by gravity action to the foundation.

Can a dam act by gravity section and as an arch at the same time?

It is obvious that the action of a curved dam is very complicated, and it probably cannot be determined accurately by any mathematical analysis. The answer to the above question, however, is unquestionably that it can act in both ways at once. If it acts by gravity alone,

each vertical strip deflects or bends like a vertical beam, on account of the unequal distribution of stress on a horizontal section. The horizontal deflection of a vertical strip will tend to bring adjoining vertical strips closer together, and must bring into action horizontal reactions. On any horizontal strip the horizontal or arch reactions must be such that at each point the deformation as an arch will equal the deformation under gravity action. On any such horizontal strip the hydrostatic pressure p may be divided into two parts, np being the part borne by the strip as an arch, and $(1 - n)p$ being the part borne by gravity action, n being a proper fraction which will vary with the depth and the width of the valley, and will probably also vary across the horizontal strip from side to side of the valley. The condition which determines n is that at every point the horizontal deflection of the horizontal strip under the arch action must equal the horizontal deflection of the vertical strip under the gravity action. It is obvious that the problem is one of extreme complexity. It cannot be further discussed here, and I, for one, do not believe it possible of solution, considering the varying width of the valley at different elevations and the varying height of the dam at different points. There will be as many values of n as there are points taken on the face of the dam, and there is a deformation equation for each point.

If the value of n were the same all along a horizontal strip, the arch thrust T (not horizontal but tangential) could be found by Navier's principle as

$$T = np \cdot r \qquad (11)$$

The radius r should be taken as the radius of the center line of the horizontal strip, though Delocre takes it at the upstream middle-third point, making the maximum arch stress on a vertical joint equal to twice the average.

Many masonry dams have been curved in plan, but probably most of them have been designed as gravity sections, the arch action being merely an additional factor of safety. This is a desirable procedure, because, if there is any uncertainty in the ability of the sides of the valley to withstand a horizontal arch thrust without yielding, the reality of the arch action is made all the more doubtful.

The arch principle is obviously most applicable in the case of a very narrow dam with steep rocky walls. In such a case it may be safe to rely on the arch action almost entirely, as in the Huacal and Upper Otay dams (Figs. 417 and 418), both of which are extremely bold.[1] Curving the arch will in any case tend to diminish the likelihood of vertical cracks.

[1] For a list of curved masonry dams, with details, see paper by H. Hawgood, on "The Huacal Dam, Sonora, New Mexico," with discussion, in *Trans. Am. Soc. C. E.*, vol. LXXVIII, pp. 564–629, 1915. The list is on p. 603.

FIG. 417.—Huacal Dam, N. M. (*Courtesy of Mr. H. Hawgood.*)

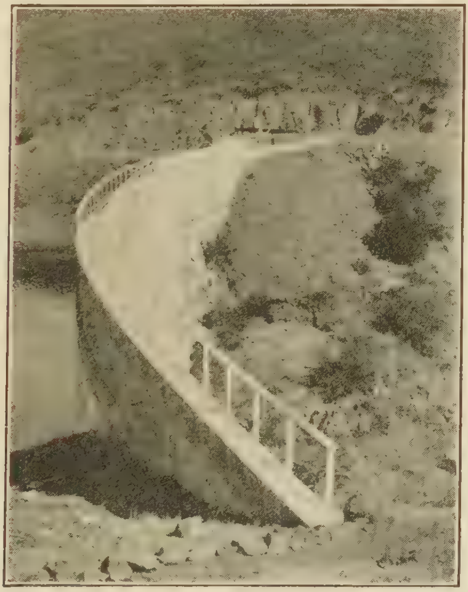


FIG. 418.—Upper Otay Dam, Cal. (*Courtesy of Mr. H. Hawgood.*)

As the width of a valley or gorge diminishes from the top of the dam downward, in almost all cases, it follows that, if the radius is the same for all horizontal strips, the angle subtended will diminish from the top downward. If the entire hydrostatic pressure is assumed resisted by arch action, $T = pr$, and since p increases from the top downward, T will increase similarly if r is constant. L. R. Jorgensen[1] has suggested that r should decrease from the top downward in such a way as to make each horizontal strip subtend a constant central angle, thus making T more nearly constant. If the gorge were of triangular shape, this principle would make T really the same for each horizontal strip. Mr. Jorgensen claims that the use of this principle will effect a saving of material, and it may clearly be made to accomplish this result. The suggestion is a good one.

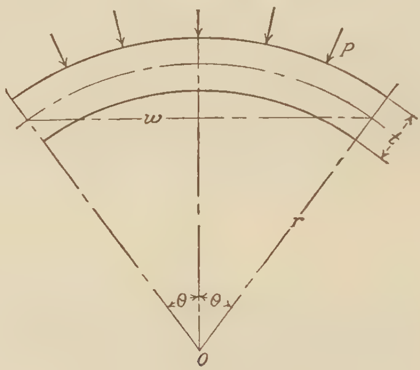


FIG. 419.

Suppose a dam acting by arch action alone in supporting the hydrostatic pressure, and let Fig. 419 represent a horizontal strip one unit in height perpendicular to the paper. Let r be the radius of the face, and t the thickness, the angle subtended being 2θ, and p the unit hydrostatic pressure. Then the thrust $T = p\left(r + \frac{t}{2}\right)$; and if the allowable pressure is f, the thickness or area will be

$$t = A = \frac{p\left(r + \frac{t}{2}\right)}{f}, \tag{12}$$

and the volume of masonry in the strip, V, will be

$$V = 2A\theta\left(r + \frac{t}{2}\right) \tag{13}$$

[1] *Trans. Am. Soc. C. E.*, p. 685, 1915, and p. 850, 1917. The same suggestion appears to have been made previously in a pamphlet (without date) on "A New Type of Dam, by F. G. Baum & Co., San Francisco.

If the chord, or width of the canyon, is w,

$$r + \frac{t}{2} = \frac{w}{2 \sin \theta}$$

Hence

$$V = \frac{pw^2\theta}{2f \sin^2 \theta} \tag{14}$$

The volume will be a minimum when $\theta/\sin^2 \theta$ is a minimum, which will be when

$$\theta = \frac{1}{2} \tan \theta$$

or when θ is about 134°. The value of V, however, changes slowly near the minimum, and is only about 1.2 per cent greater for $\theta = 60°$ than for the minimum.

An arch dam may be made of a series of horizontal segments, each of a different thickness, or in other words, stepped.

Vertical expansion joints in a curved dam, if they open, of course prevent any arch action. Some engineers advocate pouring certain vertical sections, preferably alternate sections, in cold weather, hoping that the expansion due to temperature will prevent opening of the joints, which is due to contraction, though they are called expansion joints, in this way allowing compression to be exerted on all vertical joints. But the shrinkage effect will still remain, and the joints will still open unless the compressive deformation due to temperature, plus the compressive deformation due to arch action, in a curved dam, exceeds the contraction due to shrinkage.

21a. Effect of Lateral Expansion on Arch Dams. Poisson's Ratio.—An arch dam is exposed to compression in three directions at right angles to each other. If the upstream face is vertical, these forces are:

1. A vertical pressure on a horizontal plane, due to the weight. Consider a horizontal plane at a depth H, and suppose the water level at the crest. If A is the area of the vertical cross-section above the horizontal plane, b the thickness at the horizontal plane, and w the weight of a cubic unit of masonry, the average vertical unit pressure will be

$$p_1 = \frac{wA}{b}$$

of, if b' is the average width of A

$$p_1 = \frac{wb'H}{b}$$

2. A horizontal pressure p_2 on a vertical plane parallel to the upstream face. At the upstream face, p_2 will be the hydrostatic pressure w_1H, if w_1 is the weight of a cubic unit of water. At the downstream face p_2 will be zero (atmospheric pressure neglected). Perhaps an average on the width b would be $\frac{1}{2}w_1H = p_2$.

These two pressures would tend to cause an expansion in a direction perpendicular to the two planes on which they act, that is, on a vertical transverse section. This expansion, if Poisson's ratio is m, would be

$$m\frac{p_1 + p_2}{E}.$$

If the abutments of the dam are immovable, this expansion could not occur, but there would be:

3. On a vertical transverse section, a horizontal force along the axis of the arch, or approximately along the chord; that is, an arch thrust of

$$\begin{aligned} p_3 &= m(p_1 + p_2) \\ &= mH\left(\frac{wb'}{b} + \frac{1}{2}w_1\right) \end{aligned}$$

If this action did not occur, and if r is the radius of the axis of the arch, the hydrostatic pressure on the upstream face would be resisted in two ways: first, by the arch action of the dam, which might be considered to resist a certain fraction of the pressure; second, by the action of the dam as a gravity section, or, as it is sometimes expressed, by cantilever action, the dam acting as a cantilever fixed at the bottom; this would carry the remaining part of the pressure.

The action described means that there is a certain horizontal thrust produced by the tendency to lateral expansion, *if the abutments are immovable.* It should be neglected if the abutments may yield, even a little. If it does exist, the lateral pressure due to the tendency to expand takes care of a certain part of the hydrostatic pressure, say a head H', and only the remainder is to be resisted by arch action and by gravity action (cantilever action). The lateral pressure p_3 corresponds to a hydrostatic pressure given by the equation

$$w_1H'r = p_3b = mH\left(wb' + \frac{1}{2}w_1b\right),$$

or, the part of the total head which is absorbed in producing the tendency to lateral expansion, which is resisted by unyielding abutments, is

$$H' = H \cdot m\left(\frac{wb'}{w_1r} + \frac{b}{2r}\right) \qquad (14a)$$

The value of Poisson's ratio $1/m$ may be found by direct measurement of the lateral deformation, but as these deformations will be small, a better method may be the following; from Eq. (9) of Chap. IV of "Strength of Materials,"

$$E_s = E\frac{m}{2(m+1)},$$

where E_s is the modulus of elasticity in shear, and E is the modulus of elasticity in compression. From this

$$m = \frac{2}{\frac{E}{E_s} - 2}$$

By this method, Bach found the value of m for concrete to be 5.3, or $1/m = 0.189$.

Even if m is assumed as 0.15, H' may be a considerable part of H. The value of H'/H will vary with r, h, and b'. If the radius of the arch is constant from top to bottom, the only variables will be b and b'. If the arch is one with constant central angle, r will decrease from the top down, and the value of H'/H will be much greater than if r is constant. In other words, each horizontal layer of the dam, which is a horizontal arch, will take an increasing percentage of the hydrostatic pressure on it to balance the tendency to expansion, leaving a decreasing percentage to be taken care of by arch and cantilever action.

If this is taken account of, the stresses in the dam become very complicated; but any horizontal layer can be considered by itself. General formulae will become so complicated that I will not attempt to derive them.

The constant-angle arch dam, then, is more effective in utilizing the tendency to expand under pressure, if that tendency is resisted, than the arch dam with a single center. In the constant-angle dam, as we go down, more and more of the hydrostatic pressure would be taken up in this way, leaving less and less to be resisted by arch and cantilever action; so that the deflection from a vertical should decrease from the top down. In the dam with a single center, most of the load would be taken by arch and cantilever action.

If the concrete of a dam shrinks, this may entirely counteract, or more than counteract, the tendency to expand due to the pressure, and the formulae of this article would be inapplicable. Such shrinkage may occur. The upstream face, however, and the part of the dam which is constantly wet, would expand in setting and hardening.

The subject is manifestly very uncertain. Complicated formulae give only an illusory appearance of accuracy.

22. Multiple-arch Dams.—Some dams have been built as a series of horizontal or inclined arches with piers between. Such a design has the advantage that the arches are shorter than a single arch would be, and, if the spans, rises, and the heights are the same, the horizontal thrusts on the piers will balance, since the load on each arch is the same and is constant for any stage of the water. The horizontal force on each pier is thus merely the hydrostatic pressure carried by one arch. The arch principle may clearly be more fully utilized by this construction than by

any other, as the span of the arches may be made short, and the entire hydrostatic pressure may be considered to be resisted by arch action. Just as a dam may be designed for gravity action alone, and the arch action considered as an additional factor of safety, so it may be designed for arch action alone and the gravity action considered as an additional factor of safety. Of course, the ability of the abutments at the sides of the valley to carry the thrust without yielding must be assured, the piers must be sufficient to carry the thrust by gravity action, and the foundation must be sufficient to carry the pressure. If the upstream face of the dam is vertical, pressure of the water will have no vertical component.

Fig. 420.

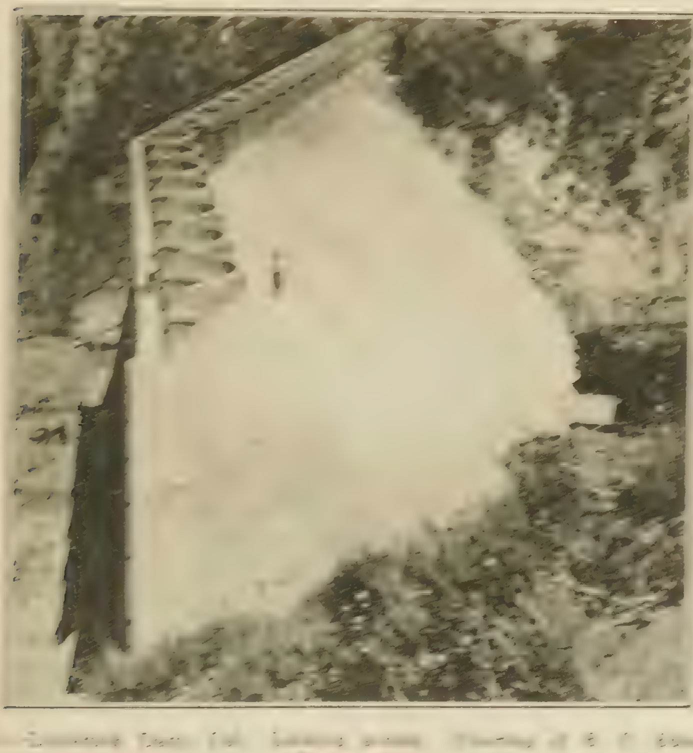
If the upstream face is inclined, the entire normal hydrostatic pressure may be taken by arch action. The weight of a ring of the arch itself will have a normal component, Fig. 420, which will be taken by arch action, and a component p along the slope, causing compression

Fig. 421.—[illegible] Dam, Cal. [illegible] (Courtesy of [illegible], Chief Engineer.)

in the arch. If properly designed for arch action, and properly constructed, such a dam should be perfectly safe. Figures 421, 421a and 421b are views of a multiple-arch dam.

The spans of such a multiple-arch dam should preferably be the same to promote uniformity in the forms. The economical span will vary with

[illegible]



[illegible]

the height. In the Stony Gorge dam of the U. S. Reclamation service, about 110 feet high, the piers were 18 feet on centers; the upper face was a continuous slab (not arch) of reinforced concrete.

Some failures of arch dams and of multiple-arch dams have occurred. The Gem Lake multiple-arch concrete dam, in California, built in 1916, had sixteen 40-foot arches and two fractional arches at the ends. The maximum height of the arch was 84 feet, and from the crest to the lowest point of the foundation was 112 feet. The upstream face made an angle of 50° with the horizontal. The thickness was 3.95 feet at the bottom and 1 foot at the crest. The buttresses were 1.85 feet thick at the top and 4.25 feet at the deepest point. This dam gave much trouble, and was rebuilt to a gravity section. The cause of the trouble appears to have been due to defective material and porous concrete, though some ascribed it to the effect of cold weather. The designer stated that about 3 years after completion, he found on the downstream face a thick layer of calcareous deposit which seemed to be increasing. The cement was evidently being dissolved out from the concrete. The concrete was 1:2:4, the sand being 75 per cent lake sand and 25 per cent crushed rock. The lake sand had 3½ per cent of clay and about 1 per cent of dirt. The aggregate is said to have contained much mica, and the concrete to have been "sloppy." Anyway, the dam was a failure, though reconstructed in time to prevent a collapse.[1]

A complete and interesting description of the failure and reconstruction of the Gem Lake dam, and an illuminating discussion of multiple-arch dams, has been given by F. O. Dolson and W. L. Huber.[2] The writers think that the failure was due to the freezing of the water which percolated into the porous concrete. In view of the difficulty of insuring waterproof concrete, I agree with the engineers who think that under such severe climatic conditions thin multiple-arch concrete dams should not be used, notwithstanding the fact that there are a number of such dams that have stood well under such conditions. It is very important that the concrete should be properly cured and great pains taken to secure a watertight face, if such dams are used.

A concrete arch dam for the water supply of Manitou, Colo., showed similar effects. It was 50 feet high and had a span of 300 feet. The concrete was poor, a layer of calcareous material ½ inch thick appeared on the lower face, the sand contained much mica, and the structure was strengthened by putting a partial fill of gravel and silt on the upper side, and a complete fill of gravel on the lower side, as shown in Fig. 422. The construction joints, where the leakage was greatest, were dug out to a maximum of 2 feet and a minimum of 6 inches, and filled with 1:2 mortar. As reconstructed, the line of resistance for reservoir empty is

[1] See *Eng. News-Record* for 1925 as follows: p. 22 July 2; July 30; Aug. 6; Sept. 3; Oct. 8.

[2] *Trans. Am. Soc. C. E.*, vol. LXXXIX, pp. 713–789, 1926.

said to fall within the middle third, and for reservoir full well within the toe, but it is not stated whether hydrostatic uplift is considered.[1]

These cases show the necessity of having a dense and impervious concrete, clean aggregates, with no mica, and also the desirability of assuming hydrostatic uplift. Concrete through which water percolates is doomed.

The most notable failure of a multiple-arch dam is that of the Gleno dam, about 30 miles northeast of Bergamo, Italy, which failed Dec. 1, 1923, about 7 weeks after completion, wiping out several towns or villages, and causing about 500 deaths and a property loss of some 150,000,000 lire. It was a reinforced concrete multiple-arch dam on a stone gravity base, and was 143 feet high above the stream and 863 feet long on top, curved in plan. The masonry base was 52½ feet high and about

FIG. 422.—Manitou dam.

250 feet long, on which rested the curved portion of the multiple-arch dam, with straight wings on both sides which rested on the smooth rock, and were not bonded to it. The base masonry was laid in lime mortar. The multiple arches were semicircular, 26 feet 3 inches center to center of buttresses and inclined about 53° to the horizontal. The plans seem to have been changed during construction, and the government ordered the work stopped till the final plans were approved, which order seems to have been ignored. The concrete appears to have been poor and porous, and the work badly executed. Other explanations, however, have been given for the failure. A Swiss engineer attributes it to weakness of the buttresses in shear. Another engineer ascribes it to upward pressures. A committee found it due to defective foundations and insufficient static resistance of the base masonry. Finally, a commission has said that the foundation rock, and probably the base masonry, had been fissured "by earthquake shocks or the like," leaving the structure

[1] See paper by J. E. FIELD, *Eng. News-Record*, Dec. 10, 1925.

in a condition of extreme susceptibilty, the immediate cause thereafter being the explosion of a charge of dynamite in the outlet tunnel.[1]

These illustrations illustrate also the many uncertainties in the use of concrete.

A concrete multiple-arch dam 286 feet high was being built in Italy in 1924, in the province of Bologna, for the State Railways of Italy.[2]

23. The Dam in Plan.—From the above it is clear that a dam may be (1) a pure arch dam, (2) a straight gravity dam, (3) a curved gravity dam.

In the pure arch dam the hydrostatic pressure is considered resisted entirely by arch action, though there is also gravity action. In the straight gravity dam there is no arch action. In the curved gravity dam there are both actions, but the arch action is not considered. It is desirable to curve a gravity dam and so obtain the advantage of some pressure on a vertical section. This action is of two kinds. The first is that due to the fact that taking the part having any chord, say w in Fig. 419, the hydrostatic pressures on each half of this portion have equal components parallel to the chord, causing compression on the vertical section at the center, this action being independent of the arch action, which would exist even if the horizontal pressure were perpendicular to the chord. The second is the true arch action.

If the dam is a straight gravity dam, the crest is straight; and if the upstream face is vertical, the upstream edge of the base will be straight in plan, while the lower edge will be curved. The downstream face will be a single curved surface generated by a horizontal line moving parallel to itself. If the upstream face is curved, both edges of the base will be curved in plan.

24. Allowable Pressures.—The maximum pressure allowed in masonry dams, computed by planar distribution on horizontal sections, has in foreign dams varied from about 85 to 200 pounds per square inch. Wegmann states that the Almanza dam in Spain has carried for 300 years about 200 pounds per square inch, or greater than in any other dam. In the New Croton dam, the maximum above the original ground surface was 16,400 pounds per square foot on the downstream face and 20,600 pounds per square foot on the upstream face, evidently following Rankine's principles; and 15 tons per square foot or 208 pounds per square inch below the original surface; and in the Olive Bridge and Kensico dams 20 tons, or 277 pounds per square inch. These pressures were computed using vertical forces alone, without Eq. (3).

In the dams of the U. S. Reclamation Service the following are maximum pressures, using Eq. (3):

[1] See *Eng. News-Record* for 1924 as follows: Jan. 3, Jan. 31, Mar. 20, Apr. 10, June 12.

[2] *Eng. News-Record*, June 19, 1924. For illustrations of arch dams in Australia see paper by L. A. B. Wade, *Proc. Inst. C. E.*, vol. CLXXVIII, pp. 1–110, 1908–1909.

Arrowrock dam.............. 38.4 tons per square foot
Elephant Butte.............. 28.3 tons per square foot
Roosevelt dam............... 34.9 tons per square foot

The largest of these, or 533 pounds per square inch, seems high. Probably other dams would show high values if Eq. (3) were used.

Coefficients of friction are given in Chap. II of "Strength of Materials." For a section through the dam, a safe value is 0.65 to 0.75. For the joint between the dam and the foundation, masonry on gravel will have a coefficient of about 0.6, on dry clay about 0.50, on sand about 0.40, and on moist clay about 0.33.

25. Examples of Masonry Dams.—Many masonry dams have been built. The oldest now existing is the Almanza dam in Spain, which was apparently in use before 1586. It is about 68 feet high, and is of rubble masonry faced with cut stone, except that the upper 20 feet is of ashlar. It is convex upstream. There are other old Spanish dams, generally with extravagant sections, the Alicante dam, built between 1579 and 1594, being nearly trapezoidal in section, with a height of about 41 meters, a width at base of about 33.5 meters and a width at the top of 20 meters, with sloping faces and six narrow steps on the downstream side, 0.4 to 1 meter in width.[1]

The first dam to be built according to scientific principles was the Furens dam near St. Etienne, France, completed in 1866, with a maximum height above foundations of about 184 feet, and a width of base of 161 feet. It was for many years the highest dam in existence. It was built of rubble masonry, and is curved with a radius of about 828 feet. The length of the chord at the top is only 328 feet. The allowed pressure is 92 pounds per square inch.

The Gileppe dam, in Belgium, is about 154 feet high, and the base is 216.5 feet wide. It somewhat resembles the old Spanish dams in its excess of material. It was computed allowing a specific gravity of the masonry of 1.3 (instead of the actual 2.3) to allow for upward pressure, although this is not the correct method of allowing for such pressure, as shown in Art. 17.

Following the Furens dam, the highest dam in Europe until recently was the Urft dam, near Aachen, completed in 1904 on the Intze plan (see Art. 18). It is 190 feet high, and curved. The width on top is 18 feet, and at the base about 166 feet.

The Habra dam, in Algiers, failed in 1881, owing partly to poor cement having an excess of free lime, and partly to a flood which raised the water level above that assumed, and increased the maximum pressure considerably.

Everyone has heard about the Assuan dam across the Nile, which was completed in 1902. It is 6,400 feet long, with a maximum height

[1] According to Wegmann.

of 96 feet, of granite rubble masonry faced with squared granite. It was built to control the flood waters of the Nile for irrigation.

The highest masonry dam in Europe, and in the world, is the Camarasa dam in the Spanish Pyrenees, built in 1918–1920 for the development of hydroelectric power. Its maximum height above the foundation is 335 feet. It is curved. The profile is a triangle, with a rectangle on top. The straight slope of the downstream face has a tangent of 0.8 with the vertical. This is found by the formula

$$\tan \alpha = \sqrt{\frac{1}{g - 1}}$$

in which g is the specific gravity of the masonry.

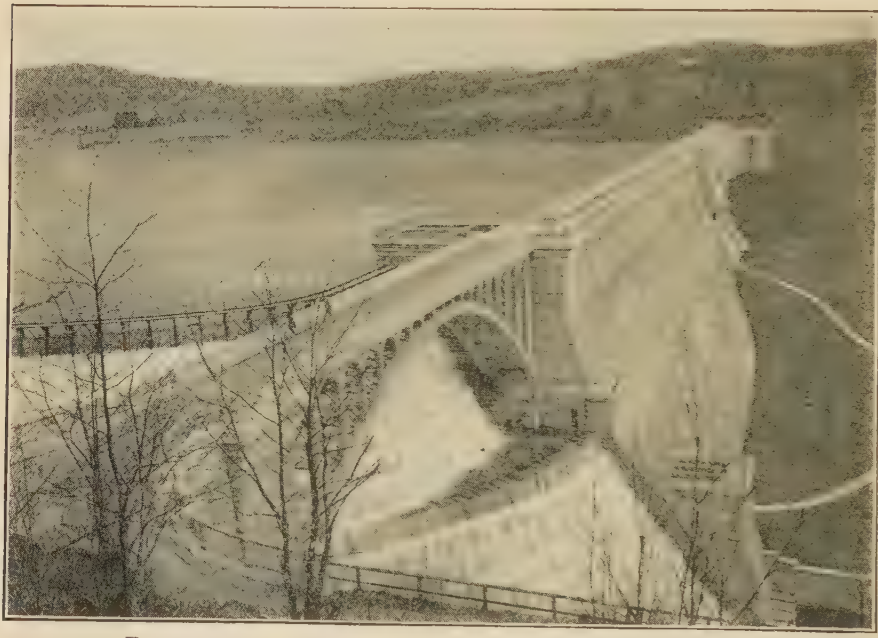


FIG. 423.—New Croton dam for water supply of New York City.

Many high and bold masonry dams have been built in the United States, for purposes of water supply, water power, and irrigation. Perhaps the boldest was the first Bear Valley dam, built in 1884 in southern California for irrigation purposes. Its maximum height is 64 feet, and it is 300 feet long on the crest. It is curved with a radius of 335 feet, and depends for its stability upon arch action, as the line of pressure lies almost entirely outside the profile. The upper 48 feet is trapezoidal with the lower face vertical, the upper face battered from a width of 3.17 feet at the top, to 8.42 feet at a depth of 48 feet. At this point

there is an offset of 2 feet on both sides, below which both faces are battered to a width of 20 feet at the base. This dam, which was of rubble faced with granite ashlar, was considered to have too small a factor of safety, and was replaced, or rather supplemented, in 1911 by a new dam built about 200 feet below it. This is a multiple-arch concrete dam with piers 32 feet center to center. The maximum height is 91.5 feet, and it is 350 feet long on the crest.

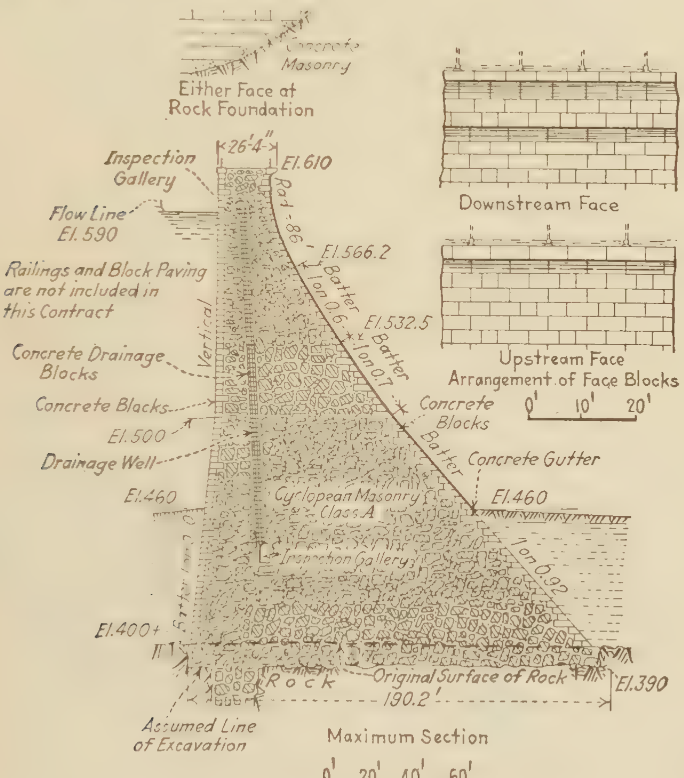


FIG. 424.—Section of Olive bridge dam for water supply of New York City.

Another bold dam is the Upper Otay dam, in California. Its maximum height is 84 feet. It is 4 feet wide on top and 14 feet at the base, built of Portland cement concrete, curved to a radius of 359 feet, and 350 feet long on the crest. It is reinforced for its upper 50 feet "by two tiers of steel plate set longitudinally in the concrete on the axis of the dam," and by 1½-inch cables placed vertically at intervals of 2 feet.

Some of the largest dams in the United States have been in connection with the water works of New York. Chief of these is the New Croton dam, which was completed in 1907, and was at the time the highest dam in the world, having a maximum height above foundations of 296 feet,

173 feet above the river bed and 240 feet above the average rock foundation. The maximum width of base was 206 feet. The foundations had to be carried deep below the surface to reach rock, and the rock itself had to be excavated to a considerable depth. The maximum depth of excavation below the river bed was 123 feet, and the maximum rock excavation about 54 feet, the average being about 20 feet. The dam was of cyclopean masonry and rubble masonry, with a facing of ashlar. The section of the overfall was stepped, the crest being 16 feet below the top of the main dam; and there are about 2 feet of permanent flashboards on this. Figure 423 shows this beautiful structure.

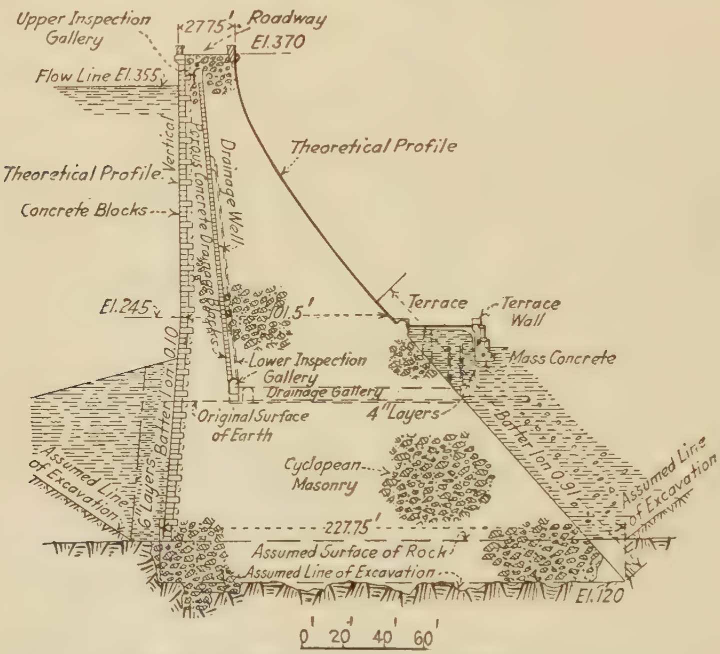


FIG. 425.—Section of Kensico dam for water supply of New York City.

Another great dam of the New York Water Works is the Olive bridge dam in the Catskill Mountains, which holds back the Ashokan reservoir on Esopus Creek. It was built in 1908–1914, and consists of a masonry dam 1,000 feet long on the crest, with an earth dam on each side having a masonry core wall. The total length is about 4,650 feet. The masonry dam has a minimum width on top of 23 feet, and is 190 feet wide at the base. Its maximum height above the foundation is 240 feet, and 210 feet above the original bed of the creek. The crest is 20 feet above the top of the overflow weir. The dam is of cyclopean masonry faced with concrete blocks. There are expansion joints at intervals of about 84 feet. The section of the dam is shown in Fig. 424.

Still another great New York dam is the Kensico dam, on the Bronx River, about 15 miles above New York City. It is 1,843 feet long on the crest, 28 feet wide on top and 235 feet at the base; and its maximum height above the foundation is 307 feet. It is built of cyclopean masonry, faced on the upstream side with precast concrete blocks 2.5 feet high and 5 to 6.5 feet long, and on the downstream side with cut stone masonry where exposed. There are expansion joints about 79 feet apart. The

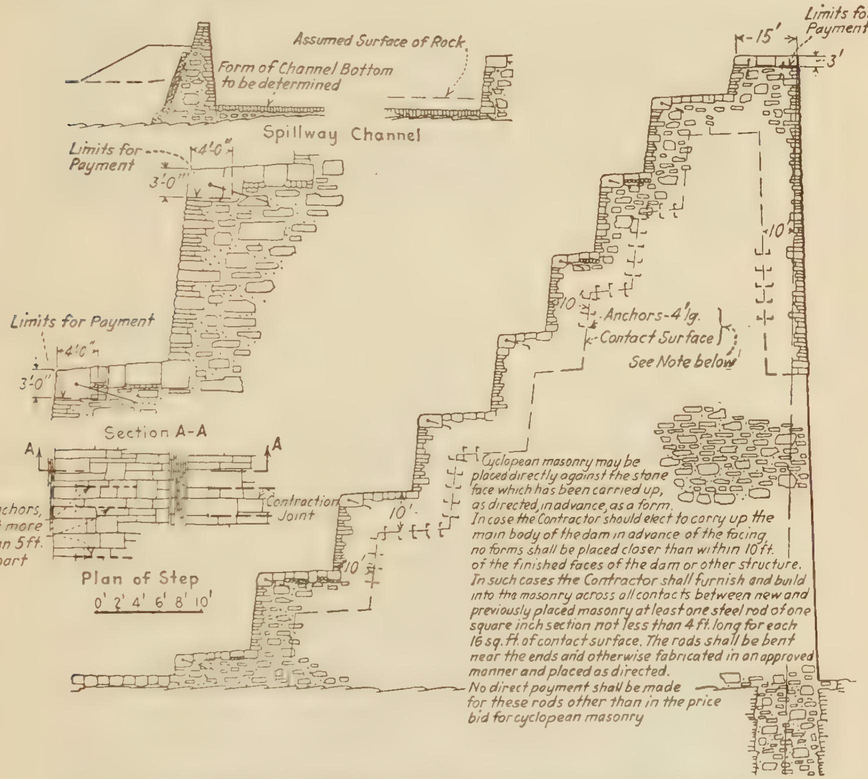


Fig. 426.—Section of Gilboa dam for water supply of New York City.

concrete blocks were in the proportions 1:3:5, and the concrete for the cyclopean masonry 1:9. The profile is shown in Fig. 425.

The last of the great New York dams to be mentioned is that at Gilboa, which forms a reservoir on Schoharie Creek, which is connected to the Ashokan reservoir by a tunnel. This dam is of cyclopean masonry with faces of cut stone. The section is shown in Fig. 426. Figure 427 is a view of this dam showing stepped down stream face, and Fig. 428 a view during construction.

The Wachusett dam of the Boston Water Works, on the south branch of the Nashua River has the section shown in Fig. 429. The

Fig. 427.—Gilboa dam. (*Courtesy of the Hugh Nawn Contracting Co.*)

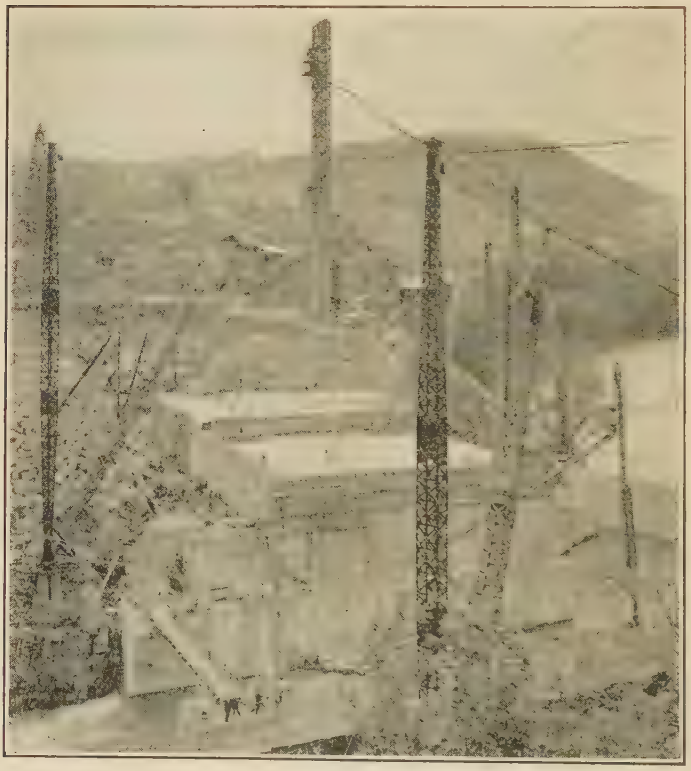


Fig. 428.—Gilboa dam during construction. (*Courtesy of the Hugh Nawn Contracting Co.*)

main dam is 971 feet long, and there is a masonry waste weir 452 feet long. The maximum height of the main dam above the lowest point of the cutoff trench is 228 feet. It has already been mentioned that upward pressure was assumed on the base of this dam. The rock was excavated to an average depth of about 13 feet, and the cutoff trench, 20 feet wide, was carried 14 feet deeper. The dam and waste weir are entirely of rubble, with ashlar facing. The top of the dam is 20 feet above "full reservoir." Ice pressure of 47,000 pounds per

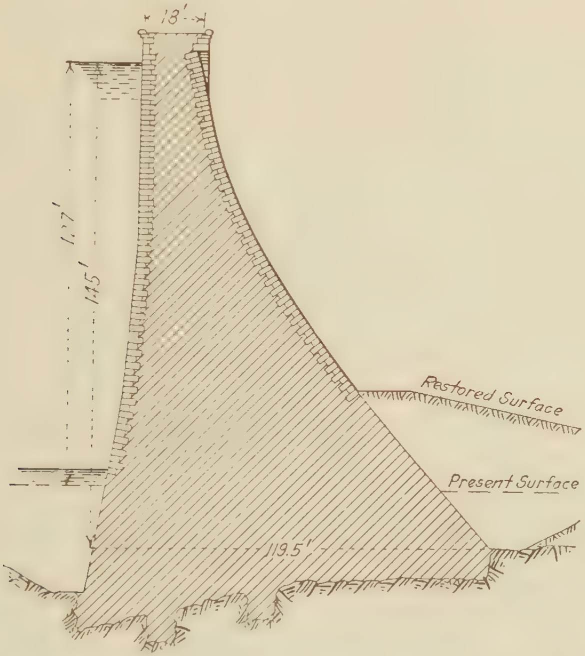


Fig. 429.—Section of Wachusett dam for Metropolitan Water District of Boston.

linear foot was allowed for. The upward pressure on the base was taken as the full head for the width of the cutoff wall, then reduced to two-thirds, and then diminished uniformly to zero at the toe. Figure 430 shows a view of the overfall section.

The dam across the Colorado River near Austin, Tex., was an overflow dam about 60 feet high. It failed in 1900 after a heavy freshet. The downstream toe had been undermined, although on a rock foundation, but of poor character, and the failure was due to sliding. The water was 11 feet above the top of the dam.[1]

Figure 431 is a view of the Wilson dam at Muscle Shoals, Tenn.

[1] Taylor, T. U., "The Austin Dam," *Water Supply and Irrigation Paper* 40, U. S. Geological Survey.

Many notable dams, some of great height, have been built by the Reclamation Service of the U. S. Geological Survey. Among the highest of these are the following:

Roosevelt dam, Arizona: height 240 feet above mean low water in the stream, and 284 feet from lowest point of foundation to parapet; width 16 feet on top and about 170 feet at the base; length on crest 1,080 feet; curved to a radius of 400 feet; built of cyclopean rubble (Fig. 432).

Fig. 430.—Wachusett dam.

Arrowrock dam, Idaho: maximum height above foundation, 354 feet; width 15.5 feet on top and 238 feet at base; length 1,100 feet on top and about 200 feet at the river bed; curved to a radius of 670 feet at the parapet, but designed without regard to arch action; has radial contraction joints about 100 feet apart.

Elephant Butte dam, Texas, across the Rio Grande: maximum height 300 feet, of which 90 feet is below the river bed; 1,200 feet long, straight; upstream batter 1:16, downstream 2:3 to the river bed and 1:1 below; top width 20 feet (Fig. 433).

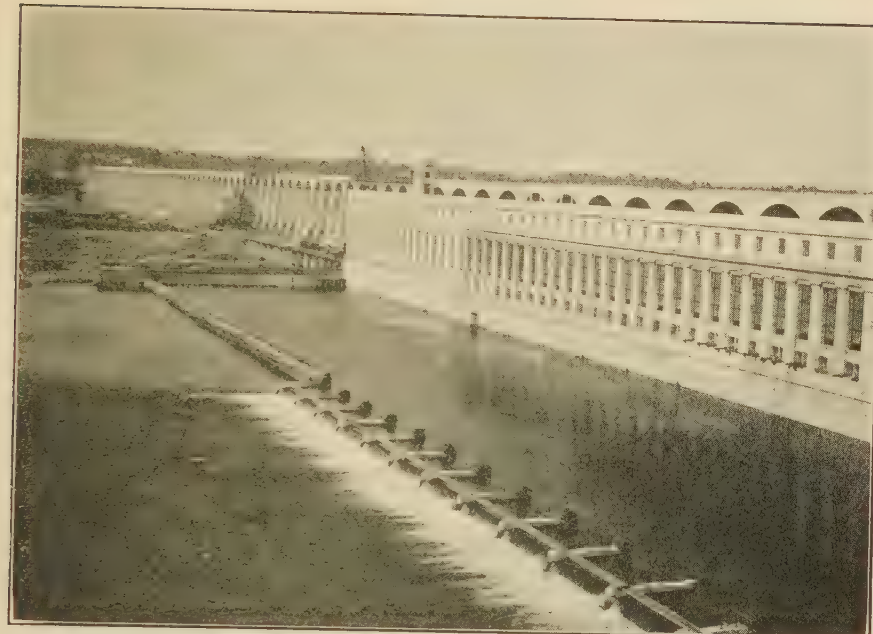


FIG. 431.—Wilson dam at Muscle Shoals, Tenn.

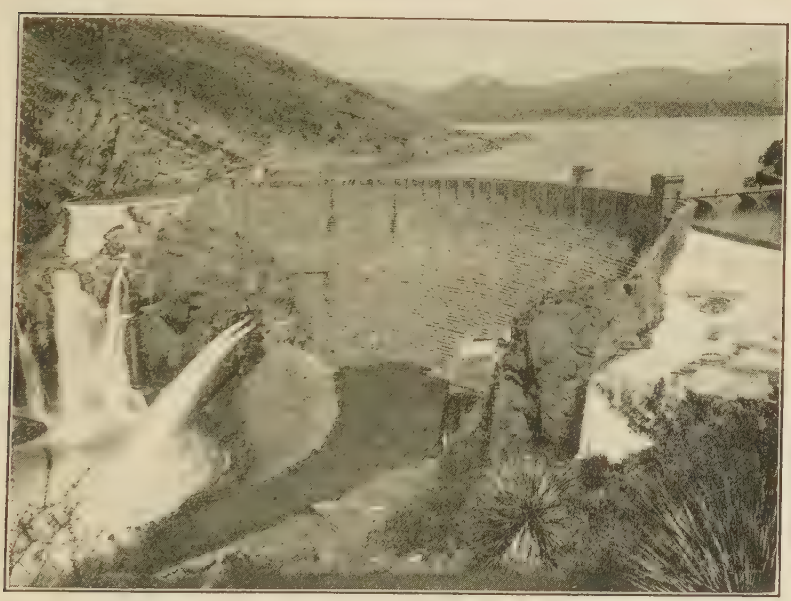


FIG. 432.—Roosevelt dam, Arizona. (*Courtesy of U. S. Reclamation Service.*)

Shoshone dam, Wyoming: maximum height from lowest point of foundation, 328.4 feet; the canyon is only 70 feet wide at the river bed and 200 feet at the crest, the granite walls being nearly vertical. The foundations were carried 85 feet below the river bed. The place is an ideal one for an arch dam, and this one is built of Portland cement concrete in which large stones up to 200 pounds in weight were imbedded, and is curved with a radius of 150 feet to the center line of the crest. The width on top is 10 feet, and 108 feet at the river bed and the same below. The flow line is 10 feet below the top. The section is a trapezoid, the upstream face battered 0.15:1 and the downstream face 0.25:1. The spillway, 300 feet long, discharges into a trench blasted out of the

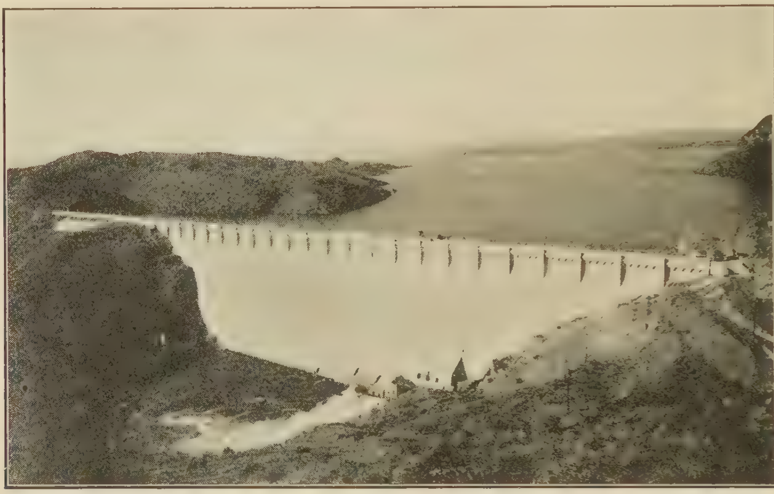


FIG. 433.—Elephant Butte dam, Texas. (*Courtesy of U. S. Reclamation Service.*)

rock walls of the canyon into a tunnel 20 by 20 feet in section and 500 feet long.

The Hetch Hetchy dam of the San Francisco water supply, called the O'Shaughnessy dam for the City Engineer, M. M. O'Shaughnessy, under whose supervision it has been built, is of the arched gravity type with a radius of 700 feet, built of cyclopean concrete with stone blocks up to 5 or 6 cubic yards in volume. It is 226.5 feet above the original stream bed, and has a maximum height above foundations of 344.5 feet. The thickness of the base is 298 feet. The upstream face has a slope of 1 in 20. There is an offset on the downstream side of 80.25 feet at the level of the stream bed, with provision for increasing the height 85.5 feet, with the same width of base. The crest is 6.5 feet above the normal water level. It has 18 siphon spillways each 4 by 10 feet at the throat, placed in three series at slightly different levels. The length of the crest is 605 feet, and the thickness 15 feet. When the height is increased as

provided for, its maximum height will be 430 feet, or higher than any existing dam.

26. Hollow Dams.—Just as a dam may consist of piers extending upstream and downstream with arches between, so, instead of arches,

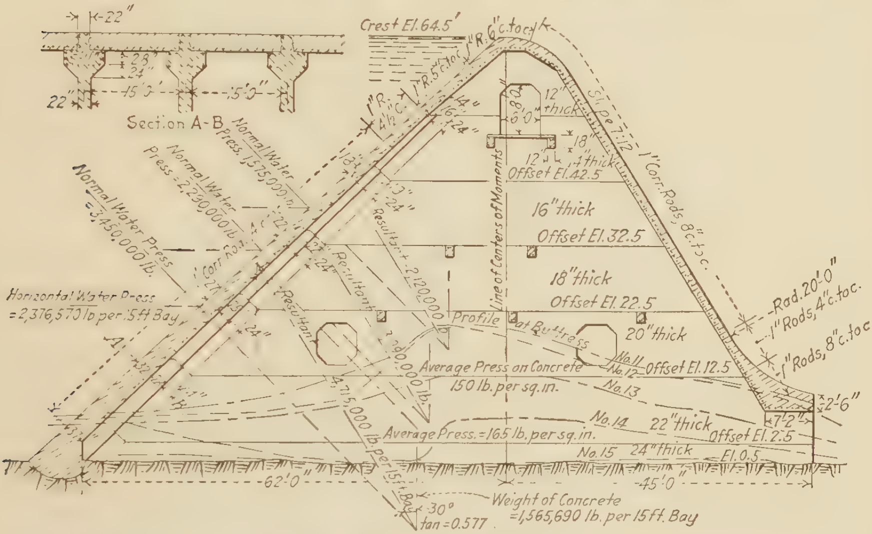


Fig. 434.—Ellsworth dam, Maine.

there may be flat concrete slabs, reinforced. Such a dam will consist of a continuous inclined slab supported by piers. There will be no bottom on which upward pressure may be exerted, except the bottom of the piers.

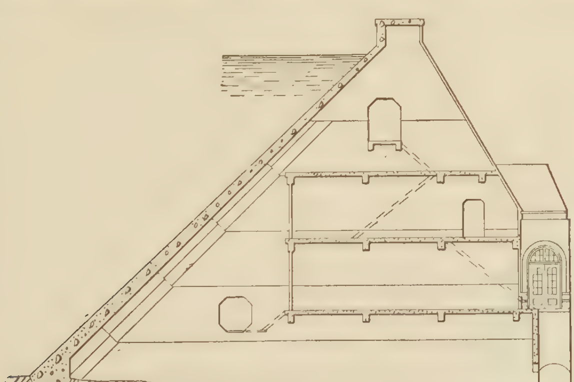


Fig. 435.—Ellsworth dam.

The latter may be, say, 12 to 30 feet on centers. Percolation under the dam, if on pervious material, may be prevented or reduced by a cutoff wall at the bottom of the slab. There may be openings in the piers,

connected by a footway supported on beams of wood or concrete. There may be horizontal platforms between the piers above the bed of the stream, which may be used for any desired purpose, even for the power machinery, the water being brought to the wheels by vertical or inclined penstocks

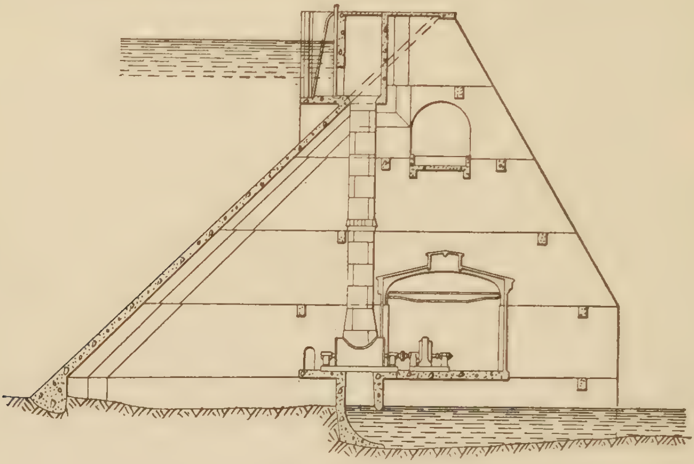


Fig. 436.—Hollow dam.

through the dam. If the dam is an overflow weir, the downstream face, instead of being open, may be covered by a curved slab of any desired shape, with an apron below.

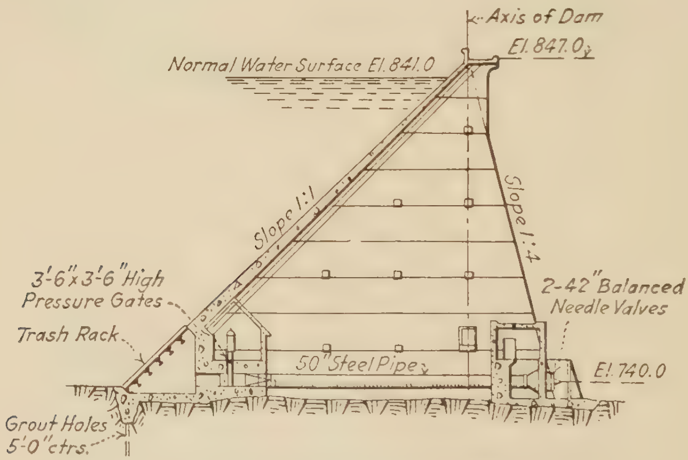


Fig. 437.—Stony Gorge dam. (*U. S. Reclamation Service.*)

This type of dam has often been provided with a base connecting the piers, on the river bed. In such case the base may be exposed to upward pressure, and on permeable soil such upward pressure should always be

assumed. Weep holes have in some dams been made through the base, to relieve the upward pressure. The trouble with weep holes, as with underdrainage, is that, if the holes or drains freeze up, the pressure is unrelieved. Floors may be built between the piers, at any desired levels, forming a series of rooms which may be lighted and used as required.

The underlying principle of this form of dam is to make the hydrostatic pressure exert a considerable downward pressure on the dam, and so have a stabilizing effect, instead of having purely an overturning effect as it does when the upper face is vertical. The principle is certainly sound. The resultant hydrostatic pressure alone may act near the center of the base, dependent upon the slope of the slab and of the downstream face. Many dams of this type were built by the Ambursen Hydraulic Construction Company, of Boston. One of these dams, at Stony Creek, West Virginia, failed due to the shallow depth of the cutoff wall, and the percolation of water beneath it, where it was not on a rock foundation.

One of the dams built by this company is that at Ellsworth, Me. It is 64 feet high, and the section of overflow is shown in Fig. 434. The foundation is solid granite. There are rooms under the dam at one end, as shown in Fig. 435. The power house is located just below this end.

A dam of this kind with the power house inside the dam is shown in Fig. 436, while Fig. 437 shows the Stony Gorge dam of the U.S. Reclamation Service.

OTHER KINDS OF DAMS

27. Earth Dams.—An earth dam[1] is merely a bank of earth. Its design is a matter of practice, not of theory. The important point is to make it tight, so that water will penetrate neither through it nor under it. The foundation must therefore be carefully prepared, by stripping off loose earth and loam, and getting down to good material, which should be compacted by rolling and wetting. There should generally be a core, of masonry, concrete or clay, carried down in a cutoff trench to an impervious stratum if possible. The width of the core should be, say, 2 to 8 feet at top, battered on both faces, to a width at the ground level of not less than one-sixth the head. In the trench the thickness may be uniform, or may diminish toward the bottom. Figure 438 gives a good section of a recent earth dam. If the core can be carried down to rock, so much the better, as in the figure. Often a line of sheet piling is driven along the heel.

[1] For details regarding earth dams, see the excellent chapter in WEGMANN's book, and the paper by J. D. JUSTIN, in *Trans. Am. Soc. C. E.*, vol. LXXXVII, p. 1, 1924.

Also the late J. D. SCHUYLER, in *Trans. Am. Soc. C. E.*, June, 1907, and in his valuable book on "Reservoirs for Irrigation, Water Power and Domestic Water Supply," John Wiley & Sons Inc., New York, 1901, gives much valuable information derived from his long practice as one of the leading engineers in the United States in this line of work.

If of puddle, the core should be placed in layers, rolled and wetted. It should best be of clayey gravel or of hardpan, not of clay alone, for clay shrinks and swells, and is dangerous if alternately wet and dry. Selected material should be used above the core, also laid in layers. A puddle core should be thicker than a concrete or masonry core. The upper side of a concrete or masonry core should be made as tight as possible. It is not designed to resist hydrostatic pressure, but to be impervious.

The top width of an earth dam may be from 10 to 30 feet, and not less than 5 feet above high-water level. The upstream slope should be paved, and is generally flatter than the downstream slope, say from 2:1 to 3:1, while the downstream slope is from 1½:1 to 2½:1. Sometimes a layer of puddle is placed on the upstream slope just under the paving.

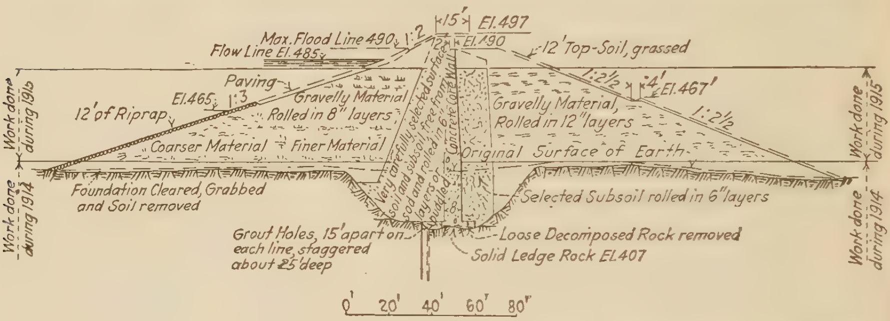


FIG. 438.—Phelps Brook dam, of Hartford water works. (*Courtesy of C. M. Soville, Chief Engineer of the Board.*)

The highest earth dam is said to be the Necaxa dam No. 2, in Mexico, 190 feet high, and 54 feet wide on top, with upstream slope of 3:1 and downstream slope of 2 :1.

In India, large reservoirs or "tanks" have been made for irrigation, with earth dams miles in length.

Discharge pipes are carried through the dam, sometimes in a conduit. Special precautions must be taken to prevent precolation along the pipes or conduit.

There have been many failures of earth dams, generally due to percolation of water and washing out of the dam or foundation, or to insufficient waste ways and consequent overtopping and washing out of the dam by a flood.

28. Hydraulic-fill dams are made by sluicing in the material for the dam in a pipe or flume, carried by a current of water at high velocity, and deposited. The process is described by J. D. Schuyler, in a paper[1] on "Recent Practice in Hydraulic-fill Dam Construction."

Allen Hazen,[2] in the latest paper on this subject says: "The hydraulic method of dam construction has great advantages, but too many dams

[1] *Trans. Am. Soc. C. E.*, June, 1907.

[2] HAZEN, ALLEN, *Trans. Am. Soc. C. E.*, vol. LXXXIII, pp. 1713–1821, 1919–1920 (with discussion).

built by it have failed." He also gives his views as to the causes of failure and the proper method of construction.

29. Rock-fill Dams.—A rock-fill dam is

. . . An embankment consisting of rock dumped loosely except at the faces, where it is laid carefully as dry slope walls. Watertightness is insured by a sheeting of boards or a facing of concrete on the water slope, or by building an earthen dam against the inner or outer slope.[1]

Leakage through a rock-fill dam, however, does no harm if the dam is on proper foundation. There have been failures, however, of rock-fill dams.

Such dams are fully described in the paper[2] by J. D. Schuyler on "Reservoirs for Irrigation," and in Wegmann's book. Rock-fill dams have been built to a height of over 100 feet. In the Lower Otay dam, near San Diego, Cal., which is 130 feet high, there is "a core of steel plates in the center of the fill, forming a web plate across the canyon."

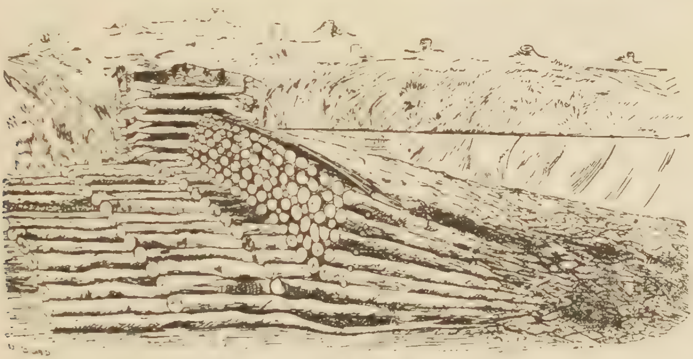


FIG. 439.—Log dam for soft or sandy bottoms.

30. Timber Dams.—Great numbers of timber dams have been built in the United States, generally of small height and for small water powers. They may be (1) of logs laid across the stream, tied back by logs laid upstream and downstream, covered by stone or earth, somewhat as in Fig. 439; (2) of more elaborate timber cribs, filled with stone, somewhat as in Fig. 440; (3) of timber frames extending upstream and downstream with the upstream face planked and backfilled with earth. There is generally a wooden apron on the downstream side to prevent erosion of the bed. Timber dams are described in the book on the "Construction of Mill Dams," published by the Leffel Turbine Water Wheel Company, and in Wegmann's book.

31. Steel Dams.—A few steel dams have been built, consisting of steel frames extending upstream and downstream, with the inclined up-

[1] WEGMANN, p. 266.
[2] *Eighteenth Annual Report*, U. S. Geological Survey, 1896–1897.

stream face covered with steel plates, either plane or curved, riveted to the frames. The foundation should be on rock. The bottom of the upstream face is made tight by being placed in a trench in the rock and surrounded with concrete. Steel dams are described in Wegmann's book, and in a paper[1] by F. H. Bainbridge on "Structural Steel Dams."

32. Movable dams are used on rivers for the control of navigation, in places where it is desirable at low stages of the water to form a pond, while at high stages it is desirable to remove the dam entirely. There are many forms of such dams. Some consist of frames, connected together, against which planks or "needles" are placed; the needles can be removed and the frames folded down on to the foundation. Some are solid wickets which can be raised or lowered as desired. In some

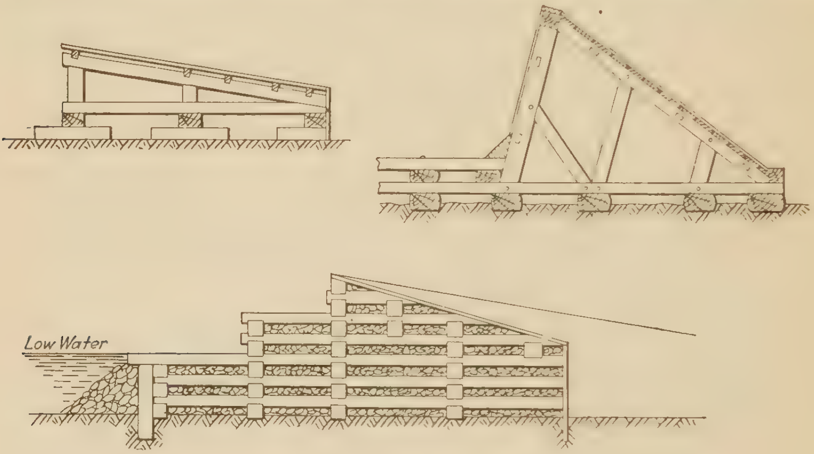


Fig. 440.—Timber crib dams.

cases there are piers, between which curtains can be rolled down, or rolling sluice gates can be raised or lowered from a bridge on top. Movable dams have been largely used on the rivers of Europe, on the Ohio River and its tributaries, on the Chicago Drainage Canal and the New York State Barge Canal, and elsewhere. The subject is a specialized one, and should be studied in the works of authorities.[2]

33. Coffer dams are dams constructed to keep the water out in building foundations, or around any area in which underwater operations are being carried on. They are either a single row of wooden or steel sheet piling, or a double row, with filling between. They are fully discussed in works on Foundations.

34. Failures of Dams.—A great many dams have failed, both in this country and abroad. In some cases these have been due to excessive

[1] *Jour. Western Soc. Eng.*, October, 1905.

[2] Vernon-Harcourt, "Rivers and Canals;" Hilgard, K. E., "Die bewegliche Wehre," "Handbuch der Ingenieurwissenschaften," third part, Der Wasserbau, Leipzig, 1912; and Wegmann, "Construction of Dams."

floods, which could not well have been foreseen, and which may be included among "Acts of God." Most failures, however, have been due to faulty design or construction, or to poor materials. In many cases adequate provision has not been made against percolation of water through the dam, or along the outlet conduits, or under the foundation, and the dam has been washed out in consequence. In some cases upward pressure should have been allowed on the base, and the failure to do this has been the cause of the catastrophe. The majority of failures have been due to the foundation.

Some cases of failure, and their causes, have been referred to in this chapter. In the *Journal of Electricity* for Mar. 15, 1920, Lars Jorgensen has given the "Record of 100 Dam Failures," and this should be consulted by the reader who wishes to pursue the subject further. Edward Godfrey, in his very interesting book on "Engineering Failures and their Lessons," 1924, has a valuable chapter on the subject, which should by all means be studied. He maintains that upward pressure should be assumed on all dams, not only on the base, but on all horizontal planes, because masonry is permeable. It certainly would be safest to follow this advice; and yet the fact that many dams are standing, with no sign of failure, in which it was not done, would seem to show that it is not necessary in all cases, notwithstanding the fact that Godfrey cites several dams which finally failed after standing a long time. Certain it is that a high masonry dam should be carried to a rock foundation, and that the core of an earthen dam should be carried to rock or hardpan if possible; and that great care should be taken to prevent percolation of water under the dam. If a dam is founded on sand or gravel, as is sometimes necessary, the head will no doubt cause percolation through the underlying strata, as discussed in Art. 17; and in such cases upward pressure on the base should be allowed, and the utmost care should be taken to diminish the rapidity of percolation to such a velocity as will not wash out the fine particles. Percolation under dams has no doubt been the cause of most failures. Whether water permeating through the masonry of a dam, with a free outlet, will cause upward pressure on any horizontal joint is doubtful. The upstream face of a dam should, however, be made as tight as practicable, by pointing the joints of stonework, and by plastering or otherwise treating a concrete surface, besides using a mortar or concrete which is as impermeable as possible.

35. References.—The outstanding work on Dams is the elaborate treatise[1] on "The Design and Construction of Dams," by Edward Wegmann, member of A.S.C.E. This is a wonderfully complete book, giving the theory, typical profiles, and complete descriptions of all kinds of dams, and of most of the important dams which have been built. It is illustrated by more than a hundred large plates. It also contains an extended

[1] Seventh ed., John Wiley & Sons, Inc., New York, 1922.

bibliography of the subject, which renders unnecessary any further references here. This book should be in the hands of every engineer who is engaged in the construction of dams. I am much indebted to it for a great deal of the information given in this chapter, and here gratefully acknowledge my indebtedness. Mr. Wegmann made the studies for the Quaker Bridge dam and for other dams, and has been widely consulted in connection with the construction of dams.

INDEX

Q

R

S

T

72567